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# COLLEGE ALGEBRA



PRENTICE-HALL MATHEMATICS SERIES

ALBERT A. BENNETT, EDITOR



# COLLEGE ALGEBRA

By

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**To**  
**E. N. and L.**







## Preface for the Instructor

In this textbook the author has attempted to combine lucid explanation of procedures with reasonable motivation for and justification of the processes, at the student's level.

*The early chapters constitute a thorough review of elementary algebra arranged as a self-contained development from first principles, with careful explanations of the most elementary matters. No accurate recollection of high school algebra is required if the book is taken from the beginning.* However, the review is presented in a more mature way than would be possible with first-year high school students; this plan enables the instructor to take up the material with a group of students having widely varying degrees of preparation without boring the better-prepared students.

*The subject matter is for the most part conventional. Chapter I on the number system seems to the author to be a much neglected necessity.* The usual theorems in the theory of equations, or even the elementary problems on the character of the roots of a quadratic, can have little meaning to a student who has not had some discussion of rational, irrational, real, and imaginary numbers such as that in Chapter I. *A suitable starting point in the book can be found for almost any class regardless of the degree of preparation. However, the inclusion of much, if not all, of Chapter I is recommended, no matter at what point in the book the course is begun.*

Technical terms are carefully defined, and ample lists of exercises are provided. *Starred exercises are more difficult than others, and starred sections may be omitted in a short course without disturbing the continuity of the book.*

*Stress is laid upon explanation of fundamental concepts and reasoning which the mathematics and science major is often over-optimistically expected to absorb by osmosis. The author is committed to the belief that a process understood will be remembered better and longer than one which is merely memorized.* The mathematics itself is, it is hoped, accurate. This does not mean that rigorous proofs are presented even when above the freshman level, but rather that bad proofs are not presented as good ones. When a correct proof is thought to be too difficult for the student, an



informal, heuristic discussion, designed to make the result plausible, is presented but not falsely called a proof.\* For a class which is going to omit a certain proof, it will be no more difficult to omit a correct proof than an inadequate one; but to teach incorrect reasoning as though it were correct can only undermine the student's future progress in and understanding of mathematics. Despite the inevitable compromise with rigor in an elementary book, the author has tried to avoid statements which would have to be unlearned by the student at a later stage.

Some topics are treated more thoroughly than is usual, as the table of contents indicates. It is hoped that this thorough treatment will increase the value of the book as a reference work for future study for the student who goes on with mathematics. For example, the student of integral calculus may be pleased to find the simple proof of the theorem on partial fractions, which is seldom found in textbooks on either college algebra or calculus.

Many practical applications are included in the exercises. But motivation does not necessarily mean introducing the subject by means of so-called "practical" problems that are either artificially constructed for the purpose or beyond the student's background; it may be purely mathematical. See, for example, the discussion preceding the standard theorem on rational roots of equations with integral coefficients. Due attention is paid to historical matters and to the relevance of the subject matter to other sciences and to civilization.

*The later chapters are arranged as far as possible to be independent of each other, to provide for maximum elasticity in the choice of material. Where earlier material is prerequisite, cross references are provided. The various chapters need not always be taken in the order of their appearance.* For example, the instructor may prefer to omit Chapter I at the start and take parts of it as needed. Similarly, the chapter on logarithms may be taken immediately after the chapter on exponents. The chapter on progressions may be inserted much earlier, provided the starred sections are omitted.

The author is indebted to Professor Harris F. MacNeish for permission to include selected Brooklyn College final examination questions among the exercises; to The Macmillan Company, for permission to include approximately eight pages of scattered, mis-

\* In avoiding certain errors which appear to have become standardized, it is possible that the author has committed others; he will be glad to have his attention called to these.



cellaneous material taken verbatim from his book *Fundamentals of Mathematics*, of which they are the publishers, and for permission to adapt Tables I, IV, V, VI, and VII from the *Macmillan Logarithmic and Trigonometric Tables*, Revised Edition, edited by E. R. Hedrick; to Miss Helen Halpern, for her able and conscientious assistance in checking the answers to exercises and typing the manuscript; and to Prentice-Hall, Inc., for their co-operation and efficiency in publishing this book expeditiously during a trying period.

M.R.







# Table of Contents

CHAPTER

|                                                                              | PAGE |
|------------------------------------------------------------------------------|------|
| PREFACE FOR THE STUDENT. . . . .                                             | 1    |
| 1. A brief lesson in logic. . . . .                                          | 1    |
| 2. Glossary of technical symbols . . . . .                                   | 5    |
| I. THE NUMBER SYSTEM OF ALGEBRA. . . . .                                     | 7    |
| 3. Introduction. . . . .                                                     | 7    |
| 4. The natural numbers. Addition and multiplication. . . . .                 | 7    |
| 5. Subtraction and division of natural numbers. . . . .                      | 12   |
| 6. Even, odd, and prime numbers. H.C.F. and L.C.M.. . . .                    | 14   |
| 7. Fractions. Multiplication and division . . . . .                          | 17   |
| 8. Fractions. Addition and subtraction. . . . .                              | 21   |
| 9. Directed numbers. Addition and multiplication. . . . .                    | 24   |
| 10. Directed numbers. Subtraction and division . . . . .                     | 28   |
| 11. The system of rational numbers. Properties of zero . . . .               | 32   |
| 12. Positive integral powers and roots . . . . .                             | 34   |
| 13. The square root of 2 . . . . .                                           | 37   |
| 14. The system of real numbers. Decimal notation . . . . .                   | 41   |
| *15. Extraction of square roots. . . . .                                     | 45   |
| 16. The system of complex numbers . . . . .                                  | 47   |
| *17. Conclusion. The axioms of algebra. . . . .                              | 51   |
| II. WHAT IS ALGEBRA? . . . . .                                               | 53   |
| 18. Algebra as a language. . . . .                                           | 53   |
| 19. Algebra as a logical science . . . . .                                   | 54   |
| 20. Algebra as a collection of techniques of calculation . . . .             | 54   |
| 21. Algebra as a branch of human endeavor. . . . .                           | 54   |
| 22. Algebra as a collection of puzzles. . . . .                              | 55   |
| III. ALGEBRAIC EXPRESSIONS . . . . .                                         | 56   |
| 23. Positive integral exponents and radicals. Order of operations . . . . .  | 56   |
| 24. Algebraic expressions . . . . .                                          | 59   |
| 25. Removal and insertion of parentheses. Combination of like terms. . . . . | 62   |
| 26. Standard form of polynomials . . . . .                                   | 64   |
| 27. Equations and identities. . . . .                                        | 66   |
| 28. Verification of identities. Reversible steps . . . . .                   | 67   |
| 29. Elementary operations with equations. . . . .                            | 69   |
| 30. Standard form of polynomial equations. Degree of an equation. . . . .    | 71   |
| IV. FUNCTIONS AND GRAPHS . . . . .                                           | 73   |
| 31. Functions . . . . .                                                      | 73   |
| 32. Functional notation. . . . .                                             | 75   |
| 33. Applications of functions in science. Formulas . . . . .                 | 76   |

\* Sections starred may be omitted without disturbing the continuity of the chapter.



IV. FUNCTIONS AND GRAPHS (*Continued*)

|                                                     |    |
|-----------------------------------------------------|----|
| 34. Rectangular coordinates. . . . .                | 77 |
| 35. The graphs of equations and functions . . . . . | 79 |
| 36. Continuous functions . . . . .                  | 81 |
| 37. Graphical solution of equations. . . . .        | 82 |
| *38. Infinities of a rational function. . . . .     | 84 |

V. ELEMENTARY OPERATIONS WITH POLYNOMIALS. . . . . 86

|                                                               |    |
|---------------------------------------------------------------|----|
| 39. Integers and polynomials. Hindu-Arabic notation . . . . . | 86 |
| 40. Addition and subtraction of polynomials. . . . .          | 87 |
| 41. Multiplication of polynomials . . . . .                   | 89 |
| 42. Division by monomials . . . . .                           | 90 |
| 43. The division algorithm . . . . .                          | 91 |

VI. FACTORING OF POLYNOMIALS . . . . . 96

|                                                     |     |
|-----------------------------------------------------|-----|
| 44. Prime factors . . . . .                         | 96  |
| 45. Special products . . . . .                      | 97  |
| 46. Simple types of factoring . . . . .             | 98  |
| 47. Factoring by grouping. . . . .                  | 100 |
| 48. Quadratic trinomials . . . . .                  | 101 |
| *49. Sum and difference of two like powers. . . . . | 102 |
| 50. H.C.F. and L.C.M.. . . . .                      | 103 |

VII. ELEMENTARY OPERATIONS WITH FRACTIONAL EXPRESSIONS . . . . . 105

|                                                                    |     |
|--------------------------------------------------------------------|-----|
| 51. Reduction to lowest terms. . . . .                             | 105 |
| 52. Multiplication and division of fractional expressions. . . . . | 106 |
| 53. Addition and subtraction of fractional expressions . . . . .   | 108 |
| 54. Complex fractions . . . . .                                    | 110 |

VIII. LINEAR EQUATIONS AND LINEAR FUNCTIONS. . . . . 114

|                                                                              |     |
|------------------------------------------------------------------------------|-----|
| 55. Solution of linear equations in one variable . . . . .                   | 114 |
| 56. Equations leading to linear equations. . . . .                           | 116 |
| 57. Literal equations and formulas. . . . .                                  | 117 |
| 58. Graph of a linear function. . . . .                                      | 119 |
| 59. Systems of two linear equations in two unknowns . . . . .                | 123 |
| 60. Graphical interpretation. Inconsistent and dependent equations . . . . . | 126 |
| 61. Systems of three linear equations in three unknowns . . . . .            | 128 |
| 62. Systems of equations reducible to linear equations . . . . .             | 130 |
| 63. Verbal problems . . . . .                                                | 133 |

IX. INTEGRAL AND FRACTIONAL EXPONENTS. . . . . 140

|                                                           |     |
|-----------------------------------------------------------|-----|
| 64. Negative integers and zero as exponents. . . . .      | 140 |
| 65. A use of powers of ten in scientific writing. . . . . | 142 |
| 66. Fractional exponents . . . . .                        | 144 |

X. RADICALS. . . . . 147

|                                                                  |     |
|------------------------------------------------------------------|-----|
| 67. Basic laws for radicals. . . . .                             | 147 |
| 68. The simplest form of a radical . . . . .                     | 148 |
| 69. Addition and subtraction of radicals. . . . .                | 150 |
| 70. Multiplication with radicals of the same index . . . . .     | 151 |
| 71. Division with radicals. Rationalizing denominators . . . . . | 152 |
| 72. A use of fractional exponents. . . . .                       | 153 |



# TABLE OF CONTENTS

xiii

CHAPTER

PAGE

## XI. QUADRATIC EQUATIONS AND QUADRATIC FUNCTIONS. . . . . 155

- 73. Solution of quadratic equations by factoring. . . . . 155
- 74. Completing the square . . . . . 156
- 75. The quadratic formula . . . . . 158
- 76. Character of the roots. . . . . 160
- 77. The graph of a quadratic function. Graphical solution of quadratic equations. . . . . 162
- 78. The sum and product of the roots . . . . . 164
- 79. Equations leading to quadratics . . . . . 166
- 80. Irrational equations. . . . . 168
- 81. Verbal problems . . . . . 169
- \*82. Maxima and minima of quadratic functions . . . . . 172

## XII. SYSTEMS OF EQUATIONS IN TWO UNKNOWNNS INVOLVING QUADRATICS. . . . . 176

- 83. The graph of a quadratic equation in  $x$  and  $y$  . . . . . 176
- 84. Graphical solution of simultaneous equations. . . . . 178
- 85. Algebraic solution when one equation is linear and one quadratic . . . . . 179
- 86. Algebraic solution when both equations are quadratic . . . 181
- 87. Verbal problems . . . . . 185

## XIII. RATIO, PROPORTION, AND VARIATION. . . . . 187

- 88. Ratio and proportion . . . . . 187
- 89. The terminology of variation. . . . . 191

## XIV. COMPLEX NUMBERS . . . . . 195

- 90. The standard form of a complex number. . . . . 195
- 91. Elementary operations with complex numbers . . . . . 197
- 92. Graphical representation of complex numbers . . . . . 199
- 93. Graphical significance of addition of complex numbers . . 200
- 94. Review of trigonometric functions . . . . . 202
- 95. Trigonometric or polar form of a complex number . . . . . 205
- 96. Graphical meaning of multiplication and division of complex numbers. . . . . 207
- 97. De Moivre's theorem . . . . . 209
- 98. The  $n$ th roots of a complex number. . . . . 210
- 99. Solution of pure equations of  $n$ th degree. . . . . 213

## XV. THEORY OF EQUATIONS . . . . . 214

- 100. The fundamental theorem of algebra . . . . . 214
- 101. The remainder theorem and the factor theorem . . . . . 216
- 102. Synthetic division . . . . . 218
- 103. The depressed equation . . . . . 221
- 104. Factorization of polynomials. The number of roots of an equation. . . . . 223
- \*105. Proof of the uniqueness of factorization . . . . . 226
- 106. Imaginary roots of equations with real coefficients . . . . 227
- 107. Rational roots of equations with integral coefficients . . . 230
- 108. Graphical solution of equations. . . . . 235



XV. THEORY OF EQUATIONS (*Continued*)

|                                                                    |     |
|--------------------------------------------------------------------|-----|
| 109. Transformation reversing the signs of the roots . . . . .     | 236 |
| 110. Upper and lower bounds for the roots . . . . .                | 238 |
| 111. Descartes' rule of signs . . . . .                            | 240 |
| 112. Real roots by successive approximations . . . . .             | 244 |
| 113. Linear interpolation . . . . .                                | 247 |
| 114. Transformations of equations . . . . .                        | 249 |
| 115. Horner's method . . . . .                                     | 252 |
| 116. Relations between coefficients and roots . . . . .            | 255 |
| 117. Algebraic solution of equations . . . . .                     | 258 |
| *118. Algebraic solution of the general cubic equation . . . . .   | 259 |
| *119. Algebraic solution of the general quartic equation . . . . . | 260 |

## XVI. DETERMINANTS AND ELIMINATION THEORY . . . . . 263

|                                                                               |     |
|-------------------------------------------------------------------------------|-----|
| 120. Solution of linear systems in two unknowns by determinants . . . . .     | 263 |
| 121. Determinants of third order . . . . .                                    | 265 |
| 122. Solution of linear equations in three unknowns by determinants . . . . . | 267 |
| *123. Area of a triangle by determinants . . . . .                            | 269 |
| 124. Determinants of order $n$ . . . . .                                      | 271 |
| 125. Some properties of determinants . . . . .                                | 273 |
| 126. Expansion by minors . . . . .                                            | 278 |
| 127. Solution of linear systems in $n$ unknowns by determinants . . . . .     | 283 |
| 128. Systems of homogeneous linear equations . . . . .                        | 285 |
| 129. Systems of $m$ linear equations in $n$ unknowns . . . . .                | 287 |
| *130. The resultant of two polynomials . . . . .                              | 290 |
| *131. Systems of higher degree in two unknowns . . . . .                      | 292 |

## XVII. PERMUTATIONS AND COMBINATIONS . . . . . 295

|                                                                         |     |
|-------------------------------------------------------------------------|-----|
| 132. Fundamental principles . . . . .                                   | 295 |
| 133. Permutations of $n$ distinct objects taken $r$ at a time . . . . . | 298 |
| 134. Permutations where some things are alike . . . . .                 | 301 |
| 135. Circular permutations . . . . .                                    | 302 |
| 136. Combinations of $n$ distinct objects taken $r$ at a time . . . . . | 303 |

## XVIII. PROBABILITY . . . . . 306

|                                                     |     |
|-----------------------------------------------------|-----|
| 137. Definition of probability . . . . .            | 306 |
| 138. Statistical probability . . . . .              | 309 |
| 139. Some theorems concerning probability . . . . . | 309 |

## XIX. MATHEMATICAL INDUCTION . . . . . 315

|                                                    |     |
|----------------------------------------------------|-----|
| 140. The $n$ th term of a sequence . . . . .       | 315 |
| 141. The axiom of mathematical induction . . . . . | 316 |
| 142. Proofs by mathematical induction . . . . .    | 317 |

## XX. THE BINOMIAL THEOREM . . . . . 322

|                                                                         |     |
|-------------------------------------------------------------------------|-----|
| 143. The binomial theorem for positive integral exponents . . . . .     | 322 |
| *144. Proof of the binomial theorem by mathematical induction . . . . . | 327 |
| 145. The binomial series . . . . .                                      | 328 |



# TABLE OF CONTENTS

xv

| CHAPTER                                                            | PAGE |
|--------------------------------------------------------------------|------|
| XXI. PROGRESSIONS . . . . .                                        | 330  |
| 146. Arithmetic progressions . . . . .                             | 330  |
| 147. Geometric progressions . . . . .                              | 333  |
| 148. Harmonic progressions . . . . .                               | 336  |
| 149. Infinite geometric series. Periodic decimals . . . . .        | 337  |
| *150. Differences of $r$ th order . . . . .                        | 339  |
| *151. Arithmetic progressions of $r$ th order. . . . .             | 341  |
| XXII. INEQUALITIES . . . . .                                       | 344  |
| 152. Inequalities . . . . .                                        | 344  |
| 153. Properties of inequalities . . . . .                          | 345  |
| 154. Solution of inequalities . . . . .                            | 346  |
| 155. Graphical significance of inequalities . . . . .              | 347  |
| XXIII. LOGARITHMS. . . . .                                         | 349  |
| 156. Definition of logarithm . . . . .                             | 349  |
| 157. Properties of logarithms. . . . .                             | 350  |
| 158. Common logarithms . . . . .                                   | 351  |
| 159. Interpolation by proportional parts. . . . .                  | 355  |
| 160. Significant figures. . . . .                                  | 356  |
| 161. Computation with logarithms . . . . .                         | 358  |
| *162. Cologarithms. . . . .                                        | 361  |
| 163. Natural logarithms. Computation of tables. . . . .            | 362  |
| 164. Exponential and logarithmic functions and equations . . . . . | 363  |
| XXIV. MATHEMATICS OF INVESTMENT. . . . .                           | 366  |
| 165. Simple interest and simple discount. . . . .                  | 366  |
| 166. Compound interest and compound discount . . . . .             | 368  |
| 167. Effective and nominal rates . . . . .                         | 371  |
| 168. Annuities . . . . .                                           | 372  |
| XXV. THE EUCLIDEAN ALGORITHM . . . . .                             | 376  |
| 169. The Euclidean algorithm for positive integers . . . . .       | 376  |
| 170. Euclidean algorithm for polynomials . . . . .                 | 379  |
| XXVI. PARTIAL FRACTIONS. . . . .                                   | 381  |
| 171. The theorem and its use. . . . .                              | 381  |
| 172. Proof of the theorem . . . . .                                | 387  |
| XXVII. INFINITE SERIES. . . . .                                    | 390  |
| 173. Introduction. . . . .                                         | 390  |
| 174. Limit of an infinite sequence. . . . .                        | 391  |
| 175. Convergence of infinite series. . . . .                       | 393  |
| 176. The comparison tests . . . . .                                | 394  |
| *177. The decimal expression for a real number . . . . .           | 397  |
| 178. The ratio tests. . . . .                                      | 399  |
| 179. Series whose terms are all negative . . . . .                 | 401  |
| 180. Alternating series. . . . .                                   | 401  |
| 181. Absolute convergence. . . . .                                 | 402  |
| 182. The general ratio tests . . . . .                             | 402  |
| 183. Power series. . . . .                                         | 403  |
| 184. The use of series in computation . . . . .                    | 405  |



TABLE OF CONTENTS

| CHAPTER                                                               | PAGE    |
|-----------------------------------------------------------------------|---------|
| XXVIII. INTERPOLATION AND CURVE FITTING . . . . .                     | 407     |
| 185. Linear interpolation. . . . .                                    | 407     |
| 186. Polynomial interpolation . . . . .                               | 409     |
| *187. Lagrange's interpolation formula . . . . .                      | 411     |
| 188. The Gregory-Newton interpolation formula . . . . .               | 412     |
| 189. Curve fitting and empirical equations. . . . .                   | 415     |
| 190. The method of differences. . . . .                               | 416     |
| 191. The method of averages. . . . .                                  | 420     |
| 192. The method of least squares . . . . .                            | 423     |
| 193. Exponential and power functions. . . . .                         | 427     |
| 194. Extrapolation . . . . .                                          | 430     |
| ANSWERS TO ODD-NUMBERED EXERCISES. . . . .                            | 433     |
| TABLES. . . . .                                                       | 457     |
| I. Powers, Roots, Reciprocals . . . . .                               | 457-458 |
| II. Trigonometric Functions. . . . .                                  | 459     |
| III. Four-place Logarithm Tables. . . . .                             | 460-461 |
| IV. Compound Interest . . . . .                                       | 462     |
| V. Compound Discount. . . . .                                         | 463     |
| VI. Amount of an Annuity. . . . .                                     | 464     |
| VII. Present Value of an Annuity . . . . .                            | 465     |
| VIII. Commissioners Standard Ordinary Mortality Table (1941). . . . . | 466     |
| INDEX. . . . .                                                        | 467     |



# COLLEGE ALGEBRA







## Preface for the Student

**1. A brief lesson in logic.** Mathematics is logical in nature, and mathematical writing is precise, compact, and logical in style. Therefore a brief review of certain elementary but fundamental logical concepts and terms is given here.

When two statements are so related that the second *must* be true *if* the first is true, we say that the second **follows from** the first, or the second is a **logical consequence** of the first, or the first **implies** the second. The first statement is called the **hypothesis** and the second is called the **conclusion**.

*Example 1. Hypothesis:  $x = 2$  and  $y = 5$ .*

*Conclusion:  $x + y = 7$ .*

If the conclusion of an argument really follows inescapably from the hypothesis, the argument is called **valid**. The process of drawing inescapable conclusions from given hypotheses is called **deduction** or **deductive reasoning**.

The proposition of example 1 may be written as: (a) *If  $x = 2$  and  $y = 5$ , then  $x + y = 7$ ;* or (b) *" $x = 2$  and  $y = 5$ " implies " $x + y = 7$ ."*

In a statement of the form "If  $A$  is true, then  $B$  is true," or " $A$  implies  $B$ ,"  $A$  is the *hypothesis* and  $B$  is the *conclusion*. This statement does not assert that  $A$  is true or that  $B$  is true. It asserts merely that *if  $A$  were true, then  $B$  would have to be true*. That is, *in a valid argument, the truth of the hypothesis guarantees the truth of the conclusion*.

*Example 2. "If today were Election Day,\* then tomorrow would be Wednesday"* is a valid argument even if today is neither Election Day nor any other Tuesday.

Note that the conclusion may be true even if the hypothesis is false. Thus, in example 2, the conclusion would be true if today were any Tuesday in the year.

Therefore, *in a valid argument, the truth of the conclusion does not guarantee the truth of the hypothesis. But the falsity of the conclusion does guarantee the falsity of the hypothesis.*

\* We assume that Election Day in the U.S.A. is, by definition, the Tuesday after the first Monday of November.



*Example 3.* Granting the axiom that "if equals are divided by equals the results are equal," we may reason as follows:

|                    |                             |                  |
|--------------------|-----------------------------|------------------|
| <i>Hypothesis:</i> | $3 = 7.$                    |                  |
| <i>Conclusion:</i> | $1 = 1.$                    |                  |
| <i>Proof:</i>      | $3 = 7$                     | (by hypothesis)  |
|                    | $3 = 7$                     | (by hypothesis). |
|                    | $\frac{3}{3} = \frac{7}{7}$ | (above axiom)    |
| Therefore,         | $\frac{3}{3} = \frac{7}{7}$ |                  |
| or,                | $1 = 1.$                    |                  |

The truth of this conclusion does not imply the truth of the hypothesis. Also, the falsity of the hypothesis does not guarantee the falsity of the conclusion.

The converse of the proposition "If  $A$  then  $B$ " or " $A$  implies  $B$ " is the proposition "If  $B$  then  $A$ " or " $B$  implies  $A$ ." That is, the converse of a proposition is formed by interchanging hypothesis and conclusion.

The converse of a correct proposition need not be correct.

*Example 4.* The converse of the proposition of example 1 is "If  $x + y = 7$ , then  $x = 2$  and  $y = 5$ ." This is not valid, for the conclusion is not inescapable; we might have  $x = 3$  and  $y = 4$ , or some other combination.

*Example 5.* The converse of the proposition of example 2 is "If tomorrow were Wednesday, then today would be Election Day," which is clearly not valid.

If both " $A$  implies  $B$ " and its converse " $B$  implies  $A$ " are valid, then the statements  $A$  and  $B$  are called **equivalent** to each other.

If  $A$  implies  $B$ , we sometimes say that  $A$  is a **sufficient condition** for  $B$ , and  $B$  is a **necessary condition** for  $A$ ; or that  $B$  is true if  $A$  is true, and  $A$  is true only if  $B$  is true. If  $A$  and  $B$  are equivalent, we sometimes say that  $A$  is a **necessary and sufficient condition** for  $B$ , or that  $A$  is true if and only if  $B$  is true.

*Example 6.* " $x = 2$  and  $y = 5$ " is a sufficient condition for " $x + y = 7$ " but not a necessary condition.

*Example 7.* "Today is Tuesday" is a necessary and sufficient condition for "Tomorrow is Wednesday." These two statements are equivalent; or "Today is Tuesday if and only if tomorrow is Wednesday."

The distinction between a proposition and its converse is of paramount importance. Another distinction which should be



carefully noted is that between the proposition (1) "Not all  $A$ 's are  $B$ 's" and (2) "All  $A$ 's are not  $B$ 's." These are often carelessly confused. The first is the denial of the proposition "All  $A$ 's are  $B$ 's" and really means "Some  $A$ 's are not  $B$ 's" or "At least one  $A$  is not a  $B$ " and is better written in the latter form. The second means "No  $A$  is a  $B$ " and is better written in the latter form.

*Every proof of a mathematical theorem must be an example of deductive reasoning; that is, the conclusion must be a necessary (inescapable) consequence of the hypothesis.*

*Note 1.* While only deductive reasoning is employed in a finished chapter of mathematics, other forms of reasoning are of great value in experimental science and in everyday life. Thus analogies and good guesses may be useful in conjecturing what may be the case, although such a conjecture should not be relied on without further proof. Of particular importance in experimental science and everyday life is **induction** or **inductive reasoning**; this is the name given to the process of coming to a probable conclusion on the basis of (many) particular instances. For example, a one-year-old scientist, after dropping objects over the side of his high chair many times, may come to the conclusion that an object released without support in the air will fall to the ground. But this conclusion is merely probable, not inescapable, as he might find out when, to his surprise, a gas-filled balloon "falls" to the ceiling. In fact, however, the young scientist's conclusion has a high probability, in the sense that, in his experience, it will prove correct most of the time. Inductive reasoning is thus the backbone of experimental science. It is used by mathematicians as well, but only as a means of suggesting what results may be proved by deduction. No mathematical theorem is considered proved on the evidence of induction alone; deductive proof is required.

*Note 2.* People whose intention is to persuade rather than to arrive at the truth, such as orators, debaters, demagogues, editors, et al., often use bad reasoning dressed up to seem good. Such reasoning may, for lack of a better term, be called *seductive reasoning*. Our only interest in it should be to detect and expose it.

## EXERCISES

1. Assuming that the conclusion follows from the hypothesis by valid reasoning, complete each of the following sentences with one of the phrases "must be true," "must be false," "may be true or false":



- (a) If the hypothesis is true, then the conclusion . . . .
- (b) If the conclusion is true, then the hypothesis . . . .
- (c) If the hypothesis is false, then the conclusion . . . .
- (d) If the conclusion is false, then the hypothesis . . . .

2. Write five correct statements of the "if . . . then . . . " type which have incorrect converses.

3. Assuming that statement (a) below is correct, decide which of the other statements necessarily follows from it:

- (a) If Jones is intelligent, then he can pass mathematics.
- (b) If Jones is not intelligent, then he cannot pass mathematics.
- (c) If Jones can pass mathematics, then he is intelligent.
- (d) If Jones cannot pass mathematics, then he is not intelligent.
- (e) Jones can pass mathematics if he is intelligent.

4. Which of the statements in exercise 3 is the converse of (a)?

5. According to the statement (a) of exercise 3, is "Jones is intelligent" a sufficient or a necessary condition for "he can pass mathematics"?

6. "If  $x = 2$ , then  $(x - 2)(x - 3) = 0$ " is correct. Is the converse necessarily correct?

7. "If  $x = -3$ , then  $x^2 = 9$ " is correct. Is the converse necessarily correct?

*Decide whether or not each of the proposed conclusions really follows from the hypothesis by valid (deductive) reasoning:*

8. *Hypothesis:* All men are vegetables; all vegetables are mortal.  
*Conclusion:* All men are mortal.

9. *Hypothesis:* No men are vegetables; no vegetables are beasts.  
*Conclusions:* (a) All men are beasts.  
(b) Some\* men are beasts.

10. *Hypothesis:* All triangles are polygons; no quadrilaterals are triangles.

*Conclusion:* No quadrilaterals are polygons.

11. *Hypothesis:* All freshmen are human; all students are human.  
*Conclusions:* (a) All freshmen are students.

(b) Some freshmen are students.

(c) Some students are not freshmen.

12. All honest congressmen voted for this bill. I voted for this bill. Therefore I am an honest congressman.

13. If the problem was worked correctly, the answer obtained was 7. I obtained the answer 7. Therefore I worked the problem correctly.

14. All good cars are expensive. This car is expensive. Therefore this car is good.

15. The sum of two even numbers is always even. Neither 7 nor 11 is even. Therefore the sum of 7 and 11 is not even.

\* "Some  $x$ 's are  $y$ 's" means "there exists at least one  $x$  which is also a  $y$ ."



16. If  $x$  is even, then  $2x$  is even. We know that  $2x$  is even. Therefore  $x$  is even.

17. Every rational number can be expressed as a quotient of two integers.  $\sqrt{2}$  cannot be expressed as a quotient of two integers. Therefore no rational number is  $\sqrt{2}$ .

2. Glossary of technical symbols. For easy reference we list the symbols used in this book.

| <i>Symbol</i>                     | <i>Definition</i>                                                                                                                                                                  |
|-----------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $+$                               | <i>plus.</i>                                                                                                                                                                       |
| $-$                               | <i>minus.</i>                                                                                                                                                                      |
| $\times$ or $\cdot$               | signs of multiplication, read <i>times</i> .                                                                                                                                       |
| $\div$ or $/$                     | <i>divided by.</i>                                                                                                                                                                 |
| $=$                               | <i>is equal to.</i>                                                                                                                                                                |
| $\neq$                            | <i>is not equal to, as <math>5 \neq 3</math>.</i>                                                                                                                                  |
| $>$                               | <i>is greater than, as <math>5 &gt; 3</math>.</i>                                                                                                                                  |
| $<$                               | <i>is less than, as <math>3 &lt; 5</math>.</i>                                                                                                                                     |
| $\geq$                            | <i>is greater than or equal to.</i>                                                                                                                                                |
| $\leq$                            | <i>is less than or equal to.</i>                                                                                                                                                   |
| $ x $                             | <i>the absolute value of <math>x</math>.</i>                                                                                                                                       |
| $( )$                             | <i>parentheses; <math>[ ]</math>, brackets; <math>\{ \}</math>, braces.</i>                                                                                                        |
| $\sqrt[n]{a}$                     | <i>the radical sign, meaning the (principal) <math>n</math>th root of <math>a</math>; in particular <math>\sqrt{a}</math> means the (principal) square root of <math>a</math>.</i> |
| $n!$                              | <i>factorial <math>n</math>.</i>                                                                                                                                                   |
| $C(n,r)$                          | <i>the number of combinations of <math>n</math> distinct things taken <math>r</math> at a time.</i>                                                                                |
| $P(n,r)$                          | <i>the number of permutations of <math>n</math> distinct things taken <math>r</math> at a time.</i>                                                                                |
| $f(x)$                            | <i>a function of <math>x</math>, or the <math>f</math>-function of <math>x</math>; similarly <math>d(x)</math> is the <math>d</math>-function of <math>x</math>, etc.</i>          |
| $(x,y)$                           | <i>denotes a point whose rectangular coordinates are <math>x</math> and <math>y</math>.</i>                                                                                        |
| $\rightarrow$                     | <i>read approaches.</i>                                                                                                                                                            |
| $\lim_{x \rightarrow a} f(x)$     | <i>read limit of <math>f(x)</math> as <math>x</math> approaches <math>a</math>.</i>                                                                                                |
| $\lim_{n \rightarrow \infty} a_n$ | <i>read limit of <math>a_n</math> as <math>n</math> increases indefinitely, or as <math>n</math> approaches infinity.</i>                                                          |
| $\Delta$                          | <i>difference of first order, or first difference.</i>                                                                                                                             |
| $\Delta^r$                        | <i>difference of <math>r</math>th order, or <math>r</math>th difference.</i>                                                                                                       |
| $s_n^-$                           | <i>amount of an annuity of 1 for <math>n</math> periods.</i>                                                                                                                       |
| $a_n^-$                           | <i>present value of an annuity of 1 for <math>n</math> periods.</i>                                                                                                                |



GREEK ALPHABET

| LETTERS  |            | NAMES   | LETTERS   |            | NAMES   | LETTERS    |            | NAMES   |
|----------|------------|---------|-----------|------------|---------|------------|------------|---------|
| A        | $\alpha$   | Alpha   | I         | $\iota$    | Iota    | P          | $\rho$     | Rho     |
| B        | $\beta$    | Beta    | K         | $\kappa$   | Kappa   | $\Sigma$   | $\sigma$   | Sigma   |
| $\Gamma$ | $\gamma$   | Gamma   | $\Lambda$ | $\lambda$  | Lambda  | T          | $\tau$     | Tau     |
| $\Delta$ | $\delta$   | Delta   | M         | $\mu$      | Mu      | $\Upsilon$ | $\upsilon$ | Upsilon |
| E        | $\epsilon$ | Epsilon | N         | $\nu$      | Nu      | $\Phi$     | $\phi$     | Phi     |
| Z        | $\zeta$    | Zeta    | $\Xi$     | $\xi$      | Xi      | X          | $\chi$     | Chi     |
| H        | $\eta$     | Eta     | O         | $\omicron$ | Omicron | $\Psi$     | $\psi$     | Psi     |
| $\Theta$ | $\theta$   | Theta   | $\Pi$     | $\pi$      | Pi      | $\Omega$   | $\omega$   | Omega   |



## CHAPTER I

### The Number System of Algebra\*

**3. Introduction.** The word *algebra* comes from the Arabic *al-jabr* signifying transposition of negative terms from one side of an equation to the other. Thus  $2x - 3 = 7$  becomes  $2x = 7 + 3$  by *al-jabr*. The ancients, such as the Egyptians\* (about 1700 B.C.), Greeks (6th century B.C. – 3rd century A.D.), Chinese (2nd–10th centuries A.D.), Hindus (5th–12th centuries A.D.), Arabs (9th–15th centuries A.D.), and others, had some rudimentary ideas of algebra. But the subject as we know it now really began to develop in the Middle Ages in Europe, beginning about the 12th century. Much of the algebra in this book dates from the 16th and 17th centuries although some of it is as recent as the 19th. During this long period of growth, the word *algebra* has come to represent an extensive body of knowledge concerning the processes of reckoning with numbers. The ancients were usually content to solve isolated problems and were usually satisfied with a single solution of a given problem. But the essence of the spirit of modern algebra is the desire to find all solutions of a given problem and especially to find general methods for coping with all problems of a given kind. This desire for generality will be seen to be the source of much of our modern progress in mathematics. As will be seen as we proceed, the superiority of algebraic methods over elementary arithmetic in solving problems springs, in large measure, from the use of letters to represent unspecified or unknown numbers. But since the letters used in elementary algebra do stand for numbers, it is necessary to have some understanding of the various kinds of numbers that they represent. In this chapter, we shall survey rapidly the long, gradual development of the concept of number, from prehistoric times to the complex number system now in use.

**4. The natural numbers. Addition and multiplication.** The earliest mathematical discovery made by a primitive man or a young

\* This discovery may be assumed to have resulted from the need for counting and referred to as needed.  
\* The knowledge of Egyptian mathematics is obtained largely from the Rhind papyrus, written about 1700 B.C., and a somewhat minutely detailed *Directions for*  
\* Following the knowledge of all such things.



child would doubtless be that two apples, two fingers, two people, etc., have something in common, and can be distinguished readily from one apple, one finger, one person, etc. Thus the first numbers to be invented were probably 1 and 2. Some primitive races are said to find it too much of a strain to engage in further numerical subtleties, and have in their language only three words of number—*one*, *two*, and *many*. This would provide a charmingly simple system of arithmetic of which lazy students might approve, but which would be inadequate for the purposes of modern science. Our prehistoric ancestors surely invented the process of counting objects and thus developed the endless succession of the so-called **natural numbers** 1, 2, 3, 4, . . . \* which are used in counting. The natural numbers are also often called **positive integers** for a reason which will be explained later. Experience in counting objects yields the tables of addition which we memorize at an early age and use in calculating sums. It is intuitively clear that the system of all the natural numbers obeys the following basic law.

**I. Law of closure for addition.** *Given any pair of numbers  $a$  and  $b$  in the system, there exists a unique† number in the system called the **sum** of  $a$  and  $b$ . The sum of  $a$  and  $b$  is denoted by  $a + b$ .*

The numbers  $a$  and  $b$ , separated by a plus sign, are themselves called **terms** of the sum.

The significance of the word *closure* is that the system of all natural numbers is *closed under addition* in the sense that the sum of two numbers in the system is itself in the system. It is as though the natural numbers are contained in an enclosure, and performing the operation of addition never takes us outside the enclosure. On the other hand, the system consisting of the natural numbers from 1 to 10 inclusive, and no others, is not closed under addition, for there are numbers in this system, 5 and 7 for example, whose sum is not in this system.

Experience also suggests that the system of natural numbers obeys the following basic law:

**II. Commutative law for addition.** *If  $a$  and  $b$  are any numbers in the system,  $a + b = b + a$ .*

This means that we may commute or interchange the order of the terms in the sum of two numbers.

Recall that parentheses, brackets, etc., are punctuation marks in the written language of algebra which are used to group together

\* Three dots occurring in succession mean "and so on."

† That is, there is one and only one such number.



whatever is in them. Thus the symbol  $3 + (4 + 5)$  indicates that we are to add 3 to the sum of 4 and 5, obtaining  $3 + 9$  or 12; while  $(3 + 4) + 5$  indicates that we are to add the sum of 3 and 4 to 5, obtaining  $7 + 5$  or 12. The fact that we get the same result both times, although the operations performed are different, suggests that the system of natural numbers obeys the following basic law.

III. Associative law for addition. *If  $a$ ,  $b$ , and  $c$  are any numbers in the system,  $a + (b + c) = (a + b) + c$ .*

*Extended statements of the three laws.* It is intuitively clear that the three laws just stated apply no matter how many terms are involved in a sum. Thus *the sum of any number of natural numbers is a natural number, the order in which they are added is immaterial, as is the manner in which they are grouped by parentheses.\** That all this is familiar is evident from the fact that shopkeepers will add a column of figures from the top down and then check by adding from the bottom up, expecting to get the same result. For, adding the column

$$\begin{array}{r} 3 \\ 4 \\ 5 \\ \hline 12 \end{array}$$

from the top down means calculating the value of the expression  $(3 + 4) + 5$ , while adding from the bottom up amounts to calculating the value of the expression  $(5 + 4) + 3$ . Hence in a sum like  $a + b + c + d$  we may freely rearrange the order of the terms and insert or remove parentheses in any way that makes sense; thus we may write

$$\begin{aligned} a + b + c + d &= (a + b) + (c + d) = [a + (b + c)] + d \\ &= a + [(b + c) + d] \end{aligned}$$

and so on.

Early in history, someone may have noticed that it was more efficient to "count by threes" than to "count by ones." This leads to the definition of multiplication as repeated addition.

DEFINITION 1. *If  $a$  and  $b$  are any natural numbers, we define the product of  $a$  and  $b$  to mean  $b + b + \cdots + b$  where there are  $a$  terms in the sum. The product of  $a$  and  $b$  is denoted by  $ab$ ,  $a \cdot b$ , or more rarely by  $a \times b$ . The numbers  $a$  and  $b$  are themselves called*

\* This can be proved by means of mathematical induction, which will be taken up in a later chapter.



*factors of the product. To multiply  $a$  and  $b$  means to find the product  $ab$ . In particular,  $1 \cdot b = b$  for any number  $b$ .*

*Examples.  $4 \cdot 3 = 3 + 3 + 3 + 3 = 12$ . Also  $3b = b + b + b$ ; in particular,  $3 \cdot 4 = 4 + 4 + 4 = 12$ .*

It is clear that the system of natural numbers is closed under multiplication. That is, it obeys the following basic law:

**IV. Law of closure for multiplication.** *If  $a$  and  $b$  are any numbers of the system, there is a unique number of the system called their product and denoted by  $ab$ .*

The examples above show that although  $4 \cdot 3$  and  $3 \cdot 4$  indicate different operations, they yield the same result. This suggests that the system of natural numbers obeys the following basic law.

**V. Commutative law for multiplication.** *If  $a$  and  $b$  are any numbers of the system,  $ab = ba$ .*

Similarly  $(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$  and  $2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$  indicate different operations but yield the same result. This suggests that the system of natural numbers obeys the following basic law.

**VI. Associative law for multiplication.** *If  $a$ ,  $b$ , and  $c$  are any numbers of the system, then  $a(bc) = (ab)c$ .*

We agree that in a succession of additions and multiplications, the multiplications are to be done first except where otherwise indicated by parentheses.

*Example.  $2 + 3 \cdot 4 = 2 + 12 = 14$ , while  $(2 + 3) \cdot 4 = 5 \cdot 4 = 20$ .*

Thus  $2 \cdot (3 + 4) = 2 \cdot 7 = 14$ , while  $2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14$ . The fact that these operations are different but yield the same result suggests that the system of natural numbers obeys the following basic law.

**VII. Distributive law.** *If  $a$ ,  $b$ , and  $c$  are any numbers of the system,  $a(b + c) = ab + ac$ .*

That is, the effect of the multiplier  $a$  is distributed between the terms  $b$  and  $c$ . This law is often referred to as "multiplying out the parentheses" or "removing the parentheses" if read from left to right, and as "taking out the common factor" if read from right to left.

A statement of equality, like  $a = b$ , is called an **equation**. The expressions on either side of the equals sign are called the **left member** and **right member** of the equation, respectively. If  $a = b$ , we may substitute  $a$  for  $b$  or  $b$  for  $a$  in any equation; this is known as the **principle of substitution**. It is related to the following obvious basic laws of equality.



VIII.  $a = a$ .

IX. If  $a = b$ , then  $b = a$ .

X. If  $a = b$  and  $b = c$ , then  $a = c$ . That is, things equal to the same thing are equal to each other.

XI. If  $a = b$  and  $c = d$ , then  $a + c = b + d$ . That is, if equals are added to equals, the results are equal.

XII. If  $a = b$  and  $c = d$ , then  $ac = bd$ . That is, if equals are multiplied by equals, the results are equal.

*Extended statements of the laws.* Just as in the case of laws I–III, the laws for multiplication and equality can be applied to any number of terms. For example, we may extend the distributive law to read

$$a(b + c + d + \cdots + k) = ab + ac + ad + \cdots + ak.$$

We shall not write out these extended statements here in full, since they are obvious.

## EXERCISES

Name the principal law or laws which justify each of the following statements. All letters represent natural numbers.

1.  $x + y = y + x$ .

2.  $x + y$  is a natural number.

3.  $x + (y + z) = (x + y) + z$ .

4.  $xy$  is a natural number.

5.  $x(yz) = (xy)z$ .

6.  $xy = yx$ .

7.  $x(y + z) = xy + xz$ .

8.  $(y + z)x = xy + xz$ .

9.  $2(x + 3) = 2x + 6$ .

10.  $2x + 3x = (2 + 3)x = 5x$ .

*Evaluate each of the following.*

11.  $3 + 4 \cdot 5$ .

12.  $(3 + 4) \cdot 5$ .

13.  $3 + 4 \cdot 5 + 6$ .

14.  $(3 + 4) \cdot 5 + 6$ .

15.  $(3 + 4) \cdot (5 + 6)$ .

16.  $3 + 4 \cdot (5 + 6)$ .

17.  $3(4 + 5[6 + 7])$ .

18.  $3(4 + 5 \cdot 6 + 7)$ .

19.  $3([4 + 5] \cdot 6 + 7)$ .

20.  $3(4 + 5)(6 + 7)$ .

Use the basic laws I–XII to prove the following theorems, in which all letters represent natural numbers.

\*21.  $(a + b) + c = a + (c + b)$ .

(Hint: use II and III.)

\*22.  $(a + b) + c = c + (a + b)$ .

\*23.  $(x + y) + z = z + (x + y)$ .

(Hint: use I and II.)

\*24.  $(ab)c = a(cb)$ .

(Hint: use V and VI.)

\*25.  $(ab)c = b(ac)$ .

\*26.  $(ab)(cd) = (cd)(ab)$ .

(Hint: use IV and V.)

\*27.  $a(b + c) = ca + ba$ .

\*28.  $a(b + c) = ba + ca$ .

\*29.  $(a + b)c = ca + ba$ .

\*30.  $(a + 3)(b + 2) = ab + 3b + 2a + 6$ .

★ May be omitted without disturbing continuity.



**5. Subtraction and division of natural numbers.** Children are usually taught to subtract 2 from 5 by asking themselves, "What must be added to 2 in order to get 5?" Formalizing this question, we define subtraction in terms of addition, as follows.

**DEFINITION 1.** *If  $a$  and  $b$  are any two numbers of the system, the difference  $a - b$  stands for the number  $x$  such that  $b + x = a$ , provided such a number exists in the system. To subtract  $b$  from  $a$  means to find  $a - b$ .*

Thus,  $5 - 2$  is found by asking, " $2 + \text{what} = 5$ ?" and looking in the "two plus" table for the answer 3. The proviso in the definition is necessary because, in the system of natural numbers, the difference does not exist in some cases. For example,  $2 - 5$  does not exist within the system of natural numbers, since there is no natural number  $x$  such that  $5 + x = 2$ .

A system of numbers is called **closed under a given operation** if performing that operation upon numbers of the system always yields a number of the system as a result. Thus, the system of all natural numbers is not closed under subtraction, although it is closed under addition and multiplication.

To prove that a first quantity,  $a$ , minus a second quantity,  $b$ , is equal to a third quantity,  $x$ , we have only to use Definition 1 and verify that the second quantity,  $b$ , plus the third,  $x$ , is equal to the first,  $a$ . This process is often called "checking" the subtraction.

**Example 1.** To verify that  $(q + p) - p = q$  it is sufficient to show that the second quantity,  $p$ , plus the third quantity,  $q$ , is equal to the first quantity,  $q + p$ . But  $p + q = q + p$  by the commutative law for addition. This completes the proof.

Example 1 shows that subtraction "undoes" addition, in an obvious sense. Therefore subtraction is called the *inverse* of addition.

**DEFINITION 2.** *If  $a$  and  $b$  are natural numbers, and if there exists a natural number  $x$  such that  $b + x = a$ , then  $a$  is said to be **greater than  $b$**  or  **$b$  less than  $a$** . In symbols,  $a > b$  or  $b < a$ .*

For example,  $5 > 2$  because  $2 + 3 = 5$ .

Similarly, children are taught to divide 6 by 2 by asking, "By what must we multiply 2 in order to get 6?" Formalizing this question, we define division in terms of multiplication, as follows.

**DEFINITION 3.** *If  $a$  and  $b$  are any two numbers of the system, the quotient  $a \div b$  stands for the number  $x$  such that  $bx = a$ , provided such a number  $x$  exists in the system. To divide  $a$  by  $b$  means to find the quotient  $a \div b$ .*



Thus  $6 \div 2$  is found by asking, “2 times what equals 6?” and looking in the “2 times” table for the quotient 3. The proviso in the definition is necessary because, in the system of natural numbers, the quotient does not exist in some cases. For example,  $5 \div 2$  does not exist within the system, since there is no natural number  $x$  such that  $2x = 5$ . Therefore *the system of natural numbers is not closed under division*.

To prove that a first quantity,  $a$ , divided by a second quantity,  $b$ , is equal to a third quantity,  $x$ , we have only to use Definition 3 and verify that the second,  $b$ , times the third,  $x$ , equals the first,  $a$ . This process is often called “checking” the division.

*Example 2.* To verify that  $(qp) \div p = q$  we have only to show that the second quantity,  $p$ , times the third,  $q$ , is equal to the first,  $qp$ . But  $qp = pq$  by the commutative law for multiplication. This completes the proof.

Example 2 shows that division “undoes” multiplication in an obvious sense. Therefore division is called the *inverse* of multiplication.

**DEFINITION 4.** If  $a$  and  $b$  are any natural numbers, and if there exists a natural number  $x$  such that  $bx = a$ , then  $b$  is called a **factor** of  $a$ ,  $a$  is said to be a **multiple** of  $b$ , or  $a$  is said to be **divisible** by  $b$ .

For example, 6 is divisible by 2, 6 is a multiple of 2, and 2 is a factor of 6 because  $2 \cdot 3 = 6$ .

In a sequence of additions, subtractions, multiplications, and divisions it is understood that multiplications and divisions are to be done before additions and subtractions, except where otherwise indicated by parentheses. For example,  $2 \cdot 5 - 2 \cdot 2 = 10 - 4$ , while  $2 \cdot (5 - 2) = 2 \cdot 3$ . The fact that both answers are equal suggests that the distributive law can be extended to include subtraction. Thus, it is possible to prove that

$$a(b + c - d - e + \cdots + k) = ab + ac - ad - ae + \cdots + ak$$

whenever these expressions have a meaning. We shall not prove this here.

## EXERCISES

1. Explain what is meant by the statement  $8 \div 2 = 4$  in terms of definition 3.

2. Explain what is meant by the statement  $8 - 2 = 6$  in terms of definition 1.



3. Which of the following expressions are meaningless within the system of natural numbers? Explain: (a)  $10 \div 2$ . (b)  $5 \div 2$ . (c)  $3 - 5$ . (d)  $3 \div 5$ . (e)  $4 \div 8$ . (f)  $8 - 2$ .

4. Assuming that  $a$  and  $b$  are natural numbers, tell whether the following expressions always represent a natural number: (a)  $a + b$  (b)  $a - b$ . (c)  $ab$ . (d)  $a \div b$ .

5. If your answer to parts of exercise 4 is "no," what restriction must be placed on  $a$  and  $b$  in order for these expressions to represent natural numbers?

*Prove each of the following, justifying each step by means of a basic law, definition, or previously proved theorem, all letters representing natural numbers.*

\*6.  $(m + n) - n = m$ .

\*7.  $(mn) \div n = m$ .

\*8. If  $a > b$  and  $b > c$ , then  $a > c$ .

\*9. If  $c$  is a factor of  $b$  and  $b$  is a factor of  $a$ , then  $c$  is a factor of  $a$ .

\*10. If  $a < b$ , then  $a + c < b + c$ .

\*11. If  $a < b$ , then  $ac < bc$ .

6. Even, odd, and prime numbers. H.C.F. and L.C.M. A natural number is called **even** if it is divisible by 2, or has 2 as a factor. That is, a natural number  $a$  is even if it can be expressed as  $2x$  where  $x$  is some natural number. For example, 6 is even since it is  $2 \cdot 3$ . But 5 is not even, for there is no natural number  $x$  such that  $5 = 2x$ . A natural number which is not even is called **odd**. It can be proved that *every odd number is one less than some even number*.

A natural number is called **prime** if it is greater than 1 and has no factors except itself and 1.

By trial we find that 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 are the first ten prime numbers. Clearly 2 is the only even prime, because any other even number has the factor 2.

All primes less than a given number may be found by a simple process known as the **sieve of Eratosthenes** (Greek, 275–194 B.C.). We illustrate the process by finding all primes less than 50. First write in succession all natural numbers from 2 up to 50:

|    |    |    |    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |    |    |    |

\* May be omitted without disturbing continuity.







*Proof.* Express each of the numbers  $a$ ,  $b$ , and  $c$  as the (unique) product of its prime factors. Since  $a$  and  $b$  have no prime factor in common and  $a$  is a factor of  $bc$ , it follows that every prime factor of  $a$  is a prime factor of  $c$  and, hence, that  $a$  is a factor of  $c$ .\*

Note the necessity for the hypothesis that  $a$  and  $b$  are relatively prime. For example, if  $a = 6$ ,  $b = 10$ ,  $c = 21$ , then  $a$  is a factor of  $bc = 210$ , but  $a$  is not a factor of either  $b$  or  $c$ ; note that neither  $a$  and  $b$  nor  $a$  and  $c$  are relatively prime.

*Note.* It may astonish the student to learn that at this seemingly elementary stage of the development of mathematics, there are already unsolved problems at hand which have resisted solution stubbornly. But this is the case concerning natural numbers and prime numbers. Among the questions which are easy to ask but remain at present unanswered are the following:†

(a) Is it true that every even number greater than 2 is the sum of two primes? It seems plausible on the basis of experiment, for  $4 = 2 + 2$ ,  $6 = 3 + 3$ ,  $8 = 3 + 5$ ,  $10 = 5 + 5$ ,  $12 = 7 + 5$ , and so on. This conjecture was made by Goldbach in 1742 but no proof has yet been given.

(b) Is the number of pairs of primes which differ by 2 limited or unlimited? For example, 3 and 5, 11 and 13, 17 and 19, and so on.

(c) Are there any natural numbers  $a$ ,  $b$ ,  $m$ , and  $n$ , where  $m$  and  $n$  are greater than 1, such that  $a^m$  and  $b^n$  differ by 1, except for  $3^2$  and  $2^3$ ?

## EXERCISES

1. Use the sieve of Eratosthenes to find all prime numbers less than (a) 100; (b) 200.

Express each of the following numbers as a product of primes:

- |         |          |          |           |         |
|---------|----------|----------|-----------|---------|
| 2. 9.   | 3. 15.   | 4. 24.   | 5. 86.    | 6. 153. |
| 7. 363. | 8. 2844. | 9. 3135. | 10. 1225. |         |

Find (a) the H.C.F., (b) the L.C.M. of each of the following sets of numbers:

- |                |                  |                     |
|----------------|------------------|---------------------|
| 11. 60 and 80. | 14. 70 and 75.   | 17. 24, 36, and 60. |
| 12. 60 and 75. | 15. 86 and 75.   | 18. 8, 20, and 24.  |
| 13. 24 and 36. | 16. 153 and 189. | 19. 60, 80, and 40. |

\* If the assumption of the unique factorization theorem is too strong for the instructor's taste, another proof of this important theorem, free of this assumption, is given in Chapter XXV, section 169, theorem 4.

† Other such unsolved problems are discussed in M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, Chapter VIII.



20. Show that if  $a$  and  $b$  are both even, then  $a + b$  is even.

21. Show that if  $a$  and  $b$  are both odd, then  $a + b$  is even.

22. Show that if  $a$  is even and  $b$  is any natural number, then  $ab$  is even.

23. Is the system of all even numbers closed under addition? Explain.

24. Is the system of all odd numbers closed under addition? Explain.

25. Is the system of all multiples of 3 closed under addition? Explain.

26. Is the system of all even numbers closed under multiplication? Explain.

27. Is the system of all multiples of 3 closed under multiplication? Explain.

28. If  $a$  is any natural number, show that the system of all multiples of  $a$  is closed under addition and multiplication.

29. A natural number is called **perfect** if it is equal to the sum of all its factors except itself. For example, 6 is perfect, since  $6 = 1 + 2 + 3$ . Show that 28 is perfect.\*

30. The numbers  $a$  and  $b$  are called **amicable** if each is the sum of all the factors of the other except the other itself. Show that 220 and 284 are amicable.

**7. Fractions. Multiplication and division.** Early in history, fractions were invented to facilitate calculations involving things divided into equal parts, such as real estate or harvests of grain. From a more sophisticated point of view we may credit motivation of the invention of fractions to our natural desire to have our system of numbers closed under all our operations. The system of natural numbers is not closed under division. That is, we cannot always divide one natural number by another and get a natural number. For example, 6 cannot be divided by 4 within the system of natural numbers. This unpleasant defect will be remedied by the introduction of fractions. Looking at it naïvely, a fraction, like  $6/4$ , is merely a symbol consisting of two natural numbers. Thus, we make the following definition:

**DEFINITION 1.** A *fraction* is a symbol  $\frac{a}{b}$  or  $a/b$  where  $a$  and  $b$  are natural numbers. We call  $a$  the **numerator** and  $b$  the **denominator** of the fraction  $a/b$ .

The symbol  $a/b$ , read " $a$  over  $b$ ," is used to represent  $a$  of  $b$  equal parts of something. Thus  $2/3$  represents 2 of 3 equal parts of something. Because of this application we wish to regard  $2/3$ ,  $4/6$ ,  $6/9$ , etc., as equal (Fig. 1), although they are recognizably different symbols. Fortunately, fractions are our own creations

\* Whether or not there exists an odd perfect number is at present unknown.



and we are free to decide what we wish to mean by equal fractions, and by the sum, product, etc., of two fractions, so long as our decision does not involve logical inconsistency. If, instead of dividing something into three equal parts we double the number of parts, we must take twice as many of the smaller parts, or 4/6, to get the same quantity as 2/3. That is,  $\frac{4}{6} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{2}{3}$ . This suggests that, in general, we adopt the following preliminary definition.

**DEFINITION 2.**  $\frac{a}{b} = \frac{ax}{bx}$  where  $a$ ,  $b$ , and  $x$  are any natural numbers. That is, a common factor may be introduced into, or removed from, the numerator and denominator of a fraction without altering its value.

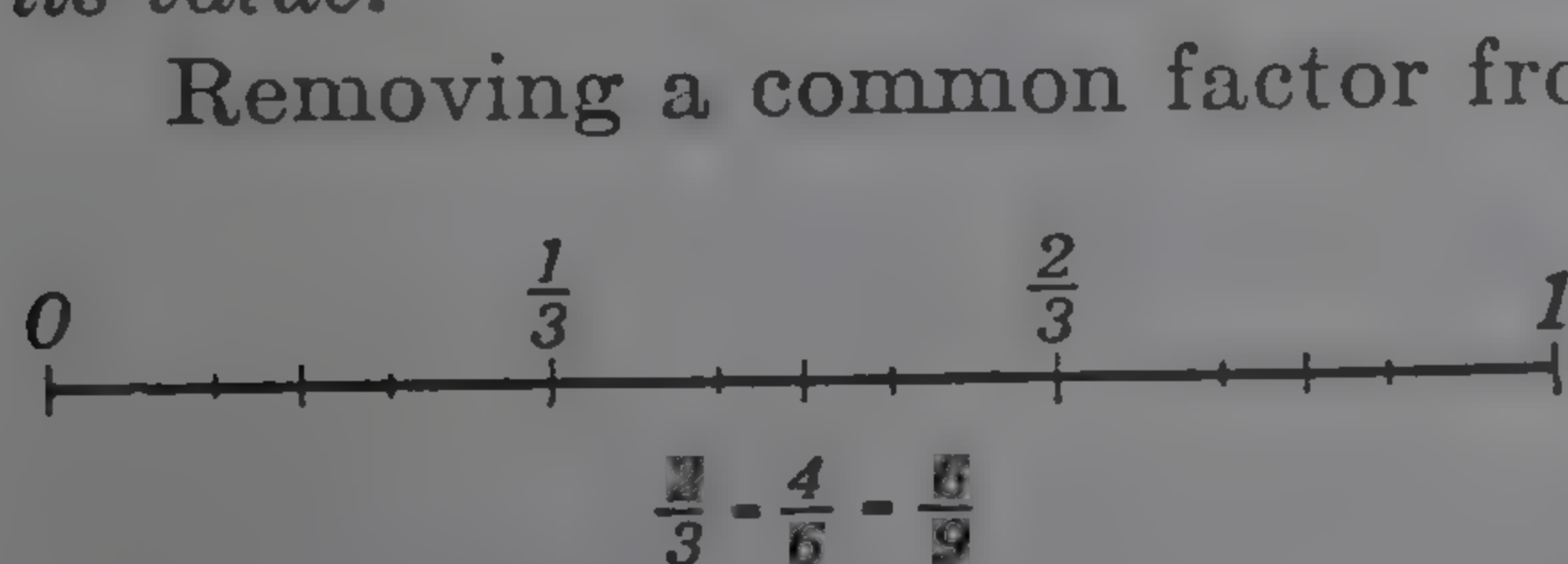


Fig. 1.

Removing a common factor from numerator and denominator of a fraction is often called **cancellation**. Note that only **factors** which are common to numerator and denominator may be cancelled. When all possible common factors have

been cancelled, the fraction is said to have been **reduced to simplest form** or **lowest terms**. For example,

$$\frac{12}{18} = \frac{2 \cdot 6}{3 \cdot 6} = \frac{2}{3}.$$

Now 2/4 and 5/10 are not related directly by definition 2 since there is no natural number  $x$  such that  $2x = 5$  or  $4x = 10$ . But both 2/4 and 5/10 can be expressed as equal fractions having a common denominator by means of definition 2. Thus

$$\frac{2}{4} = \frac{2 \cdot 10}{4 \cdot 10} = \frac{20}{40} \text{ and } \frac{5}{10} = \frac{5 \cdot 4}{10 \cdot 4} = \frac{20}{40}.$$

Since we wish the statement "things equal to the same thing are equal to each other" to remain valid for fractions, we are led to make the following general definition.

**DEFINITION 3.** Two fractions are **equal** if and only if when both are expressed as equivalent fractions having a common denominator (by definition 2 if necessary) they then have the same numerator as well.



For example,  $\frac{2}{4} = \frac{5}{10}$  since both are equal to  $\frac{20}{40}$ .

**THEOREM 1.**  $\frac{a}{b} = \frac{c}{d}$  if and only if  $ad = bc$ .

*Proof.*  $\frac{a}{b} = \frac{ad}{bd}$  and  $\frac{c}{d} = \frac{bc}{bd}$  by definition 2. Hence they are equal to each other if and only if the numerators  $ad$  and  $bc$  are equal.

The operation of passing from  $\frac{a}{b} = \frac{c}{d}$  to  $ad = bc$  is sometimes referred to as "cross-multiplying."

**DEFINITION 4.** The *product* of two fractions is the fraction whose numerator is the product of the two given numerators and whose denominator is the product of the two given denominators. In symbols

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The choice of this definition may be motivated by considering rectangular areas.\* A rectangle with length 3 inches and width

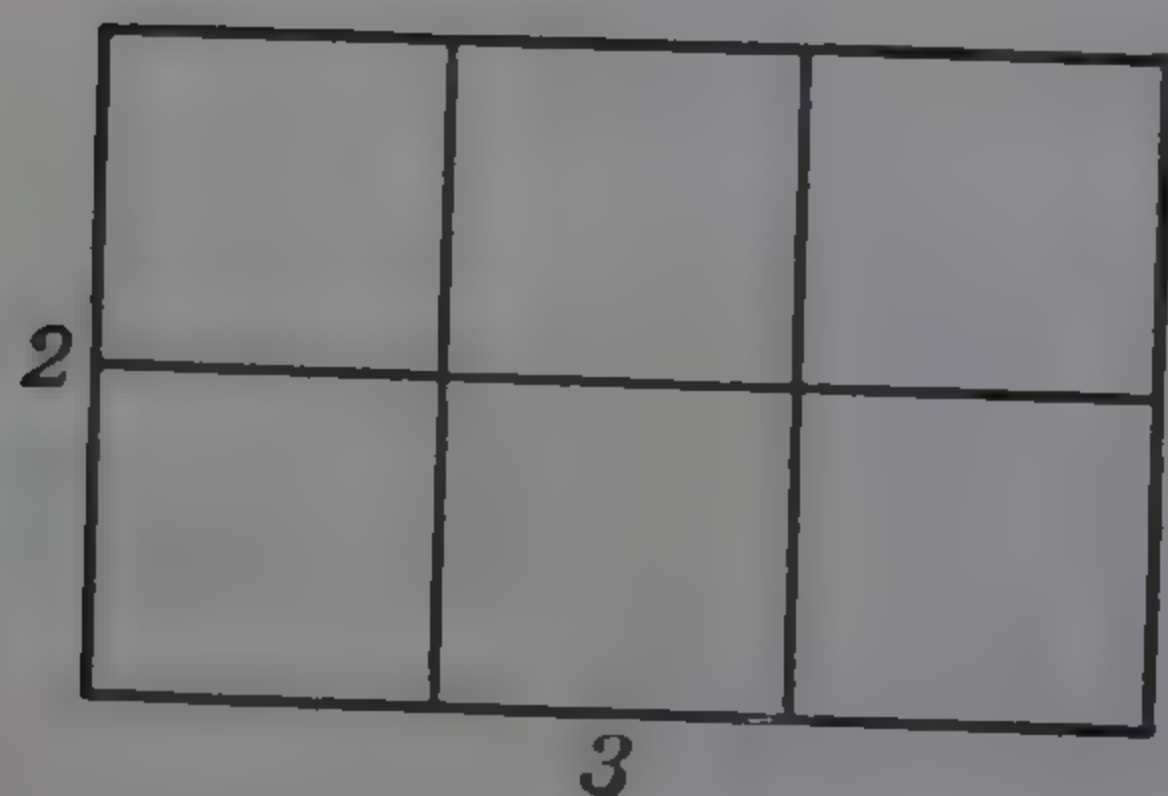


Fig. 2.

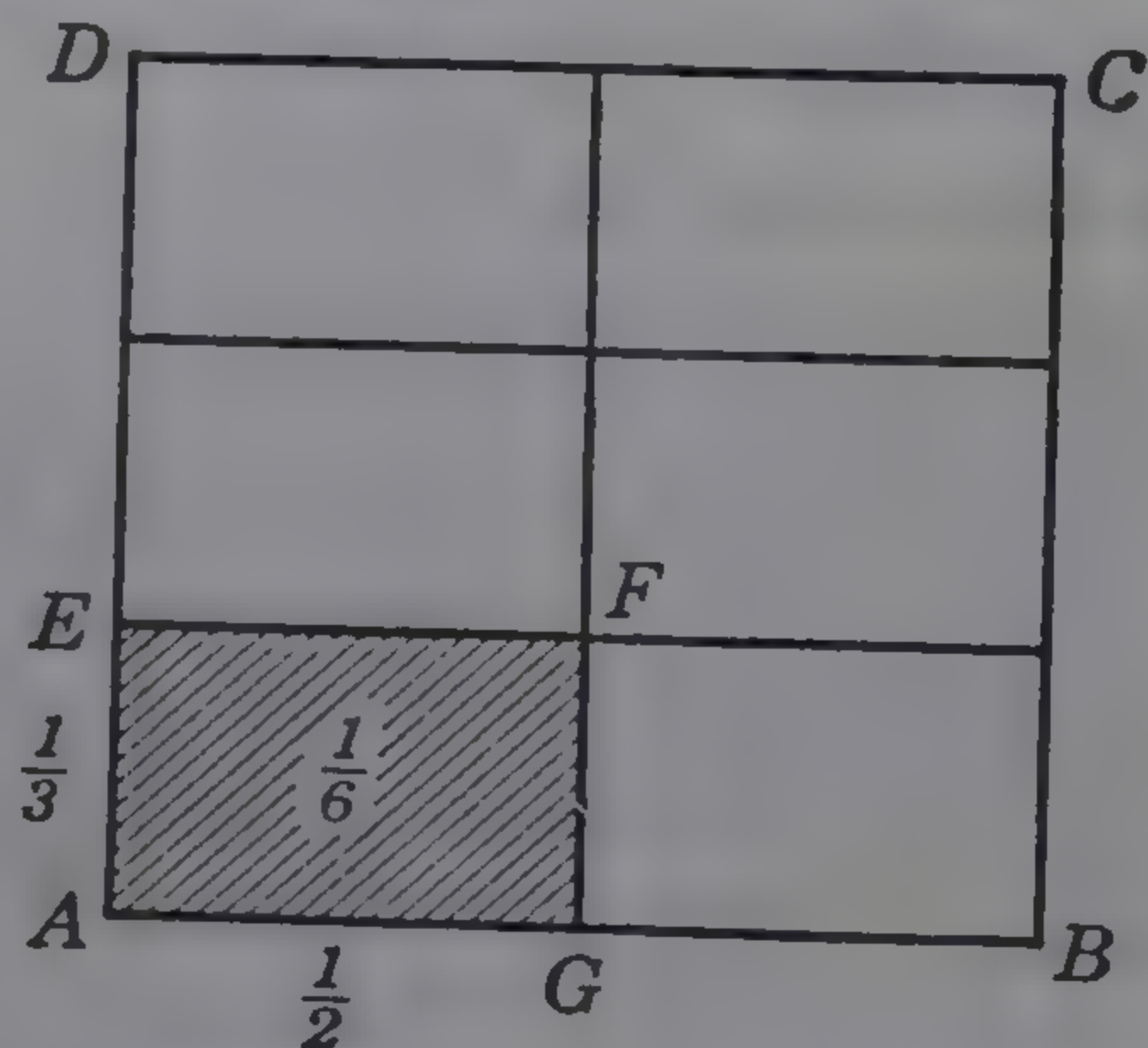


Fig. 3.

2 inches clearly has an area of 6 square inches (Fig. 2). The square  $ABCD$  with side 1 inch long has an area of 1 square inch (Fig. 3). This suggests the formula  $A = lw$  for the area of a rectangle. If we wish this formula to remain valid for fractional lengths and widths, we must define the product of two fractions as above, so that the area  $AGFE$ , for example, in Fig. 3, will be

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

\* Other considerations leading to the same definition may be found in M. Richardson, *Fundamentals of Mathematics*. Macmillan, 1941.



Whenever we introduce or invent new kinds of numbers, we want them, as far as possible, to obey the same basic laws and follow the same definitions as the preceding numbers did. Hence, for division of fractions we use definition 3 of section 5. That is, the quotient of two fractions  $\frac{a}{b} \div \frac{c}{d}$  shall be the fraction  $\frac{x}{y}$  such that  $\frac{c}{d} \cdot \frac{x}{y} = \frac{a}{b}$ , provided such a fraction exists. But we shall now prove that in the system of fractions, the quotient  $\frac{a}{b} \div \frac{c}{d}$  always exists and is the fraction obtained by inverting  $\frac{c}{d}$  and multiplying.

**THEOREM 2.**  $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

*Proof.* The right member is equal to  $\frac{ad}{bc}$  by definition 4. We have only to show that  $\frac{c}{d} \cdot \frac{ad}{bc} = \frac{a}{b}$ . But  $\frac{c}{d} \cdot \frac{ad}{bc} = \frac{cad}{dbc}$  by definition 4, and  $\frac{cad}{dbc} = \frac{a}{b}$  by cancellation. This completes the proof.

*Example.*  $\frac{2}{3} \div \frac{4}{9} = \frac{2}{3} \cdot \frac{9}{4} = \frac{18}{12} = \frac{3}{2}$

We would like now to identify the fraction  $a/b$  with the quotient  $a \div b$  so that division of one natural number by another would always be possible, according to our original aim in introducing fractions. But then we would have  $3 \div 1 = 3/1$  and also, by definition 3 section 5,  $3 \div 1 = 3$ . Thus we are led to the following agreement: we shall identify fractions  $a/1$  having denominator 1 with the natural number  $a$  in the numerator. Thus we write  $3/1 = 3$ . We can now prove the following theorem.

**THEOREM 3.** If  $a$  and  $b$  are natural numbers, then  $a \div b = a/b$ .

*Proof.* By agreement  $a = a/1$  and  $b = b/1$ . Therefore  $a \div b = \frac{a}{1} \div \frac{b}{1} = \frac{a}{1} \cdot \frac{1}{b}$  by theorem 2. Hence  $a \div b = \frac{a}{b}$  by definition 4.

We shall henceforth use the division sign  $\div$  and the fraction line interchangeably. Clearly, the system of fractions is closed under both multiplication and division.

We could now prove that the basic laws of equality, such as "if equals are multiplied or divided by equals, the results are equal," are valid for fractions.

Note that the equality  $\frac{a}{b} = \frac{ax}{bx}$  in definition 2 may now be understood in terms of multiplication. For



$$\frac{ax}{bx} = \frac{a}{b} \cdot \frac{x}{x} = \frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b} \cdot 1 = \frac{a}{b},$$

which is in harmony with the intuitive idea that multiplication of a quantity by 1 should not change its value.

### EXERCISES

*All letters represent natural numbers.*

*Reduce each of the following fractions to lowest terms:*

- |                    |                    |                    |                            |                                |
|--------------------|--------------------|--------------------|----------------------------|--------------------------------|
| 1. $\frac{10}{15}$ | 3. $\frac{21}{14}$ | 5. $\frac{34}{51}$ | 7. $\frac{38abc}{57bcd}$   | 9. $\frac{2m + 6n}{2mn}$       |
| 2. $\frac{18}{24}$ | 4. $\frac{35}{30}$ | 6. $\frac{57}{95}$ | 8. $\frac{33 xuc}{44 xcv}$ | 10. $\frac{2m + 6n}{5m + 15n}$ |

*Calculate each of the following and reduce the answer to simplest form:*

- |                                      |                                     |                                        |                                                            |
|--------------------------------------|-------------------------------------|----------------------------------------|------------------------------------------------------------|
| 11. $\frac{4}{3} \cdot \frac{5}{12}$ | 13. $\frac{5}{12} \div \frac{4}{3}$ | 15. $\frac{ab}{cd} \div \frac{b}{d}$   | 17. $\frac{xy + xz}{y} \div \frac{y + z}{2y}$              |
| 12. $\frac{2}{3} \cdot \frac{6}{4}$  | 14. $\frac{2}{3} \div \frac{6}{4}$  | 16. $\frac{2a}{3b} \div \frac{4}{3bc}$ | 18. $\frac{ax + ay}{bc + bd} \div \frac{mx + my}{2c + 2d}$ |

19. Write a fraction equal to  $3/5$  having the denominator 40.

20. Write a fraction equal to  $a/b$  having the denominator  $bd$ .

**8. Fractions. Addition and subtraction.** Since fractions are symbols invented by us, we have to decide what we want to mean by the sum of two fractions. To add coins of the same denomination we add the number of coins; for example, 2 quarters plus 1 quarter yields 3 quarters. Analogously, we adopt the following definition.

**DEFINITION 1.** *The sum of two fractions having the same denominator shall be the fraction whose numerator is the sum of the given numerators and whose denominator is the given common denominator. In symbols,*

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$

For example,

$$\frac{2}{4} + \frac{1}{4} = \frac{3}{4}.$$

To add coins of different denominations, we first express them as equivalent amounts with a common denomination, and then add as above; for example, 2 dimes and 1 quarter would be expressed as 20 cents and 25 cents, respectively, and then added, obtaining 45 cents. Analogously, we adopt the following definition.

**DEFINITION 2.** *The sum of two fractions having different denominators shall be the fraction obtained by expressing them as*

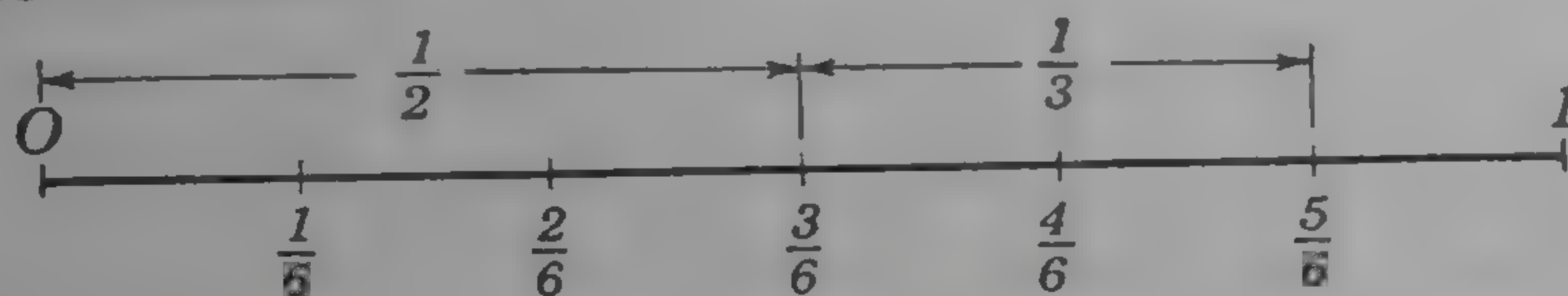


equal fractions having a common denominator (by means of definition 2, section 7), and then adding by means of definition 1.

$$\text{Example 1. } \frac{1}{2} + \frac{1}{3} = \frac{1 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 2}{3 \cdot 2} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}. \quad (\text{See Fig. 4.})$$

$$\text{Example 2. } \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$$

It is often desirable, although not essential, to use the least common denominator. By the **least common denominator** of several given fractions is meant the least common multiple of the given denominators.



$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Fig. 4.

Subtraction is defined as the inverse of addition, by means of definition 1, section 5, just as division was defined as the inverse of multiplication. That is, the **difference** of two fractions  $\frac{a}{b} - \frac{c}{d}$

shall be the fraction  $\frac{x}{y}$  such that  $\frac{c}{d} + \frac{x}{y} = \frac{a}{b}$ , provided such a fraction exists. If such a fraction does exist, we shall say that  $\frac{a}{b}$  is greater than  $\frac{c}{d}$ ; in symbols  $\frac{a}{b} > \frac{c}{d}$  or  $\frac{c}{d} < \frac{a}{b}$ .

We now prove the following theorem.

**THEOREM 1.** If  $ad > bc$ , then  $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ .

*Proof.* According to the definition of subtraction, we have only to show that

$$\frac{c}{d} + \frac{ad - bc}{bd} = \frac{a}{b}.$$

Adding the fractions in the left member, we obtain

$$\frac{bc + (ad - bc)}{bd}.$$

By definition  $(ad - bc)$  stands for a number which, when added to  $bc$ , will yield  $ad$ . Hence,  $bc + (ad - bc) = ad$ . Therefore the left member reduces to  $\frac{ad}{bd}$ , which becomes  $\frac{a}{b}$  by cancellation of the common factor  $d$ . This completes the proof.



In practice this theorem asserts that *to subtract one fraction from another we may express them as equal fractions having a common denominator and then write the difference of the numerators over the common denominator.* For example,

$$\frac{1}{2} - \frac{1}{3} = \frac{1 \cdot 3}{2 \cdot 3} - \frac{2 \cdot 1}{2 \cdot 3} = \frac{3}{6} - \frac{2}{6} = \frac{3 - 2}{6} = \frac{1}{6}.$$

It is clear that this difference  $\frac{ad - bc}{bd}$  exists if and only if  $ad > bc$ , for otherwise the numerator  $ad - bc$  has no meaning within the system of natural numbers. Hence,  $\frac{a}{b} > \frac{c}{d}$  if and only if  $ad > bc$ . Thus, *to decide whether one fraction is greater than, less than, or equal to another, we may express them as equal fractions having a common denominator, and compare the numerators.*

It can be shown\* that the system of fractions obeys the associative, commutative, and distributive laws, and the usual laws of equality and inequality. The system of fractions is closed under addition, multiplication, and division, but not under subtraction, since it is still impossible to subtract a larger number from a smaller one.

## EXERCISES

*Calculate each of the following expressions, and reduce the answer to simplest form:*

1.  $\frac{1}{3} + \frac{1}{4}$

5.  $\frac{1}{3} + \frac{5}{6}$

9.  $\frac{3}{4} - \frac{1}{8}$

2.  $\frac{1}{4} + \frac{1}{5}$

6.  $\frac{1}{3} - \frac{1}{4}$

10.  $\frac{5}{6} - \frac{1}{3}$

3.  $\frac{1}{4} + \frac{3}{8}$

7.  $\frac{1}{4} - \frac{1}{5}$

11.  $\frac{2}{3} + \left(\frac{1}{4} + \frac{5}{12}\right)$

4.  $\frac{3}{4} + \frac{1}{8}$

8.  $\frac{3}{8} - \frac{1}{4}$

12.  $\left(\frac{2}{3} + \frac{1}{4}\right) - \frac{7}{12}$

13.  $\left(\frac{1}{2} + \frac{1}{3}\right) - \left(\frac{1}{4} + \frac{5}{12}\right)$

16.  $\frac{a}{2b} + \frac{c}{2}$

14.  $\left(\frac{1}{2} + \frac{1}{3}\right) \div \left(\frac{5}{12} - \frac{1}{4}\right)$

17.  $\left(\frac{1}{x} + \frac{1}{y}\right) \div \left(\frac{2}{x} + \frac{2}{y}\right)$

15.  $\frac{1}{x} + \frac{1}{2}$

18.  $\left(\frac{1}{x} + \frac{1}{y}\right) \div \left(\frac{1}{x} - \frac{1}{y}\right)$ , where  $x < y$ .

\* See M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, for further details.



19. Is  $15/17$  equal to, less than, or greater than  $17/19$ ?
20. Is  $34/51$  equal to, less than, or greater than  $38/57$ ?
21. Decide which of the fractions  $21/23$  and  $25/29$  is greater and find their difference.
22. Decide which of the fractions  $19/23$  and  $22/29$  is greater and find their difference.
23. If  $x$  and  $y$  represent (positive) fractions, which of the expressions (a)  $x + y$ , (b)  $x - y$ , (c)  $xy$ , (d)  $x \div y$ , does not always represent a (positive) fraction? What restriction must be placed on  $x$  and  $y$  to insure the existence of this number within the system of fractions?
24. Evaluate  $5 \left( \frac{1}{5} + 3 \right)$  in two different ways.
25. Evaluate  $\frac{1}{2} \left( \frac{1}{2} + \frac{5}{6} \right)$  in two different ways.

**9. Directed numbers. Addition and multiplication.** Within the system of (positive) fractions it is not possible to deal with the expression  $a - b$  without first ascertaining that  $a > b$ . This inconvenience will be removed by the invention of directed or signed numbers. Corresponding to each number already in existence, such as 3, we invent two new symbols, such as  $+3$  and  $-3$ . The symbols preceded by a  $+$  sign are called **positive** numbers, those preceded by a  $-$  sign are called **negative** numbers. We also invent a new symbol 0, called **zero**, which is considered to be neither positive nor negative. The numbers 0,  $+1$ ,  $+2$ ,  $+3$ , . . . and  $-1$ ,  $-2$ ,  $-3$ , . . . are called **integers\*** or **whole numbers**. We speak of positive and negative integers and of positive and negative fractions. All these new numbers are called **signed numbers** or **directed numbers** because they may be conveniently interpreted in terms of direction, as follows: Choose a straight line, a unit of length, and a starting point on the line which we shall call the **origin**. To the origin we attach the number zero. Marking off the unit of length an indefinite number of times in both directions we attach the positive integers to marked points in one direction, and negative integers in the other. The other positive and negative fractions are attached to points on the line in the obvious way (Fig. 5). It is customary to use the right side of the line for positive numbers and the left side for negative numbers.† The

\* The Latin word *integer* means *whole*.

† This is, of course, merely a convention. We could equally well put the negative numbers on the right. But there is no particular advantage in flouting a harmless convention.



numbers involved in preceding sections are called **unsigned numbers** to distinguish them from the new signed numbers.

The idea of negative numbers struggled for recognition for centuries, and was accepted with considerable reluctance as late as the beginning of the 17th century, even by mathematicians. Negative numbers were often called "false" or "fictitious" numbers. Now it is true that no one ever saw  $-3$  books on a table, except possibly at a glance. But to try to interpret  $-3$  as referring to the plenty of three departed books, or to the operation of subtracting three books away from an empty table, is to miss the point completely. The point is that directed numbers are not to be interpreted as quantity or magnitude alone, but rather as quantity or magnitude together with direction. Hence they are convenient for such applications as east and west, up and down, profit and loss,

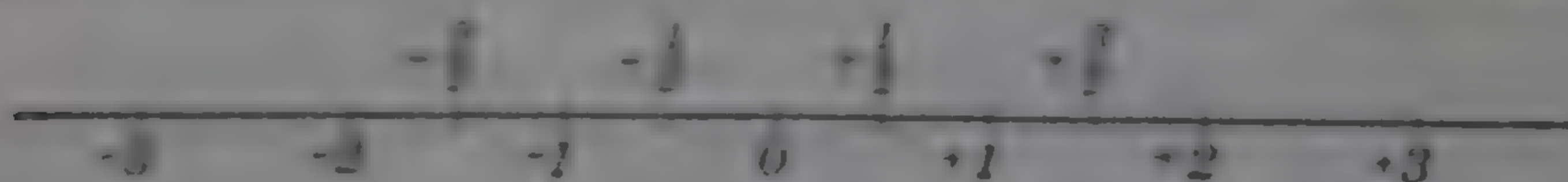


Fig. 4

down and up, temperature above and below zero, etc. They are called **numbers** largely because they obey the basic laws of calculation which all other numbers obey, such as the laws of equality, the associative, commutative, and distributive laws, etc.

Note also that the  $+$  and  $-$  signs in the symbols  $+3$  and  $-3$  are not intended to indicate the operations of addition and subtraction but are merely marks or signs to distinguish one direction from another. It would be better, from a logical standpoint, to reserve the  $+$  and  $-$  signs for the operations of addition and subtraction, and to use black and red ink, or other signs like  $_{+}3$  and  $_{-}3$  to indicate the right-hand 3 and the left-hand 3, respectively. However, we shall adhere to the traditional notation to avoid confusion.

The directed numbers, like the fractions, are our own inventions, and it is now up to us to decide what we want to mean by the sum, product, etc., of two directed numbers.

Our definition of the sum of two directed numbers can be best understood by referring to Fig. 5. Let us imagine ourselves standing at the origin, facing in the positive direction, and let us interpret each signed number as a marching order. The directed number  $+3$  shall mean "march forward, that is, to the right, 3 units," while the directed number  $-3$  shall mean "march back-



ward, that is, to the left, 3 units." Zero shall mean "do not march at all." The plus sign for addition shall be translated as "and then." The number attached to the point at which we arrive after carrying out all our marching orders is called the *sum* of the given directed numbers.

*Example 1.*  $(+3) + (+2)$  means walk right 3 units and then right 2 units, arriving at  $+5$ . Hence  $(+3) + (+2) = +5$ .

Notice that the plus signs within each pair of parentheses indicate positive numbers while the plus sign between the two pairs of parentheses indicates the operation of addition. As remarked above, it would be preferable, from a logical standpoint, to avoid this ambiguity by writing  ${}_R3 + {}_R2 = {}_R5$ , for example.

*Example 2.*  $(+3) + (-2)$  means walk right 3 units and then left 2 units, arriving at  $+1$ . Hence  $(+3) + (-2) = +1$ .

*Example 3.*  $(+2) + (-3)$  means walk right 2 units and then left 3 units, arriving at  $-1$ . Hence  $(+2) + (-3) = -1$ .

*Example 4.*  $(+3) + (-3)$  means walk right 3 units and then left 3 units, arriving at 0. Hence  $(+3) + (-3) = 0$ .

*Example 5.*  $(-3) + (-2)$  means walk left 3 units and then left 2 units, arriving at  $-5$ . Hence  $(-3) + (-2) = -5$ .

*Example 6.* Any number plus zero will be the given number again, since zero means "do not march at all." Thus  $(+3) + 0 = +3$ .

This intuitive idea of walking back and forth on the line of Fig. 5 is all that the student needs to understand in order to calculate sums of signed numbers successfully. For the sake of completeness, however, we append a formal definition, which may well be omitted on first reading, and which is cumbersome because it is necessary to use several cases in order to define the sum of signed numbers in terms of operations on unsigned numbers, which have been previously defined.

**DEFINITION 1.** If  $a$  and  $b$  are any unsigned numbers, then:

*Case 1.*  $(+a) + (+b) = +(a + b)$ .

*Case 2.*

$$(+a) + (-b) = (-b) + (+a) = \begin{cases} +(a - b) & \text{if } a > b. \\ -(b - a) & \text{if } a < b. \\ 0 & \text{if } a = b. \end{cases}$$

*Case 3.*  $(-a) + (-b) = -(a + b)$ .

*Case 4.*  $(+a) + 0 = 0 + (+a) = +a$ .

*Case 5.*  $(-a) + 0 = 0 + (-a) = -a$ .

*Case 6.*  $0 + 0 = 0$ .



*It is convenient to identify the positive (signed) numbers with the corresponding unsigned numbers. We do this henceforth. Thus we shall write 3 and +3 interchangeably.*

By the **absolute value** of a non-zero directed number is meant the corresponding positive number; thus the absolute value of 3 is 3 and the absolute value of  $-3$  is 3. The **absolute value** of zero is zero. The first three cases of definition 1 may be put as follows:

*To add two directed numbers of the same sign, add their absolute values and prefix their common sign. To add two directed numbers of opposite sign, subtract the smaller absolute value from the larger and prefix the sign attached to the number of larger absolute value.*

Our definition of multiplication of signed numbers will be motivated by our desire to have directed numbers behave as much like the previous unsigned numbers as possible. For example, we would like them to obey the basic commutative and distributive laws. By definition 1, section 4,  $3 \cdot 2$  stands for  $2 + 2 + 2$  or the sum of three 2's. Therefore we would like  $3 \cdot (-2)$  to mean the sum of three  $(-2)$ 's, or  $(-2) + (-2) + (-2)$ , or  $-6$ . Since we want to preserve the commutative law for multiplication, we want  $(-2) \cdot 3$  to mean the same thing as  $3 \cdot (-2)$ , or  $-6$ , even though  $(-2) \cdot 3$  cannot be interpreted as the sum of  $-2$  threes. Similarly, we want  $3 \cdot 0$  to mean  $0 + 0 + 0$  or 0, and, because we wish to preserve the commutative law, we want  $0 \cdot 3$  to mean 0 also. In general, we define  $0 \cdot a = a \cdot 0 = 0$  where  $a$  is any directed number.

Our desire to preserve the distributive law can be the motivation of the mysterious case  $(-3)(-2) = +6$ .\* Consider the expression  $(-3)[(-2) + 2]$ . We want the distributive law to hold so that

$$(-3)[(-2) + 2] = (-3)(-2) + (-3) \cdot 2.$$

The left member is  $(-3) \cdot [0]$  which we have already decided should be 0. The last term of the right member, as we have already decided, should be  $-6$ . Hence we have

$$0 = (-3)(-2) + (-6).$$

Thus, if the distributive law is to hold, we must define  $(-3)(-2) = +6$ . We are thus led to the following definition of the product of signed numbers.

\* Other considerations which also serve as motivations in this case will be found in M. Richardson. *Fundamentals of Mathematics*, Macmillan, 1941.



DEFINITION 2. If  $a$  and  $b$  are any two numbers, then

Case 1.  $(+a)(+b) = (-a)(-b) = +(ab)$ . (The product of two numbers with like signs is positive.)

Case 2.  $(+a)(-b) = (-b)(+a) = -(ab)$ . (The product of two numbers with unlike signs is negative.)

Case 3.  $(+a) \cdot 0 = 0 \cdot (+a) = (+a) \cdot 0 = 0$  and  $(-a) \cdot 0 = 0 \cdot (-a) = 0$ . (The product of any number with zero is zero.)

This definition is sometimes known as the rule of signs.

## EXERCISES

Calculate each of the following:

1.  $5 + (-2)$ .

2.  $3 + (-7)$ .

5.  $(+12) + (-5) + (+7) + (-15)$ .

6.  $(+16) + (-7) + (-18) + (+20)$ .

7.  $(-13) + (+7) + (-18) + (+20)$ .

8.  $(-17) + (-19) + (+12) + (-3)$ .

9.  $(+5)(-2)$ .

10.  $(-5)(-2)$ .

11.  $(-5) \cdot 0$ .

12.  $(-2)(+3)(-5)$ .

13.  $(+3)(-5)(-2)(-3)$ .

14.  $(-2)(+2)(-2)(+2)$ .

20.  $(-1)(+2)(-3) + (-2)(0)(-3) + (-2)(-3)(-4) + (-2)(-9)$

3.  $(-3) + (-5)$ .

4.  $(-4) + 0$ .

15.  $(-1)(+1)(-1)(-1)$ .

16.  $(+2)(-1)(-3) \cdot 0$ .

17.  $(-1)(-2) + (+4)(-3)$ .

18.  $(-3)(-7) + (-4)(-5)$ .

19.  $(-3)(-5) + (+4)(-3)$ .

10. Directed numbers. Subtraction and division. For subtraction we use definition 1, section 5, as before.

DEFINITION 1. If  $a$  and  $b$  are any directed numbers,  $a - b$  shall stand for the directed number  $x$  such that  $b + x = a$ , provided such a number  $x$  exists.

For example,  $(-2) - (-3)$  stands for the number  $x$  such that  $(-3) + x = (-2)$ ; hence  $x = 1$  since  $(-3) + 1 = -2$ . Similarly  $2 - (-3)$  stands for the number  $x$  such that  $(-3) + x = 2$ ; hence  $x = +5$  since  $(-3) + (+5) = (+2)$ .

It can be proved that in the system of all directed numbers  $a - b$  always exists. That is, the system is closed under subtraction. Hence the definition of  $>$  must be altered as follows.

DEFINITION 2. If  $a$  and  $b$  are directed numbers and if there exists a positive number  $x$  such that  $b + x = a$ , then we write  $a > b$  or  $b < a$ . In other words,  $a > b$  if and only if  $a - b$  is positive.



Thus  $-2 > -5$  and  $2 > -3$ . Intuitively  $a > b$  means that  $a$  is to the right of  $b$  on the line of Fig. 5.

Division is defined as in definition 3, section 5.

**DEFINITION 3.** *If  $a$  and  $b$  are any directed numbers,  $a \div b$  stands for the unique directed number  $x$  such that  $bx = a$ , provided such a number  $x$  exists.*

It can be shown that, except for the case  $b = 0$ , to be discussed in the next section,  $a \div b$  always exists.

For example,  $(-6) \div 2$  stands for the number  $x$  such that  $2x = -6$ ; hence  $x = -3$  because  $2(-3) = -6$ . Similarly  $6 \div (-2)$  stands for the number  $x$  such that  $(-2) \cdot x = 6$ ; hence  $x = -3$  because  $(-2)(-3) = +6$ . Also  $(-6) \div (-2)$  stands for the number  $x$  such that  $(-2) \cdot x = -6$ ; hence  $x = +3$  because  $(-2)(+3) = -6$ .

These examples suggest that the rule of signs for division is like that for multiplication: *the quotient of two numbers with like signs is positive, while the quotient of two numbers with unlike signs is negative.*

**DEFINITION 4.** *The **negative** of a directed number  $x$  shall be the directed number which must be added to  $x$  in order to obtain the sum 0. The negative of  $x$  is denoted by  $-x$ .*

For example, the negative of  $+3$  is  $-3$  since  $(+3) + (-3) = 0$ . Hence  $-(+3) = -3$ . Similarly, the negative of  $-3$  is  $+3$  because  $(-3) + (+3) = 0$ . Hence  $-(-3) = +3$ . That is,  $+3$  and  $-3$  are negatives of each other. This is a third use of the minus sign. It can be shown that no trouble arises from this triple ambiguity.

We could now prove that directed numbers obey all the familiar rules of algebra, such as the basic associative, commutative, and distributive laws. Also all rules for manipulating fractions carry over in an obvious way. For example,

$$(1) \qquad \frac{a}{b} = \frac{ax}{bx}$$

where  $a$ ,  $b$ , and  $x$  are any directed numbers, provided  $b$  and  $x$  are not zero. As illustrations, we shall indicate a few theorems each of which justifies a familiar but important manipulation.

**THEOREM 1.** *If  $a$  and  $b$  are any directed numbers, then  $a - b = a + (-b)$ .*

*Proof.* By definition 1 we have only to show that  $b$  plus the right member equals  $a$ . But



$$\begin{aligned}
 b + [a + (-b)] &= b + [(-b) + a] && \left( \begin{array}{l} \text{commutative law} \\ \text{for addition} \end{array} \right) \\
 &= [b + (-b)] + a && \left( \begin{array}{l} \text{associative law} \\ \text{for addition} \end{array} \right) \\
 &= 0 + a && \text{(definition 4)} \\
 &= a && \text{(definition 1, section 9).}
 \end{aligned}$$

This completes the proof.

**THEOREM 2.** *If  $a$  and  $b$  are any directed numbers, then  $a - (-b) = a + b$ .*

*Proof.* Similar to theorem 1. We leave this as an exercise for the reader.

Theorems 1 and 2 tell us that *subtracting a directed number is equivalent to adding its negative*.

A series of numbers separated by  $+$  and  $-$  signs may be called an **algebraic sum**. One of the individual numbers separated by these  $+$  and  $-$  signs, taken together with the preceding sign, constitutes a **term** of this algebraic sum. Thus in the expression

$$(1) \qquad 2 + 3 - 4 + a - b,$$

the terms are  $+2$ ,  $+3$ ,  $-4$ ,  $+a$ ,  $-b$ . By virtue of theorem 1, the expression (1) may be interpreted as  $2 + 3 + (-4) + a + (-b)$ . It follows from theorems 1 and 2 that the extended form of the commutative law for addition may be applied to algebraic sums in the following sense. *The order of the terms in an algebraic sum may be rearranged without affecting the value of the sum.*

For example, (1) may be rewritten as  $-4 + 2 - b + a + 3$ , or  $+3 - 4 - b + a + 2$ , and so on.

We could also prove\* that if  $a$  and  $b$  are any directed numbers, then

$$(2) \qquad a(-b) = -(ab) \text{ and } (-a)(-b) = +(ab).$$

This may also be called the **rule of signs for multiplication**. In particular,

$$(3) \qquad -a = (-1)a \text{ and } (-1)(-a) = a.$$

That is, *multiplication by  $-1$  reverses the sign of a directed number*.

Also,

$$(4) \qquad \frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}, \text{ while } \frac{-a}{-b} = \frac{a}{b}.$$

\* Proofs may be found in M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1911. The appropriateness of these definitions in certain geometric and practical situations is also discussed there.



This may be called the rule of signs for division.

**THEOREM 3.**  $-(a - b) = -a + b$ .

*First Proof.* By (3),  $-(a - b) = (-1)(a - b)$   
 $= (-1)(a + [-b])$   
 $= (-1)a + (-1)[-b]$   
(distributive law)  
 $= -a + b$  (by (3)).

*Second Proof.* By definition 4, we have only to prove that  $(a - b) + (-a + b) = 0$ . But the left member is  $a - a + (-b) + b = 0 + 0 = 0$ . This completes the proof.

**THEOREM 4.**  $-(a + b) = -a - b$ .

*Proof.* Left to the reader as an exercise. Compare theorem 3.

## EXERCISES

Calculate each of the following:

1.  $(-5) - (+3)$ .
2.  $5 - (-3)$ .
3.  $(-2) - (-5)$ .
4.  $(+5) - (-4) - (+3)$ .
5.  $(-7) - (-3) - (+5) - (-1)$ .
6.  $-5 - (-2) + (-5) - (+3)$ .
7.  $(-6) \div 2$ .
8.  $(-16) \div (-2)$ .
9.  $18 \div (-2)$ .
10.  $-\frac{(-5)(-8)}{-4}$ .
11.  $\frac{(-2)(+8)}{(+4)(-4)}$ .
12.  $-\left(\frac{-8}{+2} + \frac{15}{-3}\right)$ .
13.  $\frac{6 - (-6)}{2 - (-1)}$ .
14.  $\frac{16}{-2} - \frac{18}{-3}$ .
15.  $\frac{-35}{-7} - \frac{-14}{-2} + \frac{-2}{+1} - \frac{+5}{-1}$ .
16.  $\frac{16}{-2} \div \frac{22}{-2}$ .
17.  $\left(\frac{15}{-4} \div \frac{-3}{20}\right) - \left(\frac{-9}{-2} \div \frac{+3}{-10}\right)$ .
18.  $\frac{1}{2} \div \frac{-3}{2}$ .
19.  $\frac{-1}{2} + \frac{2}{5}$ .
20.  $-\frac{1}{2} - \frac{2}{5}$ .



$$21. \frac{-4 + (-3)}{(+2) + (-3)} - \frac{(-1) - 1}{(-2) - (+2)}.$$

$$22. \frac{5 - (-3)}{-2 - 3} \div \frac{8 - (-2) + (-6)}{-3 - (-13)}.$$

23. Subtract the sum of  $(-22)$  and  $(+14)$  from the product of  $(-2)$  and  $(-6)$ .

24. Subtract the quotient of  $(+16)$  divided by  $(-2)$  from the sum of  $(-3)$  and  $(+1)$ .

**11. The system of rational numbers. Properties of zero.** All the numbers introduced so far, namely zero, the positive and negative integers, and the positive and negative fractions, and no others, are called **rational numbers**. The word "rational" does not mean "reasonable"; it comes from the word "ratio." A number is *rational* if and only if it can be expressed as the quotient or ratio of two integers. Thus  $\frac{5}{2}$ ,  $\frac{3}{-7}$ , and  $\frac{2}{1}$  are rational numbers. A rational number which can be expressed with denominator 1 is an *integer*. For example,  $-2 = \frac{-2}{1}$  and  $0 = \frac{0}{1}$  are integers. Thus the system of rational numbers may be classified as follows:

*Rational numbers* (positive and negative fractions and zero)

*Integers* (rational numbers which can be expressed with denominator = 1)

*Negative integers*

*Zero*

*Positive integers or natural numbers.*

The system of rational numbers is closed under addition, subtraction, multiplication, and division, with the single exception that division by zero must be excluded. Why this exception must be made will be seen from the definition of division: the quotient  $a \div b$  stands for the unique number  $x$  such that  $bx = a$ , if such a number exists. Let us try to apply this definition if  $b = 0$ . Either  $a \neq 0$  or  $a = 0$ .

*Case 1.* Suppose  $a \neq 0$ . Then  $a \div 0$  stands for the number  $x$  such that  $0 \cdot x = a$ . But  $0 \cdot x = 0$  no matter what number  $x$  is, and  $a \neq 0$ . Hence no such number  $x$  exists. For example,  $5/0$  stands for the number  $x$  such that  $0 \cdot x = 5$ , but no number  $x$  can satisfy this requirement. Thus  $5/0$  is a meaningless symbol.

*Case 2.* Suppose  $a = 0$ . Then  $a \div 0$  or  $0 \div 0$  stands for the number  $x$  such that  $0 \cdot x = 0$ . But this equation is satisfied



no matter what number we choose for  $x$ , and is therefore quite useless.

*Therefore, division by zero is excluded in all cases.*

However,  $0 \div b$  does have a definite value if  $b \neq 0$ . It stands for the number  $x$  such that  $bx = 0$ . There is one and only one such value of  $x$ , namely zero. Hence,  $0/b = 0$  if  $b \neq 0$ . For example,  $0/3 = 0$ .

Notice that  $0/1 = 0$  while  $1/0$  is a meaningless symbol. The statement that a symbol has no meaning is very different from the statement that it has the value zero. A student who is not registered for this course receives no grade for it; a student must register for the course before he can aspire to the grade of zero.

*The system of integers is closed under addition, subtraction, and multiplication, but not division.* All definitions concerning factors, even and odd numbers, etc., may be extended from natural numbers to integers, if desired. For example, an integer  $b$  is said to be a **factor** of an integer  $a$  if there exists an integer  $x$  such that  $a = bx$ . An **even** integer is one that has 2 as a factor. For example  $+6$  and  $-6$  are even. Zero is even since  $0 = 0 \cdot 2$ ; zero has every integer as a factor.

Since the system of rational numbers is closed under division, except for division by zero, it follows that  $1/a$  is a rational number, provided  $a$  is a rational number different from zero.

**DEFINITION.** *If  $a \neq 0$ , the number  $1/a$  is called the **reciprocal** of  $a$ .*

Notice that *division by  $a$  is equivalent to multiplication by the reciprocal of  $a$ .*

The number zero has the following important property, which will be needed later:

**THEOREM 1.** *If  $ab = 0$  and  $a \neq 0$ , then  $b = 0$ .*

*Proof.* Since  $a \neq 0$ ,  $1/a$  exists. By hypothesis,

$$ab = 0.$$

Multiplying both sides by  $1/a$ , we have

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$

because if equals are multiplied by equals, the results are equal. But the right member is zero and the left member becomes  $\frac{ab}{a}$  or  $\frac{b}{1}$  or  $b$ . Hence  $b = 0$ . This completes the proof.



It follows from theorem 1 that the product of any number of non-zero numbers cannot be zero. Or, the product of several numbers is zero if and only if at least one of the factors is itself zero.

The system of rational numbers, extensive as it is, will be seen to be inadequate for the purposes of both algebra and geometry. First we need some preliminary considerations, to be taken up in the next sections.

*Note.* Amusing results can be obtained if division by zero is overlooked. (We assume for the moment that the reader recalls some elementary algebra.) For example, let Alice be  $a$  years old and let Betty be  $b$  years old, and suppose they are of the same age. Then  $a = b$ . Multiplying both sides of this equation by  $b$ , we obtain  $ab = b^2$ . Subtracting  $a^2$  from both sides, we have  $ab - a^2 = b^2 - a^2$ . Factoring, we get  $a(b - a) = (b + a)(b - a)$ . Dividing both sides by  $b - a$ , we have  $a = b + a$ . Since  $a = b$ , this implies  $a = b + b$ , by substitution. Therefore  $a = 2b$ , and Alice discovers, to her chagrin, that she is twice as old as Betty. The question is, what made Alice age so rapidly? It was not caused by overstudy of mathematics, but rather by division by  $b - a$  which is zero, since  $a = b$  by hypothesis. This absurdity can be pushed further. From the statement  $a = 2b$  above, we get  $a = 2a$ , since  $b = a$ . Dividing both sides by  $a$ , we have  $1 = 2$ . But  $3 = 2 + 1 = 1 + 1 = 2 = 1$ ,  $4 = 3 + 1 = 1 + 1 = 2 = 1$ , and so on. We thus arrive at an oversimplified system of arithmetic in which all numbers are equal to each other, and there can be no such thing as a wrong answer.

## EXERCISES

If  $a$  and  $b$  are any two rational numbers, does each of the following expressions always represent a rational number?

1.  $a + b$ .
2.  $a - b$ .
3.  $a \cdot b$ .
4.  $a/b$ .

If  $a$  and  $b$  are any two integers, does each of the following expressions always represent an integer?

5.  $a + b$ .
6.  $a - b$ .
7.  $a \cdot b$ .
8.  $a/b$ .

12. Positive integral powers and roots. The following definition is nothing more than an abbreviation.

**DEFINITION 1.** If  $n$  is a positive integer,  $x^n$  shall stand for  $x \cdot x \cdot \dots \cdot x$  with  $n$  factors.

For example,  $x^3 = xxx$ . The exponent  $n$  is merely the number of factors. We call  $x^n$  the  $n$ th power of  $x$ .  $x^2$  is called the square of  $x$ , and  $x^3$  is called the cube of  $x$ . We need the following theorems.



**THEOREM 1.** *If  $n$  is any positive integer, then  $(xy)^n = x^n y^n$ .*

*Proof.* By definition 1,  $(xy)^n = (xy)(xy) \cdots (xy)$  where there are  $n$  parentheses. By the associative law for multiplication, we may remove parentheses, so that

$$(xy)^n = xyxy \cdots xy.$$

By the commutative law for multiplication, we may rearrange the factors, writing

$$(xy)^n = xx \cdots xyx \cdots y.$$

By the associative law, we may regroup them, writing

$$(xy)^n = (xx \cdots x)(yy \cdots y).$$

By definition 1,  $(xx \cdots x) = x^n$  and  $(yy \cdots y) = y^n$ . Hence

$$(xy)^n = x^n y^n.$$

This proves the theorem.

It is clear that this theorem can be extended to more than 2 factors. Thus,

$$(1) \quad (xyzw \cdots)^n = x^n y^n z^n w^n \cdots$$

**THEOREM 2.**  $(a + b)^2 = a^2 + 2ab + b^2$ .

*Proof.* By definition 1,  $(a + b)^2 = (a + b)(a + b)$ . By the law of closure for addition, we may regard  $(a + b)$  as a single number. Hence, by the distributive law

$$(a + b)(a + b) = (a + b)a + (a + b)b.$$

Using the distributive law again, and the commutative law for multiplication, the right member becomes  $a^2 + ab + ab + b^2$ . But  $ab + ab = 2ab$ . Hence  $(a + b)^2 = a^2 + 2ab + b^2$ . This proves the theorem.

It is clear that this theorem can be extended to more than 2 terms. Thus,

$$(2) \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

**DEFINITION 2.** *If there exists a number  $x$  such that  $x^n = a$ , where  $n$  is a positive integer greater than one, then  $x$  is called an  $n$ th root of  $a$ . In particular if  $x^2 = a$ ,  $x$  is called a **square root** of  $a$ ; and if  $x^3 = a$ ,  $x$  is called a **cube root** of  $a$ .*

For example, 3 is a square root of 9 since  $3^2 = 9$ . Also,  $-3$  is a square root of 9 since  $(-3)^2 = 9$ . Both 2 and  $-2$  are fourth



roots of 16 since  $2^4 = 16$  and  $(-2)^4 = 16$ . A cube root of 8 is 2 because  $2^3 = 8$ ; a cube root of  $-8$  is  $-2$  because  $(-2)^3 = -8$ .

It is clear that a number may have more than one  $n$ th root; in fact, in a later chapter it will be shown that, in the system of complex numbers, every number has  $n$  distinct  $n$ th roots; that is, two square roots, three cube roots, and so on. To avoid the unpleasant ambiguity attached to the idea of  $n$ th root, we agree to reserve the radical sign  $\sqrt[n]{a}$  to denote a single one of these  $n$ th roots, to be called the **principal  $n$ th root**. If  $a$  is positive, the principal  $n$ th root will be the positive one. If  $a$  is negative and  $n$  is odd, the principal  $n$ th root will be the negative one. No other cases are needed here. For example,  $\sqrt[3]{9}$  stands for 3 but not for  $-3$ ; if we wish to indicate  $-3$ , we write  $-\sqrt[3]{9}$ . Also  $\sqrt[3]{8} = 2$  while  $\sqrt[3]{-8} = -2$ . This agreement is made to avoid the confusion that would arise if we permitted the symbol  $\sqrt[3]{9}$  to stand for either 3 or  $-3$  ambiguously. The principal  $n$ th root of a number is uniquely determined. The number  $n$  written above the radical sign is called the **index** of the root.

We make the linguistic agreement that powers shall take precedence over multiplication and division, except where otherwise indicated by parentheses. For example,  $5 \cdot 3^2 = 5 \cdot 9 = 45$  while  $(5 \cdot 3)^2 = 15^2 = 225$ . The radical sign, however, acts as a parenthesis; all indicated operations under the radical sign are to be done before extracting the root. For example,  $\sqrt{9 + 16} = \sqrt{25} = 5$ , while  $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$ .

By definition  $\sqrt[n]{a}$  stands for a number which yields  $a$  when raised to the  $n$ th power; hence

$$(3) \quad (\sqrt[n]{a})^n = a.$$

**THEOREM 3.** If  $a$  and  $b$  are positive, then  $\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}$ .

*Proof.* Let  $x = \sqrt[n]{a}$  and  $y = \sqrt[n]{b}$ . By (3),  $x^n = a$  and  $y^n = b$ . By theorem 1,  $(xy)^n = x^n y^n$ . Hence  $(xy)^n = ab$ . By definition,  $xy = \sqrt[n]{ab}$ , since  $xy$  is also positive. By substitution,  $\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}$ . This completes the proof.

For example,  $\sqrt{4} \sqrt{9} = \sqrt{4 \cdot 9} = \sqrt{36}$ , or  $2 \cdot 3 = 6$ .

Note that this theorem is not true for negative  $a$  and  $b$ . For example,  $\sqrt{-2} \sqrt{-2}$  is not  $\sqrt{4}$ , because  $\sqrt{-2} \sqrt{-2} = (\sqrt{-2})^2 = -2$  by (3), while  $\sqrt{4} = +2$ .

It is obvious that if  $a = b$ , then  $a^2 = b^2$ . But the converse is not true. From  $a^2 = b^2$  we cannot infer that  $a = b$ ; for example, from  $(-3)^2 = (+3)^2$  we cannot infer that  $-3 = +3$ . It is cor-



and the fact, however, that if  $a^2 = b^2$ , then  $a$  is equal to either  $b$  or  $-b$ . But from  $a^2 = b^2$  we may conclude that  $a = \pm b$ .

# EXERCISES

By inspection, find the value of each of the following:

1.  $\sqrt{25}$       2.  $\sqrt{-121}$       3.  $\sqrt{16}$       4.  $\sqrt{81}$       5.  $\sqrt{100}$
6.  $\sqrt{49}$       7.  $\sqrt{36}$       8.  $\sqrt{64}$       9.  $\sqrt{144}$       10.  $\sqrt{-100}$

11. Is it true that  $\sqrt{a^2 + b^2} = a + b$ ? Explain. (Of what  $a, b$  is this square root?)

12. Is it true that  $\sqrt{a^2 - b^2} = a - b$ ? Explain. (Of what  $a, b$  is this square root?)

13. Is it true that  $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$ ? Explain.

14. Is it true that  $\sqrt{x - y} = \sqrt{x} - \sqrt{y}$ ? Explain.

Compute each of the following, remembering that all letters represent positive numbers.

15.  $(\sqrt{16})^2$       16.  $\sqrt{16} \cdot \sqrt{9}$       17.  $\sqrt{16} \cdot \sqrt{25}$
18.  $(\sqrt{25})^2$       19.  $\sqrt{25} \cdot \sqrt{16}$       20.  $\sqrt{25} \cdot \sqrt{36}$
21.  $(\sqrt{36})^2$       22.  $\sqrt{36} \cdot \sqrt{25}$       23.  $\sqrt{36} \cdot \sqrt{49}$
24.  $(\sqrt{49})^2$       25.  $\sqrt{49} \cdot \sqrt{36}$       26.  $\sqrt{49} \cdot \sqrt{64}$
27.  $(\sqrt{64})^2$       28.  $\sqrt{64} \cdot \sqrt{49}$       29.  $\sqrt{64} \cdot \sqrt{81}$
30.  $(\sqrt{81})^2$       31.  $\sqrt{81} \cdot \sqrt{64}$       32.  $\sqrt{81} \cdot \sqrt{100}$

33. Find the relation to the following "proof" that all numbers are equal to each other:

Let  $a$  and  $b$  be any two unequal numbers. Let  $c$  be their average, that is,  $c = \frac{a+b}{2}$ . Then  $2c = a + b$ . Multiplying both sides by  $a - b$ , we have  $2c(a - b) = (a + b)(a - b)$ , or  $2ac - 2bc = a^2 - b^2$ . Transposing, we get  $b^2 - 2bc + c^2 = a^2 - 2ac + c^2$ . Adding  $c^2$  to both sides, we get  $b^2 - 2bc + c^2 = a^2 - 2ac + c^2$ , or  $(b - c)^2 = (a - c)^2$ . Therefore  $b - c = a - c$ . Hence  $b = a$ .

12. The square root of 2. We shall prove the following surprising theorem.

**THEOREM 1.** No rational number is a square root of 2.

Before proving the theorem, let us see why it is surprising. It is clear that there is not a square root of 2. Figure 1-10 is the proof of this fact. But we must restrict our attention to positive rational numbers. Recall that a rational number is one that can be expressed as a quotient of two integers. Hence we can arrange



all the positive rational numbers in the following array, placing all those with denominator 1 in the first horizontal row, all those with denominator 2 in the second row, and so on:

|               |               |               |               |     |
|---------------|---------------|---------------|---------------|-----|
| 1             | 2             | 3             | 4             | ... |
| $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | ... |
| 1             | 2             | 3             | 4             | ... |
| $\frac{1}{2}$ | $\frac{2}{2}$ | $\frac{3}{2}$ | $\frac{4}{2}$ | ... |
| 1             | 2             | 3             | 4             | ... |
| $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{3}{3}$ | $\frac{4}{3}$ | ... |
| 1             | 2             | 3             | 4             | ... |
| $\frac{1}{4}$ | $\frac{2}{4}$ | $\frac{3}{4}$ | $\frac{4}{4}$ | ... |
| .             | .             | .             | .             | .   |
| .             | .             | .             | .             | .   |
| .             | .             | .             | .             | .   |

This scheme, extending endlessly to the right and down, surely includes all positive rational numbers; for if you name any positive

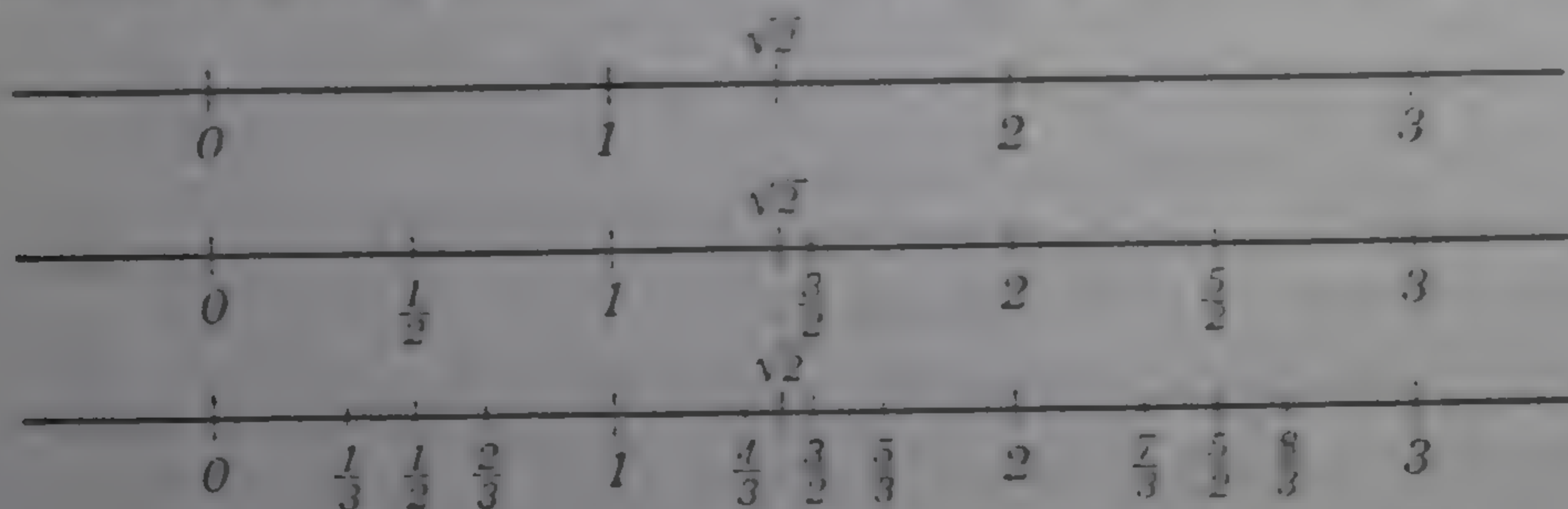


Fig. 6.

rational number, we can say exactly where it is in this array. For example,  $\frac{159}{357}$  is to be found in the 159th vertical column and the 357th horizontal row.

Let us imagine the points corresponding to these numbers marked off on a line, one horizontal row at a time, the number attached to each point representing its distance from the origin or zero point. As each succeeding row is used, the marked points clearly become more and more crowded together (*densely distributed* is a technical term for this). See Fig. 6. By the time the thousandth row is reached, we are marking off  $\frac{1}{1000}$ ,  $\frac{2}{1000}$ ,  $\frac{3}{1000}$ , and so on. Now imagine all the rational numbers already marked off. Your intuition might tell you that surely all points on the line must have been marked off, *but this is not the case*. There are still points which are not marked off by any rational number. One such point may be constructed as follows. Take



an isosceles right triangle with legs 1 unit long (Fig. 7). Then, by the Pythagorean theorem, the length of the hypotenuse is  $\sqrt{1^2 + 1^2} = \sqrt{2}$ . This length may be laid off from the origin, and the point thus marked off cannot have been marked by any rational number if our theorem is correct. If you recall that all rational numbers are supposed to be marked off, including all millionths, all trillionths, etc., it may seem remarkable that there are still unmarked points left (Fig. 6).

To prove our theorem we might at first try squaring all the rational numbers, one at a time, to see if any of them will yield 2 when squared. Of course, if we started out along the first horizontal row we would never get out of the first row, since it is end-

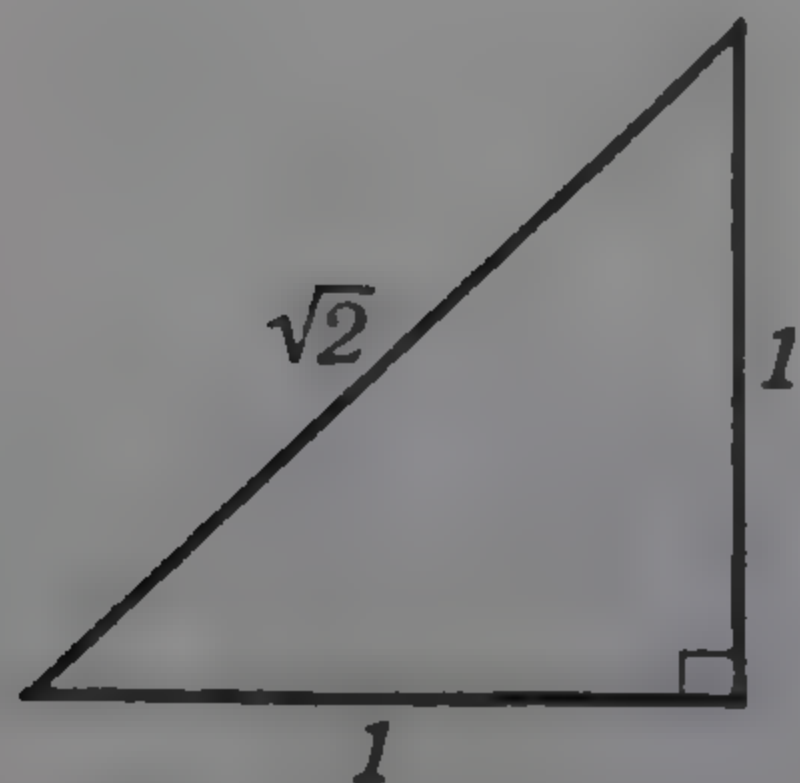


Fig. 7.

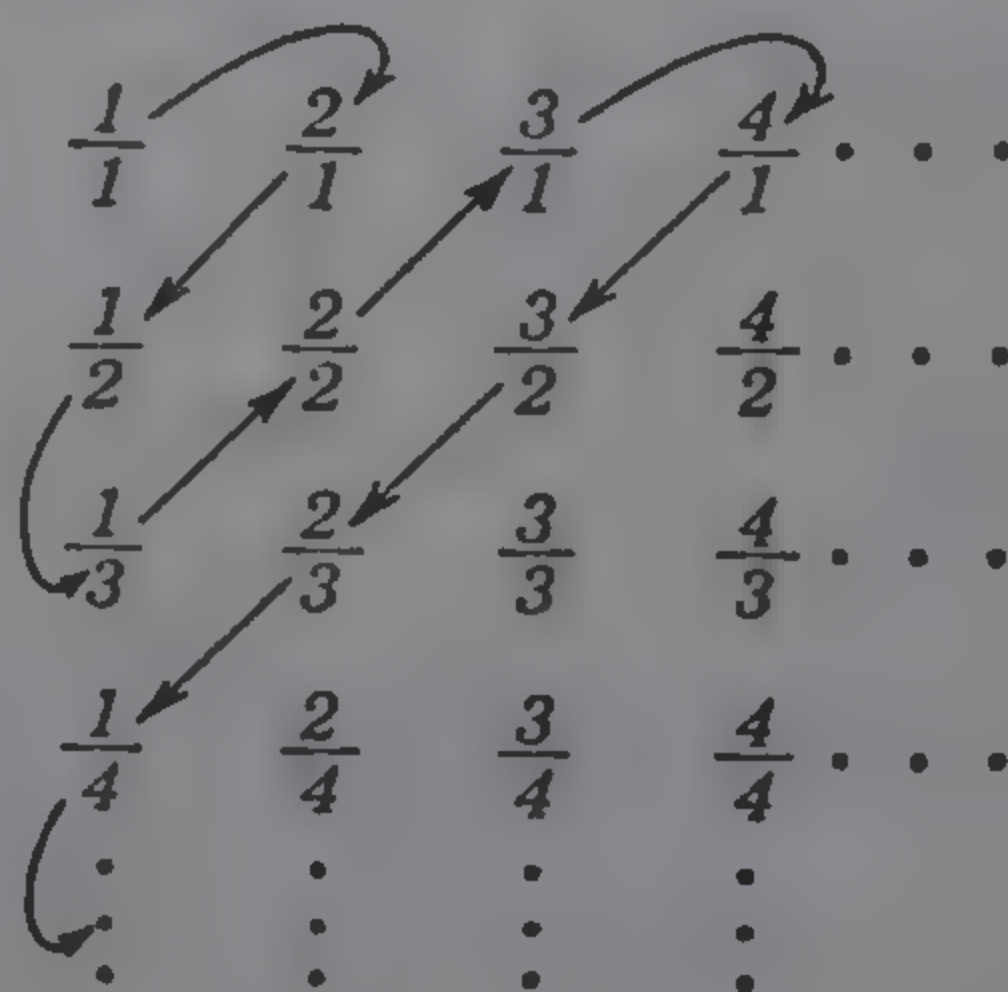


Fig. 8.

less. Therefore we might try to sweep up and down the diagonals, following the arrows in Fig. 8; that is, we would try squaring the rational numbers in the following order:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{2}, \frac{1}{1}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{1}{2}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \dots$$

If only the number of rational numbers were limited, we could indeed establish our theorem by this naïve method. However, since there is no end to the sequence of rational numbers, this method is absolutely hopeless. For if we squared the first 1000 rational numbers in this sequence, and failed to obtain 2 as the result, this would not prove that we might not get 2 as the result at some later trial. Therefore we must do something more clever to prove the theorem. We shall need the following preliminary theorem.

**THEOREM A.** *If a natural number  $n$  is the product of the prime factors  $p, q, r, \dots$ , then  $n^2$  is the product of the same prime factors, each occurring twice as often.*



*Proof.* Suppose  $n = p \cdot q \cdot r \cdot \cdot \cdot$ . Then, by (1), section 12,  $n^2 = p^2 q^2 r^2 \cdot \cdot \cdot = p \cdot p \cdot q \cdot q \cdot r \cdot r \cdot \cdot \cdot$ . This completes the proof.

For example,  $12 = 2 \cdot 2 \cdot 3$  and  $12^2 = 144 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ .

*Proof of Theorem 1.* Suppose there were a (positive) rational number  $p/q$  (where  $p$  and  $q$  are natural numbers), whose square is 2. Then  $p^2/q^2 = 2$ , or, multiplying both sides by  $q^2$ ,

$$(1) \quad p^2 = 2q^2.$$

By theorem A, both  $p^2$  and  $q^2$  have 2 as a factor an even number of times, if at all. But then the right member of (1) has 2 as a factor an odd number of times while the left member has it an even number of times. This is absurd because it contradicts the fact that a natural number can be expressed as a product of prime factors in only one way (section 6). Hence the supposition that there is a rational number whose square is 2 is false. This completes the proof.

The fact\* that there is no rational number whose square is 2 was probably known to Pythagoras in the 6th century B.C. It is said to have disturbed him so much that he tried to suppress the information lest it discredit mathematicians in the eyes of the general public.

*Note.* In the proof of theorem 1 we used the method of **indirect proof** or **reductio ad absurdum** which you first met in your study of plane geometry. That is, we show that if the supposition that statement  $S$  is true leads logically to a false conclusion, then  $S$  must be false.

## EXERCISES

*Prove that no rational number is equal to:*

1.  $\sqrt{3}$ .

3.  $\sqrt{7}$ .

5.  $\sqrt[3]{2}$ .

2.  $\sqrt{5}$ .

4.  $\sqrt{p}$ , where  $p$  is a prime.

6.  $\sqrt[3]{3}$ .

7.  $\sqrt[n]{p}$ , where  $p$  is a prime.

\*8.  $\sqrt{m}$  where  $m$  is a positive integer which is not the square of an integer.

\*9.  $\sqrt[n]{m}$  where  $m$  is a positive integer which is not the  $n$ th power of an integer.

\* For the relation between this fact and the so-called incommensurable magnitudes of geometry, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941. Another proof of the theorem above may be found there as well.

\* May be omitted without disturbing continuity.



**14. The system of real numbers. Decimal notation.** The fact that no rational number is exactly a square root of 2 means, from the point of view of algebra, that the system of rational numbers is not closed under the operation of taking square roots. That is, we cannot apply the operation of taking square roots to any number of the system, 2 for example, and always get a number of the system. From the point of view of geometry, it means that there are lengths, such as the hypotenuse of the isosceles right triangle with unit legs, which are not exactly described by any rational number. From either point of view, the system of rational numbers, extensive though it is, is insufficient for our needs. We therefore invent **irrational numbers**, such as  $\sqrt{2}$ . The word "irrational" must not be taken to mean "unreasonable"; it means not expressible as a ratio of two integers. We shall give no accurate definition of irrational number here because of the technical difficulties involved. Indeed, while the fact that  $\sqrt{2}$  is irrational was probably known to Pythagoras (6th century B.C.), a completely logical treatment of irrational numbers was not presented until about 1870, when G. Cantor and R. Dedekind, two German mathematicians, straightened out the difficulties by different methods. However, we can discuss three of the essential characteristics of irrational numbers.

The first point is that the irrational numbers fill up all the "gaps," that is, unmarked points, on the line (Fig. 6), so that every point on the line has a number attached to it, either rational or irrational. All these numbers, attached to points on the line, are called **real**. The system of real numbers is closed under addition, multiplication, subtraction, and division, except for division by zero, and obeys all the basic laws of algebra such as the associative, commutative, and distributive laws.

The second point is that irrational numbers are by no means rare, even though, in the preceding section, we proved only that one specimen,  $\sqrt{2}$ , was irrational. In fact, it can be shown that the square root of any positive integer which is not a perfect square (that is, the square of an integer) is irrational; for example,  $\sqrt{3}$ ,  $\sqrt{5}$ ,  $\sqrt{6}$ , etc. Furthermore,  $\sqrt[n]{a}$  is irrational if  $a$  is a positive integer which is not the  $n$ th power of an integer. For example,  $\sqrt[3]{2}$ ,  $\sqrt[3]{7}$ ,  $\sqrt[3]{9}$ , etc. There are many other irrational numbers, such as  $\pi = 3.14159265 \dots$ , which cannot be expressed in terms of  $n$ th roots of integers at all. To demonstrate further the plentifulness of irrational numbers, we prove the following two theorems.



**THEOREM 1.** *If a rational number  $R$  is added to an irrational one  $I$ , the sum  $S$  is again irrational.*

*Proof.*  $S = R + I$  implies  $I = S - R$ . Now  $S$  is real and therefore either rational or irrational. If  $S$  were rational then  $S - R$  would be rational since the system of rational numbers is closed under subtraction. But then  $I$  would be rational, contrary to hypothesis. Therefore  $S$  must be irrational.

*Examples.* It follows that  $2 + \sqrt{2}$ ,  $3 + \sqrt{2}$ ,  $-\frac{1}{2} + \sqrt{2}$ , etc., are irrational.

**THEOREM 2.** *If an irrational number  $I$  is multiplied by a rational number  $R \neq 0$ , then the product  $P$  is irrational.*

*Proof.*  $P = RI$  implies  $I = P/R$  since  $R \neq 0$ . Now  $P$  is real and therefore either rational or irrational. If  $P$  were rational, then  $P/R$  would be rational since the system of rational numbers is closed under division except for division by zero. Then  $I$  would be rational contrary to hypothesis. Therefore  $P$  is irrational.

*Examples.*  $2\sqrt{2}$ ,  $3\sqrt{2}$ ,  $\frac{1}{2}\sqrt{2}$ ,  $\frac{1+\sqrt{2}}{2}$ , etc., are irrational.

The third point concerns the decimal expression of real numbers. Recall that the usual decimal notation for numbers is based on powers of 10. For example, 3425.678 really stands for

$$3 \cdot 10^3 + 4 \cdot 10^2 + 2 \cdot 10 + 5 + \frac{6}{10} + \frac{7}{10^2} + \frac{8}{10^3}.$$

Let us try to express  $\sqrt{2}$  as a decimal. If  $x = \sqrt{2}$ , then  $x^2 = 2$ . Hence we try  $x = 1$  and find it too small since  $x^2 = 1 < 2$ ; but  $x = 2$  is too large since  $x^2 = 4 > 2$ . Hence  $1 < \sqrt{2} < 2$ , or  $\sqrt{2} = 1. \dots$ . That is, we have located  $\sqrt{2}$  between 1 and 2.

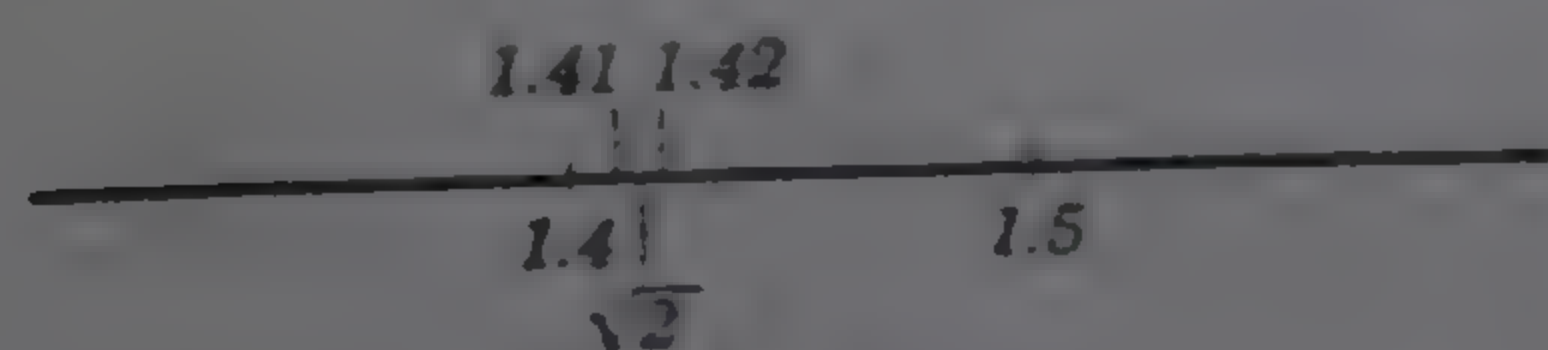


Fig. 9.

We then get a better approximation by splitting the interval between 1 and 2 into tenths and trying them. We find  $(1.1)^2 = 1.21$ ,  $(1.2)^2 = 1.44$ ,  $(1.3)^2 = 1.69$ ,  $(1.4)^2 = 1.96$  all too small,

while  $(1.5)^2 = 2.25$  is too large. Hence  $1.4 < \sqrt{2} < 1.5$ , or  $\sqrt{2} = 1.4 \dots$ . (See Fig. 9.) Then we split the interval from 1.4 to 1.5 into hundredths and find by trial that  $(1.41)^2 = 1.9881 < 2 < (1.42)^2 = 2.0164$ . Hence  $\sqrt{2} = 1.41 \dots$ . Then trying successive thousandths, we find  $(1.414)^2 = 1.999396 < 2 < (1.415)^2 = 2.002225$ ; hence  $\sqrt{2} = 1.414 \dots$ . At each stage of this



successive approximation we fail to get an exact square root of 2. Thus the decimal expression for  $\sqrt{2}$  does not seem to stop. The mere fact that we find that it doesn't stop for the first thousand decimal places, however, does not prove that it will not stop thereafter. But we can, in fact, prove that it never stops. For if it did stop, we would have a terminating decimal expression for  $\sqrt{2}$ . Now the following theorem is obvious.

**THEOREM 3.** If a decimal terminates, it represents a rational number.

For example, 3.426 is a rational number since it is a quotient of two integers, namely  $3426/1000$ .

But  $\sqrt{2}$  is not rational. Hence its decimal expression cannot terminate. However, each of the stages of our process of successive approximations yields a rational number whose square becomes closer and closer to 2. That is,  $\sqrt{2}$  can be approximated as closely as we please (that is, to the nearest thousandth, millionth, etc.) by rational numbers, by carrying the process far enough. This is one essential characteristic of irrational numbers.

The converse of theorem 3 is false, for a number may be rational without having a terminating decimal expression. For example we find by division, that

$$\frac{2}{3} = .666 \dots$$

and

$$\frac{1}{7} = .142857142857142857 \dots$$

However, both of these decimal expressions "repeat in blocks"; in technical language they are **periodic** or **recurring**. In the case of  $1/7$ , notice that the only remainders which can occur in the process of long division are 0, 1, 2, 3, 4, 5, 6. If 0 occurred, the process would terminate. If 0 does not occur, then, after enough steps (seven or fewer), we must get one of the remainders 1, 2, 3, 4, 5, 6 back again for the second time, whereupon the whole process repeats itself exactly, thus producing a periodic decimal. In the same way, the following theorem can be proved.

**THEOREM 4.** *The decimal expression of a rational number either terminates or, if it is non-terminating, is periodic.*

After the subject of geometric progressions has been taken up, it will be possible for us to prove the following.

**THEOREM 5.** *Every periodic decimal represents a rational number.*



The preceding theorems imply that the following is true:

**THEOREM 6.** *An irrational number has a decimal expression which is neither terminating nor periodic.*

*Every real number can be expressed as a decimal,\* approximating or not, and, conversely, every decimal represents a real number.* Whether a real number is rational or not does not matter to the practical engineer directly, since measurements are never more than approximate and an irrational number can be approximated as closely as we please by terminating decimals. But the concept of irrational numbers is essential to any logical theory of geometric magnitudes, and to the logical derivation of many theorems of algebra and calculus, upon which much practical work is based.

*Every real number is either positive, negative, or zero according as it lies to the right of, to the left of, or at the origin.*

Note that the simple, although tedious, method of successive approximation by trials, which we used above to find the decimal expression for  $\sqrt{2}$  can be used not only for square roots, but also for cube roots, fourth roots, etc. It resembles the process of running down a base runner caught trying to steal second base in a baseball game, except that while we continue to pinch the irrational number between narrower and narrower limits, we never tag it exactly.

**Note 1.** The word "fraction" is often used for any indicated quotient such as  $\sqrt{2/3}$  which is not a fraction in the sense of section 7. To avoid confusion, we shall henceforth refer to the fractions of section 7 as *positive rational numbers*.

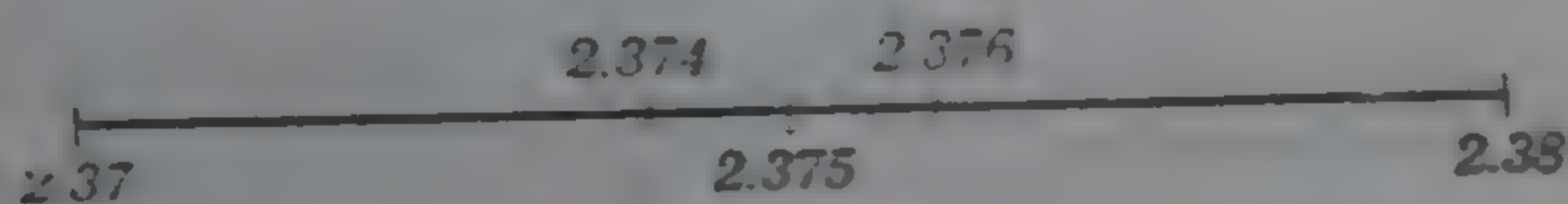


Fig. 10.

**Note 2.** The meaning of the phrase *correct to the nearest hundredth* is self-explanatory if one thinks of the geometric picture (Fig. 10). For example, 2.374 becomes 2.37 when corrected or rounded off to the nearest hundredth, while 2.376 becomes 2.38 when corrected to the nearest hundredth. If a number is exactly in the middle of an interval, we agree to assign it to the larger end: thus 2.375 is rounded off to the nearest hundredth as 2.38.

\* This will be proved in Chapter XXVII.



## EXERCISES

(a) Approximate as far as three decimal places, that is, locate between successive thousandths; (b) write the answer rounded off to the nearest hundredth:

1.  $\sqrt{3}$ .    2.  $\sqrt{5}$ .    3.  $\sqrt[3]{2}$ .    4.  $\sqrt[3]{3}$ .    5.  $\sqrt[3]{5}$ .    6.  $\sqrt[3]{10}$ .

Show that each of the following numbers is irrational:

7.  $3 + \sqrt{2}$ .    8.  $5\sqrt{2}$ .    9.  $\sqrt{2}/5$ .    10.  $\frac{1 + \sqrt{2}}{3}$ .    11.  $\frac{1 - \sqrt{2}}{3}$ .

12. (a) Show that  $22/7$  approximates  $\pi = 3.14159 \dots$  to the nearest hundredth. (b) If we take the decimal expression for  $22/7$  beyond the hundredths place, will more places yield a closer approximation to  $\pi$ , or not?

Tell whether each of the following is true or false:

13. The decimal expression for  $\sqrt{4/9}$  is neither terminating nor periodic.

14. The decimal expression for  $\sqrt{5}/3$  is neither terminating nor periodic.

15. Show that  $3\frac{10}{71} < \pi < 3\frac{1}{7}$ . ( $\pi = 3.14159 \dots$ )

\*15. **Extraction of square roots.** While the method of trials used in the preceding section to obtain the decimal expression for  $\sqrt{2}$  has the advantage of applying also to cube roots, etc., it is admittedly tedious. The student may recall the following quicker scheme for extracting square roots.

*Example 1.*

$$\begin{array}{r} 3 \ 5. \\ \sqrt{12'25.} \\ 9 \\ \hline 65 \overline{) 3 \ 25} \\ \underline{3 \ 25} \\ 0. \end{array}$$

But it is likely that the student memorized this process without ever understanding how or why it works. We shall try to explain the basis for this peculiar procedure, including the "doubling" of the partial answer.

\* This section may be omitted without disturbing the continuity of the chapter.



Suppose first that the number  $N$ , whose square root we wish to extract, is a perfect square, that is, the square of an integer. Note that  $1^2 = 1$ ,  $9^2 = 81$ ;  $10^2 = 100$ ,  $99^2 = 9801$ ;  $100^2 = 10000$  and  $999^2 = 998001$ ; and so on. Hence we see that a one- or two-digit square has a one-digit square root, a three- or four-digit square has a two-digit square root, a five- or six-digit square has a three-digit square root, and so on. Therefore, *the first step in the process is to mark off pairs of digits in the number  $N$  to the left of the decimal point, as far as possible*: thus,  $12'25$ . in example 1. Let us separate the answer, which we know will consist of two digits, in the case of example 1, into tens and units, denoting the tens by  $t$  and the units by  $u$ . That is, we write the answer as  $t + u$ ; in example 1, it turns out that  $t = 30$  and  $u = 5$ , so that  $t + u = 35$ . Our task is to find  $t$  and  $u$ . We know that  $N = (t + u)^2$ . By theorem 2, section 12, this means that  $N = t^2 + 2tu + u^2$ . The significance of the steps in the process of determining  $t$  and  $u$  will be understood by studying the following analysis of example 1.

$$\begin{array}{rcl}
 3 \quad 5. & = & 30 + 5 = t + u \\
 \sqrt{12' \quad 25.} & = & N \\
 9 \quad (00) & = & t^2 \\
 2t + u = 65 \left[ \begin{array}{rcl} 3 \quad 25 & = & N - t^2 \\ 3 \quad 25 & = & (2t + u)u = 2tu + u^2 \\ \hline 0 & = & N - t^2 - 2tu - u^2. \end{array} \right.
 \end{array}$$

From the last line we see that  $N = t^2 + 2tu + u^2$ , or  $N = (t + u)^2$ . Thus  $t + u = 35 = \sqrt{1225}$ .

*Example 2.* Similarly, let  $h$  represent the hundreds in the three-digit answer for  $\sqrt{54756}$ . Then

$$\begin{array}{rcl}
 2 \quad 3 \quad 4. & = & 200 + 30 + 4 = h + t + u \\
 \sqrt{5' \quad 47' \quad 56.} & = & N \\
 4 \quad (00 \quad 00) & = & h^2 \\
 2h + t = 43(0) \left[ \begin{array}{rcl} 1 \quad 47 \quad 56 & = & N - h^2 \\ 1 \quad 29 \quad (00) & = & (2h + t)t = 2ht + t^2 \\ 2h + 2t + u = 464 \left[ \begin{array}{rcl} 18 \quad 56 & = & N - h^2 - 2ht - t^2 \\ 18 \quad 56 & = & (2h + 2t + u)u = 2hu \\ & & + 2tu + u^2 \\ \hline 0 & = & N - h^2 - 2ht - t^2 - 2hu \\ & & - 2tu - u^2. \end{array} \right. \end{array} \right.
 \end{array}$$



From the last line we have

$$N = h^2 + t^2 + u^2 + 2ht + 2hu + 2tu = (h + t + u)^2$$

by (2), section 12. Hence  $h + t + u = \sqrt{N}$ , or  $234 = \sqrt{54756}$ .

*The zeros written in parentheses, above, are usually omitted.*

The student should see for himself that this process can be continued in the usual way to the right of the decimal point, to obtain (approximate) square roots of numbers which may not be perfect squares. Thus

$$\begin{array}{r} 1.414 \dots \\ \sqrt{2.0000} \\ 1 \\ \hline 24 \overline{) 100} \\ 96 \\ \hline 281 \overline{) 400} \\ 281 \\ \hline 2824 \overline{) 11900} \\ 11296 \\ \hline 605 \\ \dots \end{array}$$

*It will be unnecessary to develop a similar scheme for cube roots, etc., since a general method for performing such computations will be taken up in sections 112 and 115. See also table I.*

## EXERCISES

*Using the usual scheme, calculate the (principal) square root of each of the following:*

- |             |             |            |             |              |
|-------------|-------------|------------|-------------|--------------|
| 1. 625.     | 2. 576.     | 3. 729.    | 4. 64516.   | 5. 237169.   |
| 6. 450241.  | 7. 498436.  | 8. 651249. | 9. 76.3876. | 10. 87.9844. |
| 11. 4.7089. | 12. 9.4864. |            |             |              |

*Approximate the (principal) square root of each of the following to the nearest thousandth.*

- |           |           |           |           |
|-----------|-----------|-----------|-----------|
| 13. 366.  | 14. 5.35. | 15. 56.5. | 16. 77.6. |
| 17. 78.9. | 18. 2.27. | 19. 29.2. | 20. 683.  |

16. The system of complex numbers. At the beginning of section 14, two motives were presented for the introduction of irrational numbers, one geometric and the other algebraic. Our geometric objective has been accomplished, and hence real numbers are all that are necessary for most geometric purposes. But the system of real numbers is not closed under the operation of



taking square roots. This can be seen from the following theorem.

**THEOREM 1.** *No negative number  $N$  has a square root among the real numbers.*

*Proof.* Let  $x$  be any real number. It is either positive, zero, or negative. If  $x$  is positive, then  $x^2$  is positive and cannot be equal to  $N$ . If  $x$  is zero, then  $x^2$  is zero and cannot be equal to  $N$ . If  $x$  is negative, then  $x^2$  is positive and cannot be equal to  $N$ . This completes the proof.

We therefore introduce the symbols  $\sqrt{N}$  where  $N$  is negative and call them **pure imaginary numbers**. Thus  $\sqrt{-4}$  and  $\sqrt{-5}$  are pure imaginary numbers. We agree to write  $\sqrt{-4} = \sqrt{4}\sqrt{-1}$  and  $\sqrt{-5} = \sqrt{5}\sqrt{-1}$ , etc. With this agreement we see that every pure imaginary number can be written as the product of a real number and  $\sqrt{-1}$ . We call  $\sqrt{-1}$  the **imaginary unit** and denote it by the letter  $i$ ; thus

$$(1) \quad i = \sqrt{-1} \text{ or } i^2 = -1.$$

Thus we may write  $\sqrt{-4} = 2i$  and  $\sqrt{-5} = i\sqrt{5}$ . Hence, *every pure imaginary number may be written in the form  $bi$  where  $b$  is real and  $i = \sqrt{-1}$  ( $b \neq 0$ ).*

The system consisting of all real numbers and pure imaginary numbers together would not be closed under addition, for the sum of a real number and a pure imaginary number would be neither real nor pure imaginary; for example,  $2 + 3i$ . Symbols of the form  $a + bi$ , where  $a$  and  $b$  are real and  $i = \sqrt{-1}$ , are called **complex numbers**. We call  $a$  the **real part** and  $bi$  the **imaginary part** of the complex number  $a + bi$ . We define equality, addition, and multiplication of complex numbers as follows:

**DEFINITION 1.**  $a + bi = c + di$  if and only if  $a = c$  and  $b = d$ .

**DEFINITION 2.**  $(a + bi) + (c + di) = (a + c) + (b + d)i$ .

*Example 1.*  $(3 + 2i) + (5 + 4i) = 8 + 6i$ .

**DEFINITION 3.**  $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$ .

*Example 2.*  $(3 + 2i)(5 + 4i) = (15 - 8) + (12 + 10)i = 7 + 22i$ .

Multiplication of complex numbers may be carried out according to the usual laws, with the additional step that  $i^2$  is to be replaced by  $-1$ . Thus

$$\begin{aligned} (3 + 2i)(5 + 4i) &= (3 + 2i)5 + (3 + 2i)4i \\ &= 15 + 10i + 12i + 8i^2 \\ &= 15 + 22i + 8(-1) \\ &= 7 + 22i. \end{aligned}$$



Subtraction of  $c + di$  can be defined as before (definition 1, section 5), and may be done by addition of  $-(c + di)$  as before. Hence

$$(2) \quad (a + bi) - (c + di) = (a - c) + (b - d)i.$$

*Example 3.*  $(5 + 4i) - (3 + 2i) = 2 + 2i.$

We defer further study of the arithmetic of complex numbers until it is needed.

The system of complex numbers includes all previous kinds of numbers. For example, if  $b = 0$ , the complex number  $a + bi$  is real; for example,  $2 + 0i = 2$  is real. If  $b \neq 0$ , the complex number is called **imaginary**; thus  $2 + 3i$  is imaginary. If  $b \neq 0$  and  $a = 0$ , the complex number is called **pure imaginary**; for example,  $0 + 3i = 3i$  is pure imaginary. We may classify the entire complex number system as follows:

*Complex numbers* ( $a + bi$  where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$ ).

*Imaginary numbers* ( $b \neq 0$ ). *Example:*  $2 + 3i$ .

*Pure imaginary numbers* ( $b \neq 0, a = 0$ ). *Example:*  $0 + 3i$ .

*Real numbers* ( $b = 0$ ). *Example:*  $2 + 0i$ .

*Irrational.* *Examples:*  $\sqrt{2}, \sqrt{3}, \sqrt[3]{2}, \pi, \dots$

*Rational.* (Expressible as  $p/q$  where  $p$  and  $q$  are integers.) *Example:*  $2/3$ .

*Integers.* (Expressible with denominator 1.) *Example:*  $2/1 = 2$ .

*Negative integers.* *Examples:*  $-1, -2, \dots$

*Zero.*

*Positive integers or natural numbers.*

Note that we have proceeded a long way from the natural numbers with which we started. At each stage, new kinds of numbers were introduced because the system at hand was not closed under all operations. But it will not be necessary to invent still more numbers now because it can be shown that the system of complex numbers is closed under all the operations studied in this book, such as addition, subtraction, multiplication, division (except for division by zero), and taking  $n$ th roots. It can also be shown that the basic laws of algebra are obeyed, such as the associative, commutative, and distributive laws. No useful concept of greater or less can be applied to imaginary numbers.



Hence  $>$  and  $<$ , and positive and negative will be used only in connection with real numbers.

The word "imaginary" does not mean that these numbers are somehow not genuine, or impractical or fictitious, although it did have such connotations in the 17th century, when imaginary numbers were not well understood. In fact, a logically satisfactory theory of complex numbers was first given by the Irish mathematician, Sir William Rowan Hamilton, in 1835. In a later chapter we shall give a perfectly "real" graphical interpretation of complex numbers. They are used practically in advanced engineering, the theory of electricity and magnetism, the theory of elasticity and plasticity, the theory of aerodynamics and aeronautical engineering, etc. While other types of numbers exist, the system of complex numbers suffices for much of applied mathematics, and it is with this system of numbers that we shall be concerned.

## EXERCISES

*Express each of the following in the form  $bi$  where  $b$  is real:*

- |                              |                    |                    |                   |
|------------------------------|--------------------|--------------------|-------------------|
| 1. $\sqrt{-9}$ .             | 2. $\sqrt{-49}$ .  | 3. $\sqrt{-5}$ .   | 4. $\sqrt{-17}$ . |
| 5. $\sqrt{-c^2}$ , $c > 0$ . | 6. $2\sqrt{-16}$ . | 7. $3\sqrt{-25}$ . | 8. $5\sqrt{-4}$ . |

*Express each of the following in the form  $a + bi$  where  $a$  and  $b$  are real:*

- |                                       |                                        |                        |
|---------------------------------------|----------------------------------------|------------------------|
| 9. $3 + \sqrt{-4}$ .                  | 10. $\sqrt{25} + \sqrt{-64}$ .         | 11. $4 + 2\sqrt{-9}$ . |
| 12. $3 + 4\sqrt{-16}$ .               | 13. $2 + 3i + 7 + i$ .                 |                        |
| 14. $5 + \sqrt{-4} + 2 + \sqrt{-9}$ . | 15. $1 + 3\sqrt{-4} + 2 - \sqrt{-9}$ . |                        |
| 16. $4 - 3i + 2 + 3\sqrt{-25}$ .      | 17. $(2 - 3i) - (5 + 2i)$ .            |                        |
| 18. $5 + 7i - (2 - 3i)$ .             | 19. $(6 + i) - (-2 + 3i)$ .            |                        |
| 20. $(2 + 3i) - (5 + i)$ .            | 21. $4i(2 + 3i)$ .                     |                        |
| 22. $2i(5 + 3i)$ .                    | 23. $(2 + 3i)(5 + i)$ .                |                        |
| 24. $(2 + 5i)(3 + 2i)$ .              | 25. $(2 - 3i)(4 + 2i)$ .               |                        |
| 26. $(3 + 2i)(5 - 3i)$ .              | 27. $(2 - 3i)(5 - 2i)$ .               |                        |
| 28. $(4 - i)(2 - 3i)$ .               | 29. $(2 - 3i)(1 + i)(2 - i)$ .         |                        |
| 30. $(1 - 2i)(3 + i)(2 + 5i)$ .       | 31. $(4 + i)(3 - 2i)(1 + 2i)$ .        |                        |
| 32. $(5 - 2i)(4 - 3i)(1 + i)$ .       |                                        |                        |

*List all the adjectives "complex, imaginary, pure imaginary, real, irrational, rational, integral, positive, negative, zero, natural" which apply to each of the following numbers:*



33.  $2 + 3\sqrt{-4}$ .    34.  $2 + 3i$ .    35.  $4\sqrt{-2}$ .    36.  $4i$ .  
 37.  $3\sqrt{5}$ .    38.  $\sqrt{2}/3$ .    39.  $-2/3$ .    40.  $7/5$ .  
 41.  $(1 - \sqrt{2})/3$ .    42.  $(1 + \sqrt{9})/2$ .    43.  $(2 - \sqrt{4})/3$ .

44. What was our motive in introducing (a) fractions; (b) negative numbers; (c) irrational numbers; (d) pure imaginary numbers; (e) complex numbers?

**\*17. Conclusion. The axioms of algebra.** In the course of this chapter we sketched the evolution of the number system and found that some of the familiar algebraic manipulations were logical consequences of a few simple definitions and basic laws. We list these basic laws here for easy reference, *all letters representing arbitrary numbers of the system.*

I. *Law of closure for addition.* Given any pair of numbers  $a$  and  $b$ , there exists a unique number in the system called the sum of  $a$  and  $b$ , denoted by  $a + b$ .

II. *Commutative law for addition.*  $a + b = b + a$ .

III. *Associative law for addition.*  $(a + b) + c = a + (b + c)$ .

IV. *Law of closure for multiplication.* Given any pair of numbers  $a$  and  $b$ , there exists a unique number in the system called the product of  $a$  and  $b$ , denoted by  $ab$ .

V. *Commutative law for multiplication.*  $ab = ba$ .

VI. *Associative law for multiplication.*  $(ab)c = a(bc)$ .

VII. *Distributive law.*  $a(b + c) = ab + ac$ .

VIII. There exists a unique number  $0$ , called *zero*, such that: (a)  $0 + a = a$  for any number  $a$ ; (b)  $0 \cdot a = 0$  for any number  $a$ ; (c) if  $a \neq 0$  and  $b \neq 0$ , then  $ab \neq 0$ .

IX. *Law of closure for subtraction.* Given any numbers  $a$  and  $b$ , there exists a unique number  $x$  such that  $b + x = a$ . This number  $x$  is denoted by  $a - b$ .

X. *Law of closure for division.* Given any numbers  $a$  and  $b$  ( $b \neq 0$ ), there exists a unique number  $x$  such that  $bx = a$ . This number  $x$  is denoted by  $a/b$ .

XI.  $a = a$ .

XII. If  $a = b$ , then  $b = a$ .

XIII. If  $a = b$  and  $b = c$ , then  $a = c$ . (*Things equal to the same thing are equal to each other.*)

XIV. If  $x = y$  and  $u = v$ , then (a)  $x + u = y + v$ ; (b)  $x - u = y - v$ ; (c)  $xu = yv$ ; (d)  $x/u = y/v$  provided  $u$  and  $v$  are

\* May be omitted without disturbing continuity.



not zero. That is, if equals are (a) added to, (b) subtracted from, (c) multiplied by, (d) divided by equals, the results are equal.

There is a subdivision of the complex number system, called the *real numbers*, which have the following properties:

XV. If  $a$  and  $b$  are any real numbers, one and only one of the relations  $a > b$ ,  $a = b$ , or  $a < b$  holds. If a real number  $x > 0$ , it is called *positive*: if  $x < 0$ , it is called *negative*.

XVI. *Law of signs for multiplication.*

(a) If  $a > 0$  and  $b > 0$ , then  $ab > 0$ .

(b) If  $a < 0$  and  $b < 0$ , then  $ab > 0$ .

(c) If  $a < 0$  and  $b > 0$ , then  $ab < 0$ .

XVII. If  $a < b$  and  $b < c$ , then  $a < c$ .

XVIII. If  $a < b$ , then  $a + c < b + c$ .

XIX. If  $a < b$ , then  $-b < -a$ .

XX. If  $a < b$  and  $c > 0$ , then  $ac < bc$ .

XXI. Every real number can be expressed as a decimal, terminating or non-terminating, and, conversely, every decimal represents a real number.

It would be possible to use these basic laws, plus a few others not stated here, as axioms or postulates in a strictly logical treatment of algebra.\* This is not our purpose here, because such a task is best left for more advanced courses. We shall occasionally indicate, however, how these basic laws justify our manipulations.

The natural numbers may be regarded as the basis of all algebra. All the different kinds of numbers can be strictly defined in terms of the natural numbers, although we have not done this here completely. For example, a fraction was defined as a symbol consisting of a pair of natural numbers. L. Kronecker (German, 1823–1891) is said to have remarked that “the whole number was created by God, everything else is man’s handiwork.” The fundamental character of the natural numbers, alluded to by Kronecker, has led to the extensive development not only of the *theory of numbers*, one of the oldest branches of mathematics, but also of mystical *numerology*, one of the oldest varieties of nonsense.†

\* For more detailed indications of the role of axioms in algebra, or, indeed, in any branch of mathematics, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, and references listed there. The axioms given above are neither independent nor complete.

† See E. T. Bell, *Numerology*, Williams and Wilkins, Baltimore, 1933, for an entertaining account of numerology.



## CHAPTER II

### What Is Algebra?

Like a well-cut jewel, algebra presents many facets. It is at the same time a language, a mathematical (logical) science, a collection of techniques of calculation, an important element in the history of human endeavor, and a collection of puzzles.

**18. Algebra as a language.** The symbols and technical terms of algebra have been evolved over a long period of history, and have proven themselves most apt for the consideration of quantitative relationships and problems in all fields, such as physical sciences, social sciences, statistics, business, etc. The earliest advances over ordinary prose were largely abbreviations. Thus, in his *Ars Magna de Rebus Algebraicis* (1545), Cardan wrote "cubus p. 6 rebus aequalis 20," which we would express as  $x^3 + 6x = 20$ . The  $=$  sign is found in print for the first time in *The Whetstone of Witte*, by Robert Recorde, published in 1557. He writes that he uses two parallel lines for the equals sign because "noe two thynges could be moare equalle." In the early 17th century, we still find *aaaaa* written for  $a^5$ . The ancients had no adequate algebraic symbolism and it is likely that this handicapped their development of algebraic ideas. One has only to try to solve a relatively simple problem without using any algebraic symbolism to realize the necessity for some such compact language. A similar need can be seen in music; imagine an orchestra trying to play a symphony if all directions for playing were in prose instead of the compact musical symbolism. In algebra, as in music, the student should master as soon as possible the symbolism and technical terminology in order to leave his mind free for the interesting things that are expressed in this language. A definition in this book will be clearly recognizable whether or not it is marked "Definition" because the term being defined will be printed in bold-faced (heavy) type. In algebra, as in music, considerable effort and drill is needed for this purpose. Benjamin Franklin wrote,\*

\* *On the Usefulness of Mathematics, Complete Works of Benjamin Franklin*, vol. I, p. 421. G. P. Putnam's Sons. The Knickerbocker Press (1887).



“Whatever may have been imputed to some other studies under the notion of insignificance and loss of time, yet these (mathematics), I believe, never caused repentance in any, except it was for their remissness in the prosecution of them.”

**19. Algebra as a logical science.** Any logical science should begin with axioms or assumptions and deduce all further propositions from them by strictly logical argument. The assumptions ought to be as simple, as few, and as plausible as possible. Such a logical organization was first attempted by the ancient Greeks for geometry. A partial set of axioms for algebra is given in section 17, but no attempt to develop algebra completely from these axioms will be made here because such a task presents technical difficulties best left for more advanced courses.\*

**20. Algebra as a collection of techniques of calculation.** This aspect of algebra is doubtless the most familiar to the student, since techniques or processes of calculation are the reasons for much of the usefulness of algebra in applied science and are heavily stressed in elementary teaching. Nevertheless, if one is asked the embarrassing question, “How do you know this technique or process always works?” the only answer is to justify the technique by a reasonable argument. Thus in section 15, the familiar process for extracting square roots has been justified by tracing it back to the simple algebraic fact that  $(t + u)^2 = t^2 + 2tu + u^2$ , which in turn may be derived from the axioms themselves as in section 12. Techniques and processes will not only be explained clearly in this book, but will also be justified by reasonable arguments as far as possible. It is the author’s belief that *a process understood will be remembered better and longer than one which is merely memorized.*

**21. Algebra as a branch of human endeavor.** The search for truth has certainly been, throughout history, one of the most distinctively human of human activities. In it, mathematics has played a central role. The manifold applications of mathematics to astronomy, physics, chemistry, biology, statistics, the social sciences, engineering, etc., would not be possible without algebra, which is basic to trigonometry, analytic geometry, calculus, and higher mathematics. Thus, the industrial revolution in the 18th and 19th centuries, and whatever analogous social changes result from the modern scientific advances in aeronautics, radiation, and

\* For a more complete discussion of the role of axioms in mathematics, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941.



the release of atomic energy, are in considerable measure related to progress in mathematics.

From a personal rather than a social point of view, algebra requires and stimulates clear and persistent thinking and clear understanding of concepts and precise definitions.

**22. Algebra as a collection of puzzles.** To the student, algebra presents the appearance of a collection of puzzles, especially in the exercises on which the student must cut his mental teeth. Needless to say, the significance of algebra is far deeper than could possibly be possessed by a mere collection of puzzles. Nevertheless, the puzzle-motive is a universal urge and is surely present in algebra. Solving an algebraic problem yields satisfaction of the same kind as that obtained by solving puzzles of a less useful sort, and at the same time gives the deeper satisfaction of furthering one's mathematical and scientific education. The student, to be successful, should be as reluctant to give up an algebraic problem as he was to give up a useless puzzle when he was very young and full of the pure intellectual curiosity that is evident in almost all children, at least until they are exposed to the educational system. It should also be mentioned that there are many historic puzzles which appear useless on the surface but which have proved to be very fruitful in the advancement of science, if only because the attempts to solve them led to the development of new methods which were useful in other problems.\*

\* For further details of some puzzles of this sort, see M. Richardson, *Fundamentals of Mathematics*, Chapter VIII, Macmillan, 1941.



## CHAPTER III

### Algebraic Expressions

**23. Positive integral exponents and radicals.** Order of operations. If  $n$  is any positive integer, we have defined the  $n$ th power of  $x$  to mean the product of  $n$  factors, each equal to  $x$ . In symbols,  $x^n = x \cdot x \cdot \dots \cdot x$  where there are  $n$  factors on the right. The exponent  $n$  indicates the number of factors. For example,  $x^3 = xxx$ . Clearly this definition can apply only to *positive integral* exponents. The following theorems are probably familiar to the student.

**THEOREM I.**  $x^a x^b = x^{a+b}$ .

*Example:*  $x^2 x^3 = (xx)(xxx) = x^5$ .

**THEOREM II.**  $(x^a)^b = x^{ab}$ .

*Example:*  $(x^2)^3 = (x^2)(x^2)(x^2) = (xx)(xx)(xx) = x^6$ .

**THEOREM III.**  $(xy)^n = x^n y^n$ .

*Example:*  $(xy)^3 = (xy)(xy)(xy) = xxxxyyy = x^3 y^3$ .

**THEOREM IV.**  $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ .

*Example:*  $\left(\frac{x}{y}\right)^3 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x^3}{y^3}$ .

**THEOREM V.** If  $x \neq 0$ ,

$$\frac{x^a}{x^b} = \begin{cases} x^{a-b} & \text{if } a > b. & \text{Example: } \frac{x^5}{x^2} = \frac{xxxx\cancel{x}}{\cancel{x}} = x^3. \\ \frac{1}{x^{b-a}} & \text{if } a < b. & \text{Example: } \frac{x^2}{x^5} = \frac{\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}\cancel{x}\cancel{x}} = \frac{1}{x^3}. \\ 1 & \text{if } a = b. & \text{Example: } \frac{x^2}{x^2} = 1. \end{cases}$$

These are the fundamental rules for the manipulation of exponents. Their proofs, for positive integral exponents, are clearly indicated by the examples above.

*Proof of Theorem I.*  $x^a = xx \cdot \dots \cdot x$  with  $a$  factors on the right, while  $x^b = xx \cdot \dots \cdot x$  with  $b$  factors on the right. Hence,  $x^a x^b = (xx \cdot \dots \cdot x)(xx \cdot \dots \cdot x) = xx \cdot \dots \cdot x$  where there are  $a + b$  factors on the right. Hence,  $x^a x^b = x^{a+b}$ .

*Proof of Theorem II.*  $(x^a)^b = (x^a)(x^a) \cdot \dots \cdot (x^a)$  where there are  $b$  pairs of parentheses on the right. But each pair of paren-



theses on the right encloses the product of  $a$   $x$ 's. Hence there are  $ab$   $x$ 's on the right. Hence,  $(x^a)^b = x^{ab}$ .

*Proof of Theorem III.* See section 12, Theorem 1.

*Proof of Theorem IV.*  $\left(\frac{x}{y}\right)^n = \frac{x}{y} \cdot \frac{x}{y} \cdot \dots \cdot \frac{x}{y}$ . By multiplying fractions, we get  $\frac{xx \cdot \dots \cdot x}{yy \cdot \dots \cdot y} = \frac{x^n}{y^n}$ .

*Proof of Theorem V. Case I.* Suppose  $a > b$ . Then

$$\frac{x^a}{x^b} = \frac{\overbrace{(xx \cdot \dots \cdot x)}^{b \text{ factors}} \overbrace{(xx \cdot \dots \cdot x)}^{a-b \text{ factors}}}{\underbrace{(xx \cdot \dots \cdot x)}_{b \text{ factors}}} = \overbrace{(xx \cdot \dots \cdot x)}^{a-b \text{ factors}} = x^{a-b}.$$

*Case II.* Suppose  $a < b$ . Then

$$\frac{x^a}{x^b} = \frac{\overbrace{(xx \cdot \dots \cdot x)}^{a \text{ factors}}}{\underbrace{(xx \cdot \dots \cdot x)}_{a \text{ factors}} \underbrace{(xx \cdot \dots \cdot x)}_{b-a \text{ factors}}} = \frac{1}{\underbrace{(xx \cdot \dots \cdot x)}_{b-a \text{ factors}}} = \frac{1}{x^{b-a}}.$$

*Case III.* Suppose  $a = b$ . Then  $\frac{x^a}{x^b} = \frac{x^a}{x^a} = 1$ .

Note that in all cases we must suppose  $x \neq 0$  because division by zero is excluded (see section 11).

If  $n$  is a positive integer greater than 1, and  $x^n = a$ , then  $x$  is called an  $n$ th root of  $a$ . If  $x^2 = a$ ,  $x$  is called a square root of  $a$ ; if  $x^3 = a$ ,  $x$  is called a cube root of  $a$ . For example,  $+3$  and  $-3$  are both square roots of 9, since  $(+3)^2 = 9$  and  $(-3)^2 = 9$ ; 2 is a cube root of 8, since  $2^3 = 8$ ; and  $-2$  is a cube root of  $-8$ , since  $(-2)^3 = -8$ . It will be shown in a later chapter that every number  $a$  has  $n$  distinct  $n$ th roots. To avoid the confusion that would arise if we allowed the radical sign  $\sqrt[n]{a}$  to designate any of these  $n$ th roots ambiguously, we reserve the radical sign for a particular one of them, called the principal  $n$ th root. If  $a$  is positive, the principal  $n$ th root  $\sqrt[n]{a}$  shall be the positive  $n$ th root; if  $a$  is negative and if  $n$  is odd, the principal  $n$ th root  $\sqrt[n]{a}$  shall be the negative  $n$ th root; no other cases are needed for the present. For example,  $\sqrt{9} = 3$ ,  $\sqrt[3]{8} = 2$ ,  $\sqrt[3]{-8} = -2$ . If we wish to designate



$-3$  as a square root of 9, we write  $-\sqrt{9}$ . Obviously, by definition, we have

$$(1) \quad (\sqrt[n]{a})^n = a.$$

*Order of operations.* Parentheses ( ), brackets [ ], braces { }, and sometimes the vinculum  $\overline{\phantom{x}}$ , are used to group together the quantities enclosed by them. *Operations within parentheses (or other symbols of grouping) are to be done first.* Parentheses are thus merely punctuation marks in the written language of algebra, used to prevent ambiguity and misunderstanding, as are punctuation marks in prose. Thus  $8 \div 4 \div 2$  might mean either  $(8 \div 4) \div 2 = 2 \div 2 = 1$  or  $8 \div (4 \div 2) = 8 \div 2 = 4$ . Similarly the phrase "beautiful little girls' school" might mean "beautiful (little girls' school)" signifying "beautiful school for little girls," or "(beautiful little girls') school," signifying "school for beautiful little girls," or "(beautiful little) (girls' school)" signifying "girls' school which is beautiful and little."

We agree that *powers take precedence over multiplications and divisions, which in turn precede additions and subtractions, except where otherwise indicated by parentheses.* Thus,  $2 \cdot (3 + 4) = 2 \cdot 7 = 14$  while  $2 \cdot 3 + 4 = 6 + 4 = 10$ . We may also write  $2 \cdot (3 + 4)$  as  $2[3 + 4]$  or  $2\{3 + 4\}$  or  $2 \cdot \overline{3 + 4}$ . Also  $2 \cdot 5^2 = 2 \cdot 25 = 50$ , while  $(2 \cdot 5)^2 = 10^2 = 100$ . Thus,  $2 \cdot 5^2 + (2 \cdot 5)^2 = 50 + 100 = 150$ .

It is understood that *all operations indicated under a radical sign are to be done before extracting the root.* For example,  $\sqrt{9 + 16} = \sqrt{25} = 5$  is correct, but  $\sqrt{9} + 16 = 3 + 4 = 7$  is wrong. That is, a radical sign acts as a pair of parentheses, grouping together the quantities under it. In fact, it may be conjectured that the radical sign originates from a script  $r$  (for *radix*, Latin word meaning *root*) and the vinculum (Latin word meaning *yoke* or *bond*). Thus  $\sqrt{9 + 16} = r(9 + 16) = 5$ .

## EXERCISES

1 to 29. Do exercises 1–29 of section 12.

*Simplify each of the following:*

30.  $a^3 \cdot a^4$ .

31.  $a^{10}/a^2$ .

32.  $(a^3)^4$ .

33.  $a^2/a^8$ .

34.  $\frac{a^3b^4c^2}{a^5bc^2}$ .

35.  $(2a^3b)(3ab^3)^2$ .

36.  $\frac{(x^3y)^2}{(xy^2)^3}$ .

37.  $\frac{2(a^2b^3)^2}{(2a^3b^2)^3}$ .

38.  $-2(x^2y^3)^4$ .

39.  $(-2x^2y^3)^4$ .



40.  $\left(\frac{a^3}{b^4}\right)^5$

41.  $\left(\frac{2x^2}{y}\right)^3 \left(\frac{y^4}{2x^2}\right)^2$

42.  $\frac{(ab^2)^3(a^2b^3)^2}{(a^5b^2)^2}$

43.  $\frac{(2a^5b^4)^3}{(2a^2b^3)^2(3ab)^3}$

44.  $8^9/8^7$

45.  $(-3)^2(-2)^3$

46.  $(-x^2y^3)^2$

47.  $(-x^2y^3)^3$

48.  $(2^3)^4/(2^4 \cdot 2^7)$

49.  $8^4/2^{11}$

Tell whether each of the following is true or false, and, if false, correct the right member:

50.  $3^2 \cdot 3^3 = 9^5$

51.  $3^2 \cdot 3^3 = 3^6$

52.  $3^2 \cdot 3^3 = 3^8$

53.  $3^2 \cdot 3^3 = 9^6$

54.  $x^3 + x^3 = 2x^3$

55.  $3^2 + 3^3 = 3^5$

56.  $5^3 + 5^3 = 5^6$

57.  $5^3 + 5^3 = 10^3$

58.  $3^8/3^3 = 3^4$

59.  $(3^2)^3 = 3^5$

60.  $(4^2)^3 = 4^8$

61.  $6^3/2^3 = 3^3$

Assuming that all letters represent positive numbers, simplify each of the following:

62.  $\sqrt{x^4y^2}$

63.  $\sqrt{9x^8y^{10}}$

64.  $\sqrt[3]{x^6y^9}$

65.  $\sqrt[3]{8x^{12}y^6}$

**24. Algebraic expressions.** A **variable** or **unknown** is a letter which is permitted to have different values (that is, to stand for different numbers) during the same discussion. A **constant** is a symbol which is permitted to stand for just one particular number during the discussion, even if we do not specify which number it stands for. It is customary to use the later letters of the alphabet, such as  $x, y, z, t, u$ , for variables, and the early letters of the alphabet, such as  $a, b, c, d$ , for constants. Of course, specified numbers like  $2, -\frac{1}{2}, \sqrt{2}, 3 + \sqrt{-2}$ , are constants. We shall also use subscripts and primes to distinguish different quantities. For example,  $a, a'$  (read " $a$  prime"),  $a''$  (read " $a$  double prime"),  $a'''$  (read " $a$  triple prime") may be used to represent four constants with arbitrary values. Similarly,  $a_0$  (read " $a$  sub-nought" or " $a$  sub-zero"),  $a_1$  (read " $a$  sub-one"),  $a_2$  (read " $a$  sub-two") may be used to represent three constants with arbitrary values. Be careful not to confuse subscripts with exponents.

By an **algebraic expression** is meant an expression which can be built up from a limited number of constants and variables connected by a limited number of additions, subtractions, multiplications, divisions, radical signs, and parentheses (or other symbols of grouping).

*Examples of algebraic expressions:*

(1)  $(2x + \sqrt{5}) \sqrt[3]{4x - 2} + \frac{5a}{x^3}$

(2)  $1 + \sqrt{x}$



(3)  $\sqrt{x + y}$ .

(4)  $\frac{1}{x}$ .

(5)  $\frac{\sqrt{3}x^2 - 5x + b}{4x + 6}$ .

(6)  $\frac{1}{x + 2} + \frac{2}{x + 3}$ .

(7)  $\frac{1}{x + y} + \frac{1}{x - y}$ .

(8)  $2x^2y + 3xy^2 + cxy + dx + 4$ .

(9)  $5x^2 - \frac{1}{3}x - 2$ .

(10)  $(x + 2)(x + 3)$ .

(11)  $3x$ .

(12)  $3$ .

*Note.* Powers with positive integral exponents amount to nothing more than multiplication; thus  $x^3 = xxx$ . Hence  $x^3$  is an algebraic expression. On the other hand, the "exponential" expression  $3^x$  (where  $x$  is a variable), and  $\log x$ , and  $\sin x$  are not algebraic expressions. We shall have little to do with non-algebraic expressions in this book.

An algebraic expression which can be written without any variables occurring under radical signs is called a **rational expression**. An algebraic expression which necessarily involves variables occurring under radical signs is called an **irrational expression**.

Thus, examples (1), (2), and (3), above, are irrational expressions, while the rest are rational expressions.

A rational expression which can be written without any divisions by divisors (or denominators) involving variables is called a **rational integral expression**, or simply a **polynomial**.

Thus, examples (8), (9), (10), (11), and (12) are polynomials or rational integral expressions.

A rational expression which is not a polynomial, that is, in which denominators (or divisors) involving variables necessarily occur, is called a **rational fractional expression**, or simply a **fractional expression**.

Thus, examples (4), (5), (6), and (7) are fractional expressions.

If an algebraic expression is written as a succession of quantities, or partial expressions, each separated from its neighbors by a plus or a minus sign, then each quantity or partial expression, taken together with the preceding plus or minus sign, is called a **term**. Such an algebraic expression may be called the **algebraic sum** of its terms. Compare section 10.

Thus, example (9) above has 3 terms, namely  $5x^2$ ,  $-\frac{1}{3}x$ , and  $-2$ . Example (8) has 5 terms. Examples (3), (4), (5), (10),



(11), and (12) consist of 1 term each. Examples (1), (2), (6), and (7) consist of 2 terms each.

A term consisting only of the product of positive integral powers of variables and any constants whatever is called a **monomial**.\*

Thus, examples (11) and (12) are monomials. Example (9) is an algebraic sum of 3 monomials. Example (8) is an algebraic sum of 5 monomials.

The constant factor in a monomial is called its **coefficient**.

For example, the coefficient of the monomial  $-3x^2$  is  $-3$ . The coefficients of the terms of example (8) are 2, 3,  $c$ ,  $d$ , and 4. The coefficients of the terms of example (9) are 5,  $-\frac{1}{3}$ , and  $-2$ .

Monomials which differ at most in their coefficients are called **like terms**.

*Examples.*  $2x^3$  and  $-5x^3$  are like terms. Also  $3x^2y$ ,  $\frac{1}{2}x^2y$ , and  $ax^2y$  are like terms.

The degree of a monomial with non-zero coefficient which involves variables is the sum of the exponents attached to those variables. A monomial with non-zero coefficient which involves no variables is said to be of degree zero. Monomials with coefficient zero do not have any degree.

For example, the degree of the monomial  $5x^2y$  is  $2 + 1$  or 3. The degree of  $-2xy^2$  is 3. The degree of  $3x^2y^2$  is 4. The degree of  $3x$  is 1. The degree of  $5x^3$  is 3. The degree of 3 is zero. The degree of  $3a$  is also zero, since  $a$  is a constant.†

If an algebraic expression, a rational expression, or a polynomial, etc., involves no variables other than  $x$ , we shall refer to it as an algebraic expression in  $x$ , a rational expression in  $x$ , or a polynomial in  $x$ , etc., as the case may be. Similarly, a polynomial in  $x$  and  $y$  shall mean a polynomial involving no variables other than  $x$  and  $y$ .

Thus, example (8), above, is a polynomial in  $x$  and  $y$ . Example (9) is a polynomial in  $x$ .

\* The word *monomial* is sometimes used to refer to any algebraic expression with one term; in that case, our monomials would be called *rational integral monomials*, since they are special cases of rational integral expressions or polynomials. Since rational integral monomials are the only ones we shall be concerned with, we adopt the usage in the text above. Thus, *an algebraic sum of monomials is a polynomial*. An algebraic sum of terms which are not (rational, integral) monomials may be called a **multinomial**. An expression with 2 terms may be called a **binomial**; with 3 terms, a **trinomial**, etc. Thus,  $1 + \sqrt{x}$  is a binomial and a multinomial but not a polynomial in  $x$ .

† We could, if we wished, consider also the degree of a monomial with respect to literal constants, but there is nothing to be gained by doing so.



## EXERCISES

In each of the following algebraic expressions (a) list all the terms (A) irrational expression, (B) rational expression, (C) polynomial, or rational integral expression which apply to it; (b) find the value of it when  $x = 3$  and  $y = 2$ :

- |                                      |                                    |                                      |
|--------------------------------------|------------------------------------|--------------------------------------|
| 1. $\sqrt{3x + 7}$ .                 | 2. $12/x$ .                        | 3. $2x^3 - 3x^2 - 20$ .              |
| 4. $2x + 3y$ .                       | 5. $\frac{7x - 3}{x + 6}$ .        | 6. $\frac{5x^2 - 3x + 1}{2 - x^2}$ . |
| 7. $\frac{1}{3}x^3 + x + \sqrt{2}$ . | 8. $\frac{\sqrt{x} - 2}{2x + 1}$ . | 9. $x$ .                             |
| 10. $\sqrt{x^2 + 16}$ .              |                                    |                                      |

Name (a) the coefficient, and (b) the degree of each of the following monomials.

- |               |                           |                             |
|---------------|---------------------------|-----------------------------|
| 11. $-2x^3$ . | 12. $5x^2$ .              | 13. $-x^3y^2$ .             |
| 14. $5xy^3$ . | 15. $-\frac{1}{2}x^3yz$ . | 16. $-\frac{1}{3}x^2yz^3$ . |

**25. Removal and insertion of parentheses. Combination of like terms.** We have the following rule for removal of parentheses (or any other symbol of grouping):

If parentheses immediately follow a minus sign, they may be removed, together with the minus sign, provided the sign of every term within the parentheses is reversed. If parentheses immediately follow a plus sign, they may be removed without altering the signs of terms within the parentheses.

*Example 1.*  $a - (2b - 3c + 4d) = a - 2b + 3c - 4d$ .

*Example 2.*  $a + (2b - 3c + 4d) = a + 2b - 3c + 4d$ .

*Example 3.*  $a - (-2b + 3c) = a + 2b - 3c$ .

*Example 4.*  $a + (-2b + 3c) = a - 2b + 3c$ .

Clearly, parentheses may be inserted according to the same rule. Thus, the right members of the above examples may be changed to the left members, if desired.

The justification of this rule is found in theorems 1, 2, 3, and 4 of section 10.

In an algebraic sum, like terms may be combined by adding their coefficients, as follows:  $2x^2 + 3x^2 = 5x^2$ , and  $5x^2y - 2x^2y = 3x^2y$ .

The justification of this rule is found in the distributive law, for  $2x^2 + 3x^2 = (2 + 3)x^2 = 5x^2$ , and  $5x^2y - 2x^2y = (5 - 2)x^2y = 3x^2y$ .



Parentheses preceded by a multiplier or factor may be removed by multiplying every term within the parentheses by the multiplier, as in the following examples:

*Example 5.*  $3(2x - 5y) = 6x - 15y.$

*Example 6.*  $(a + b)(x + y) = (a + b)x + (a + b)y = ax + bx + ay + by.$

*Example 7.*  $3x^2(2x^3y - 5y) = 6x^5y - 15x^2y.$

The justification of this rule is to be found in the distributive law.

To simplify a rational integral expression means to remove all parentheses enclosing variables and to combine like terms, as in the following examples:

*Example 8.*  $(x + 2)(x + 3) = (x + 2)x + (x + 2)3 = x^2 + 2x + 3x + 6 = x^2 + 5x + 6.$

*Example 9.* Removing the innermost symbols of grouping first, we have:

$$\begin{aligned} & 3x^3 + (5x^2 - [3x^2 - 4x\{x - 2\} + 3] - 2) \\ &= 3x^3 + (5x^2 - [3x^2 - 4x^2 + 8x + 3] - 2) \\ &= 3x^3 + (5x^2 - 3x^2 + 4x^2 - 8x - 3 - 2) \\ &= 3x^3 + 5x^2 - 3x^2 + 4x^2 - 8x - 3 - 2 \\ &= 3x^3 + 6x^2 - 8x - 5. \end{aligned}$$

We could also proceed by removing the outermost symbols of grouping first, as follows:

$$\begin{aligned} & 3x^3 + (5x^2 - [3x^2 - 4x\{x - 2\} + 3] - 2) \\ &= 3x^3 + 5x^2 - [3x^2 - 4x\{x - 2\} + 3] - 2 \\ &= 3x^3 + 5x^2 - 3x^2 + 4x\{x - 2\} - 3 - 2 \\ &= 3x^3 + 5x^2 - 3x^2 + 4x^2 - 8x - 3 - 2 \\ &= 3x^3 + 6x^2 - 8x - 5. \end{aligned}$$

## EXERCISES

*Simplify each of the following expressions:*

1.  $3x + 4y - 2x + 3y.$
2.  $3x^2 - 2x + 5 - x^2 + 5x - 2.$
3.  $x^2 - 2xy + 3y^2 - 3x^2 + 5xy - y^2.$
4.  $x + y + z - 4x + 5y - 6z + 3z - y + 5x.$
5.  $(2x - 3y) + (3x - y).$
6.  $(5x + 3) + (2x - 1).$
7.  $(2x + y - z) - (3x - 2y - 2z).$
8.  $(3x - 5y + 3) - (x - 2y + 5).$
9.  $(5x^2 - 2x + 5) - (2x^2 - 3x - 2).$



10.  $(-2x^2 + x - 3) - (-5x^2 - 3x - 1)$ .  
 11.  $2x - (3x + [x - 2y] - 3y)$ . 12.  $3x - (2y - [5x - y] + x)$ .  
 13.  $5x^2 - (4x + [2x^2 - x] - [x - 3] - 5)$ .  
 14.  $(2x^2 - [4x - 3x^2] + 5) - (x - 3)$ .  
 15.  $x - (x - [x - \{x - y\}])$ .  
 16.  $x - (2x - [3x - \{4x + 3\}])$ .  
 17.  $3 - (x - 2 + [3x - 4y] - [y - \{x + 1\}])$ .  
 18.  $-(2x + [3x - y + \{-x - 2y\}]) + (x + y)$ .

(a) Insert parentheses preceded by a plus sign about the first two terms and about the second two terms; (b) insert parentheses preceded by a minus sign about the first two terms and about the second two terms:

$$19. -x + y + 2a - 3b.$$

$$20. 2x - 3y - 4a + 5.$$

**26. Standard form of polynomials.** The standard form of a polynomial or rational integral expression in  $x$  is obtained by removing all parentheses enclosing variables, combining like terms, and arranging terms in order of descending powers of  $x$ .

*Example 1.* To put the polynomial  $(x + 2)(x + 3)$  in standard form, we write  $(x + 2)(x + 3) = (x + 2)x + (x + 2)3 = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$ . The last expression is the standard form of the polynomial.

*Example 2.* The following polynomials in  $x$  are already in standard form:

- (1)  $3x^4 + 2x^2 - 5,$   
 (2)  $\frac{1}{2}x^2 - \sqrt{5}x + (3 + \sqrt{-2}),$   
 (3)  $ax^2 + (2 + b)x - (c + \sqrt{2}).$

A polynomial in more than one variable may be put into a **standard form** by removing all parentheses, combining like terms, and arranging terms in order of descending powers of one of the variables.

*Example 3.*  $x^2(2y^3 - xy^2) - (5x^2y^3 - 2x^3y^2 - 4x + 3) = 2x^2y^3 - x^3y^2 - 5x^2y^3 + 2x^3y^2 + 4x - 3 = x^3y^2 - 3x^2y^3 + 4x - 3$ .

The latter expression is in standard form, arranged according to descending powers of  $x$ . If descending powers of  $y$  are used, we have the alternative standard form  $-3x^2y^3 + x^3y^2 + 4x - 3$ .

It is intuitively clear, and it can be proved, that every polynomial can be written in standard form.

The **degree of a polynomial** is the degree of the term or monomial\* of highest degree occurring in it when the polynomial is

\* The degree of a monomial is defined in section 24.



expressed in standard form. When a polynomial is written in standard form, the coefficients of its terms are called the **coefficients of the polynomial**.

For example, the polynomial  $(x + 2)(x + 3)$  of example 1 has degree 2. The polynomial (1) above is of degree 4, (2) is of degree 2, and (3) is of degree 2. The polynomial of example 3 has degree 5.

*Example 4.* The degree of  $(x - 2)(x + 5) - x^2$  is 1, since the standard form of this polynomial is  $3x - 10$ .

Polynomials of degree 1, 2, 3, 4, and 5 are called **linear, quadratic, cubic, quartic,\*** and **quintic**, respectively.

By the **general linear polynomial in  $x$**  is meant the expression

$$(4) \quad ax + b$$

where  $a$  and  $b$  are arbitrary constants,  $a \neq 0$ . Similarly, the **general quadratic polynomial in  $x$**  is the expression

$$(5) \quad ax^2 + bx + c$$

where  $a$ ,  $b$ , and  $c$  are arbitrary constants ( $a \neq 0$ ). The **general cubic polynomial in  $x$**  is the expression

$$(6) \quad ax^3 + bx^2 + cx + d \quad (a \neq 0).$$

The **general polynomial in  $x$  of  $n$ th degree**, where  $n$  is any positive integer, is the expression

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l \quad (a \neq 0)$$

or, in a better notation,

$$(7) \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \quad (a_0 \neq 0).$$

For example, using subscripts as in (7), the general polynomials of degrees 1, 2, and 3 may be written  $a_0x + a_1$ ,  $a_0x^2 + a_1x + a_2$ , and  $a_0x^3 + a_1x^2 + a_2x + a_3$ , respectively. An arbitrary constant such as  $a$  or  $a_0$  may be considered to be the general polynomial of degree zero.

*Note.* It can be shown that every rational fractional expression can be expressed as a quotient of two polynomials. For example, we may write

$$\begin{aligned} \frac{1}{x+2} + \frac{2}{x+3} &= \frac{1 \cdot (x+3)}{(x+2)(x+3)} + \frac{2 \cdot (x+2)}{(x+2)(x+3)} \\ &= \frac{3x+7}{x^2+5x+6} \end{aligned}$$

\* Biquadratic is sometimes used synonymously with *quartic*.



by means of the definition of addition of fractions (definition 2, section 8). Thus *polynomials, or rational integral expressions, and rational expressions are very analogous to integers and rational numbers, respectively*. Many similarities between the properties of integers and the properties of polynomials will be brought out in later chapters.

## EXERCISES

Write each of the following polynomials in standard form, and state its degree:

1.  $5x^2 - 4x + 5 - 2x^2 + x - 1$ .

2.  $3 - x^3 + 4x^2 - 5x^3 - x^2 + x - 1$ .

3.  $(2x^3 + 5x - 1) - (x^3 + x - 3) - x^2 + 2$ .

4.  $(-x^3 + 2x^2 - 3) - (-x^3 - x^2 + 2x - 1)$ .

5.  $(x^4y + x^2y^2) - (x^3y + 2x - 3)$ .

6.  $x^2 + xy - y^2 - (3 - xy)$ .      7.  $(x + 1)(x + 2) - (x^2 + 1)$ .

8.  $(x + 3)(x - 1) - (2x - 1)$ .

9. If integers are substituted for the variables in a polynomial whose coefficients are all integers, what kind of number must the resulting value of the expression be? Explain and illustrate.

10. If rational numbers are substituted for the variables in a rational expression consisting of the quotient of two polynomials with rational coefficients, what kind of number must the resulting value of the expression be, if it exists at all? Explain and illustrate.

11. Will the value of an irrational expression for any values of the variables always be an irrational number? Explain and illustrate.

**27. Equations and identities.** An algebraic expression which contains variables has no definite numerical value unless definite values are assigned to the variables. For example,  $2x + 6 = 8$  if  $x = 1$ , but  $2x + 6 = 10$  if  $x = 2$ , and so on. Hence, the meaning of statements such as

(1)  $2(x + 3) = 2x + 6$

and

(2)  $2x + 3 = 11,$

each of which seems to say that two algebraic expressions are equal, requires explanation.

Equation (1) is true regardless of what value we substitute for  $x$ , since, by virtue of the distributive law,  $2(x + 3) = 2 \cdot x + 2 \cdot 3 = 2x + 6$  for any number  $x$  whatever. For example, setting  $x = 1$ , (1) becomes  $2 \cdot (1 + 3) = 2 \cdot 1 + 6$  or  $2 \cdot 4 = 2 + 6$ ; setting  $x = 2$ , (1) becomes  $2 \cdot (2 + 3) = 2 \cdot 2 + 6$  or  $2 \cdot 5 =$



$4 + 6$ ; and so on. Equation (2) is of a quite different character. It becomes false for  $x = 1$ , since  $5 \neq 11$ . In fact, it becomes false if we substitute for  $x$  anything but 4; for  $x = 4$  it becomes true. We distinguish between these two kinds of equations by means of the following definitions.

An equation which becomes a true statement for all allowable values of the variables is called an **identical equation** or simply an **identity**.

For example, (1) is an identity. By allowable values of the variables is meant values for which all the expressions involved have a definite meaning. The reason for the word "allowable" in the definition is that some values of  $x$  may have to be excluded from consideration. For example,  $\frac{1}{x} = \frac{3}{3x}$  is an identity, but the value  $x = 0$  has to be excluded from consideration, since division by zero cannot be permitted (see section 11). Similarly,  $\frac{1}{x-2} + \frac{1}{x-3} = \frac{2x-5}{x^2-5x+6}$  is an identity, but the values  $x = 2$  and  $x = 3$  must be excluded from consideration for the same reason. Both these equations are true, however, for all allowable values of  $x$ .

An equation which becomes false for some allowable value(s) of the variable(s) is called a **conditional equation** or simply an **equation**.

For example, (2) is an (conditional) equation.

If an (conditional) equation involves only one variable, a value of the variable for which the equation becomes true is called a **root** or a **solution** of the equation, and is said to **satisfy** the equation. To solve an equation in one variable means to find all its roots.

For example, the only root of (2) is 4.

If an (conditional) equation involves more than one variable, a set of values for the variables which makes the equation true is called a **solution** of the equation and is said to **satisfy** the equation.

For example, the equation  $x + y = 7$  has the solutions  $(x = 1, y = 6)$ ,  $(x = 2, y = 5)$ ,  $(x = 3, y = 4)$ ,  $(x = -1, y = 8)$ , etc.

The subject of solving (conditional) equations will occupy several later chapters.

**28. Verification of identities. Reversible steps.** The subject of verifying identities is simpler, however. Note that an equation in one variable can be shown to be conditional, and hence not an identity, by merely exhibiting one particular allowable value of the



variable for which it becomes a false statement; but a thousand particular values of the variable for which the equation becomes true will not suffice, unassisted by further information, to prove that it is an identity, for they will provide no guarantee that the equation will become true for the 1001th value substituted. (See, however, corollary 2, section 104.)

*The simplest way to verify an identity is to reduce one or both members by steps of substitution of equals for equals until they look alike.*

*Example 1.* Verify that  $x(x + 3) + 6 = x^2 + 3(x + 2)$  is an identity. Since  $x(x + 3) = x^2 + 3x$  and  $3(x + 2) = 3x + 6$ , we may substitute on both sides, obtaining  $x^2 + 3x + 6 = x^2 + 3x + 6$ .

*If steps other than substitution of equals for equals are used, one must take care that all steps are reversible.*

*Example 2.* Verify that

$$(3) \quad \frac{x + y}{2} + \frac{z}{2} = \frac{x}{2} + \frac{y + z}{2}.$$

We might reason as follows. If (3) is true, then what is obtained by multiplying both sides by 2 will be true (since if equals are multiplied by equals, the results are equal). Thus, if (3) is true, then

$$(4) \quad x + y + z = x + y + z$$

which is clearly true. Does this enable us, without further consideration, to assert that (3) is true? Certainly not, for a true conclusion may be obtained by correct reasoning from a false hypothesis. Thus, "if (3) is true, then (4) is true, and (4) is true" does not guarantee that (3) is true. What we really need is the converse proposition "if (4) is true, then (3) is true," for if the hypothesis of a valid argument is true, then the conclusion must be true. Now we have so far proved only that "if (3) is true, then (4) is true," and we know that the converse (which is now needed) of a valid argument may not be valid. To prove the converse, we may try to deduce (3) from (4). But this may be done by merely reversing the steps of the argument by which we deduced (4) from (3), *provided those steps were all reversible*. In our present example this is the case, for since (4) was derived from (3) by multiplying both sides of (3) by 2, we can reverse the argument by dividing both sides of (4) by 2.



That one must really be careful not to use non-reversible steps, will be clearly shown by the next two examples.

*Example 3.* Let us set out to prove that  $3 = 7$ . We reason that if  $3 = 7$ , then  $3 \cdot 0$  would be equal to  $7 \cdot 0$ , because if equals are multiplied by equals the results are equal. But  $3 \cdot 0 = 7 \cdot 0$ , since  $0 = 0$ . May we then conclude that  $3 = 7$ ? Of course not! The step of multiplying both sides by 0 cannot be reversed, because to reverse it would require us to divide both sides by 0, which cannot be done (see section 11).

*Example 4.* Let us set out to prove that

$$(5) \quad x = -x.$$

If equals are squared, the results are equal; that is, if  $a = b$ , then  $a^2 = b^2$ . Hence, squaring both sides of (5), we obtain  $(x)^2 = (-x)^2$  or  $x^2 = x^2$ , which is true. May we conclude that therefore  $x = -x$ ? Certainly not! For the converse of the italicized statement above is not true (compare section 12) and *the step of squaring both sides cannot be reversed*.

*The step of substituting equals for equals is always reversible; therefore the method first suggested is always safe, if it is feasible.*

## EXERCISES

*Verify the identities:*

$$1. \quad x^2 - 3(x - 4) = x(x - 3) + 12.$$

$$2. \quad \frac{1}{x} + \frac{2}{x} = \frac{3}{x}.$$

$$3. \quad (x + 2)(x + 3) = x^2 + 5x + 6.$$

$$4. \quad \frac{1}{x-2} + \frac{1}{x-3} = \frac{2x-5}{x^2-5x+6}.$$

*Decide whether each of the following is an identity or a conditional equation:*

$$5. \quad 5(x - 2) = 5x - 10.$$

$$6. \quad \frac{1}{x} + \frac{2}{2x} = \frac{3}{3x}.$$

$$7. \quad 7 - 3x = 4x. \text{C}$$

$$8. \quad \sqrt{x^2 + 9} = x + 3. \text{C}$$

$$9. \quad \frac{7x}{4} - \frac{3x}{4} = x.$$

$$10. \quad \frac{2+x}{2y} = \frac{1+x}{y}.$$

\*11. Do exercise 30, section 12.

29. Elementary operations with equations. In the last section we pointed out the desirability of using reversible steps. We shall

\* May be omitted without disturbing continuity.



have definite, agreed, useful operations on equations which are identical or conditionally which are always reversible.

I. A term may be transferred from one side of an equation to the other, provided the sign is changed. This operation is called transposition.

Thus,  $2x + 3 = 11$  becomes  $2x = 11 - 3$ , or  $2x = 8$ , by transposition.

The justification for transposing a term is based on the axiom that if equals are added to, or subtracted from, equals, the results are equal. Such a step is always reversible, since, to make an addition, we have only to subtract, and vice versa.

II. Both sides of an equation may be multiplied or divided by a non-zero constant.

Example 2.  $x = 4$  becomes  $x = 4$  by dividing both sides by 2 (or multiplying by 4).

This is clearly reversible, since to undo a multiplication we have only to divide, and vice versa; if we reverse from multiplying by zero, we shall have no trouble with division by zero.

Now that these steps have sufficed to solve the equation  $2x + 3 = 11$ . For we have proved that if  $x$  is such that  $2x + 3 = 11$ , then  $x = 4$ . Since all steps in the argument are reversible, the converse is also true, namely, if  $x = 4$ , then  $2x + 3 = 11$ . Hence it is a root and the only root.

Another useful operation with equations is based on the following proposition (see section 12):

III. If  $a^2 = b^2$ , then  $a = \pm b$ .

This is a reversible step, since the converse is also true, namely, if  $a = \pm b$ , then  $a^2 = b^2$ . Note that the  $\pm$  sign in III is necessary.

Thus  $x^2 = 9$  implies  $x = \pm 3$ , and conversely. Hence  $\pm 3$  are the only roots of  $x^2 = 9$ .

Steps I, II, and III often suffice to solve simple equations.

Two conditional equations are called equivalent if they have exactly the same set of solutions.

The reversible operations above clearly result in equivalent equations. The following operations do not always result in equivalent equations and therefore must be used with caution.

IV. Multiplying both members by a quantity involving unknown

Example 1. Solve  $\frac{2x}{x-2} = \frac{4}{x-2}$ . Multiply both sides by  $x-2$ , we obtain  $2x = 4$ . Hence  $x = 2$ . This proved that if  $x$  were a root of the given equation, then  $x$  would only be 2. But 2



does not satisfy the given equation, since division by zero is excluded. Hence the given equation has no root at all, and is certainly not equivalent to the derived equation  $2x = 4$ .

V. *Squaring both members of an equation.* (See the preceding section.)

*Example 2.* Solve  $\sqrt{x-3} = -3$ . Squaring both sides yields  $x-3 = 9$ , or  $x = 12$ . Hence if the original equation had a root, it could only be 12. But 12 does not satisfy the original equation, since  $\sqrt{12-3} = \sqrt{9}$  stands for  $+3$  but not for  $-3$ . Hence the given equation is not equivalent to the derived equation  $x-3 = 9$ .

VI. *Dividing both members by a quantity involving unknowns.*

*Example 3.*  $(x-1)(x-2) = 0$  has the roots 1 and 2, as can be seen by substitution. If we divide both sides by  $(x-1)$ , we get  $x-2 = 0$ , which has only the root 2. Hence  $x-2 = 0$  is not equivalent to the given equation.

Operations IV and V may lead to equations which have roots not possessed by the original equation; such roots are often called **extraneous**. They can always be detected by checking by substitution in the original equation.

Operation VI may lead to equations which do not have roots which are possessed by the original equation. These lost roots are not easily detected.

We give only a few exercises here, since much later work illustrates these considerations.

## EXERCISES

*Find the roots of the following equations.*

$$1. 2x - 3 = x + 2. \quad 2. 3x + 2 = x + 8. \quad 3. \frac{3x}{x-3} = \frac{9}{x-3}.$$

$$4. \frac{5x}{x-2} = \frac{10}{x-2}. \quad 5. \sqrt{x+2} = -3. \quad 6. \sqrt{x+2} = 3.$$

7. Do exercise 30, section 12.

30. **Standard form of polynomial equations.** Degree of an equation. We shall be principally concerned with polynomial equations, or rational integral equations, that is, with (conditional) equations both of whose members are polynomials or rational integral expressions. For example,

$$(1) \quad x^3 + 5x^2 - x + 3 = x^3 + x^2 + x + 2$$

is a polynomial equation. By transposing and simplifying, we can write such an equation with zero on the right of the equals



sign, and a polynomial in standard form (see section 26) on the left; when this is done, the equation is said to be in **standard form**. For example, (1) may be written in the standard form

$$(2) \quad 4x^2 - 2x + 1 = 0.$$

When a polynomial equation is written in standard form, the degree of the polynomial on the left of the equals sign is called the **degree** of the equation. Hence equation (1) has the degree 2, since its standard form is (2). Equations of degree 1, 2, 3, 4, and 5 are called **linear**, **quadratic**, **cubic**, **quartic**,\* and **quintic** equations, respectively. For example,  $2x^3 + x - 5 = 0$  is a cubic equation in  $x$ ; and  $x^2y - 2xy^2 + xy - 5x + 4y - 3 = 0$  is a cubic equation in  $x$  and  $y$ . It will be seen in later chapters that the solution of polynomial equations increases in difficulty as the degree increases.

By the **general equation in  $x$  of a given degree** we mean the equation obtained by setting the general polynomial (see section 26) in  $x$  of that degree equal to zero. Hence the **general linear equation in  $x$**  may be written as

$$(3) \quad ax + b = 0 \quad (a \neq 0), \text{ or } a_0x + a_1 = 0 \quad (a_0 \neq 0)$$

where  $a$ ,  $b$ ,  $a_0$ ,  $a_1$  are arbitrary constants. Similarly, the **general quadratic equation in  $x$**  may be written as

$$(4) \quad ax^2 + bx + c = 0 \quad (a \neq 0), \text{ or } a_0x^2 + a_1x + a_2 = 0 \quad (a_0 \neq 0).$$

The **general cubic equation in  $x$**  may be written as

$$(5) \quad ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0), \\ \text{or } a_0x^3 + a_1x^2 + a_2x + a_3 = 0 \quad (a_0 \neq 0).$$

The **general equation of  $n$ th degree in  $x$**  may be written as

$$(6) \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad (a_0 \neq 0).^\dagger$$

## EXERCISES

Write each of the following equations in standard form and state its degree:

$$1. \quad 3x^4 + 2x^3 - x^2 + 3 = 3x^4 - x^3 + 2x - 1.$$

$$2. \quad x^3 - x + 1 = x^3 - 3x + 4.$$

$$3. \quad 2x^4 + x^2(3x - 1) + 2 = 2x^4 + 3x^2(x - 1) - x - 5.$$

$$4. \quad x(x^2 - 1) = x(x + 1) - 2.$$

$$5. \quad x(2x^2 + 1) = x + 16.$$

$$6. \quad \text{Write the general equations of degree 4 and 5.}$$

\* Or biquadratic.

† If  $a_0 = 0$ , the equation would have degree lower than  $n$ .



## CHAPTER IV

### Functions and Graphs

**31. Functions.** Consider two variables  $x$  and  $y$ . The variable  $y$  is said to be a **function** of  $x$  when any scheme or rule (relation, or correspondence) is given whereby to each value of  $x$  there correspond one or more definite values of  $y$ .

A simple way to define  $y$  as a function of  $x$  is to let  $y$  equal an algebraic expression in  $x$ .

*Example 1.* The rule  $y = 2x + 1$  defines  $y$  as a function of  $x$ . From this rule one can calculate the value of  $y$  corresponding to any value of  $x$ ; thus, if  $x = 2$ ,  $y = 5$ ; and if  $x = 3$ ,  $y = 7$ , and so on.

*Example 2.* The rule  $y = \frac{12}{x-2}$  defines  $y$  as a function of  $x$ . If  $x = 8$ ,  $y = 2$ ; if  $x = 4$ ,  $y = 6$ ; if  $x = 5$ ,  $y = 4$ ; if  $x = 7$ ,  $y = 12/4$  and so on.

*Example 3.* The rule  $y = \pm \sqrt{x}$  defines  $y$  as a function of  $x$ . If  $x = 4$ ,  $y = \pm 2$ ; if  $x = 9$ ,  $y = \pm 3$ ; if  $x = 2$ ,  $y = \pm \sqrt{2}$ , and so on.

If the scheme or rule assigns just one value of  $y$  to each value of  $x$ , the function is called **single-valued**; otherwise, **many-valued**. Thus, the functions given in examples 1 and 2 are single-valued functions, but the function given in example 3 is a many-valued function.

If  $y$  is a function of  $x$ , we call  $x$  the **independent variable**, and  $y$  the **dependent variable**, and we say that  $y$  depends on  $x$ . It may sometimes be necessary or desirable to restrict the possible values of the independent variable somewhat; if this is done, the totality or set of values which  $x$  may assume is called the **range** of  $x$ . If nothing is said to the contrary, the range of  $x$  will be assumed to be all numbers for which there is a corresponding value of  $y$ . In example 2, the range of  $x$  must not include the number 2, since division by zero is excluded. In example 3, if we wish  $y$  to be a real number, then the range of  $x$  must be restricted to non-negative real numbers. A function is said to be **defined** for every value of  $x$  in the range.



We may also consider functions of two or more independent variables. Thus, the variable  $z$  is said to be a function of  $x$  and  $y$  if a scheme or rule is given whereby to every given pair of values of  $x$  and  $y$  there corresponds one or more definite values of  $z$ . For example, the rule  $z = x^2 + y$  defines  $z$  as a single-valued function of  $x$  and  $y$ . If  $x = 2$ , and  $y = 3$ , then  $z = 7$ , and so on.

If a function is defined by letting the values of the dependent variable be obtained by substituting values of the independent variable(s) in a polynomial, or rational integral expression, in the independent variable(s), we say that the function is a **polynomial** or **rational integral function** of the independent variable(s). If a rational expression is used, we call the function a **rational function** of the independent variable(s). If a linear polynomial is used, we call the function a **linear function** of the independent variable(s). If a quadratic polynomial is used, we speak similarly of a **quadratic function**.

Thus, example 1 defines  $y$  as a linear function of  $x$ . Example 2 defines  $y$  as a rational function of  $x$ . Also,  $z = 2x + 3y - 4$  defines  $z$  as a linear function of  $x$  and  $y$ ; and  $y = x^2 - 2x + 3$  defines  $y$  as a quadratic function of  $x$ .

Note that the word "function" is used in mathematics in a technical sense quite different from its everyday meaning as in the statements "pumping blood is the function of the heart," "passing bills is a function of the legislature," or "the Senior Prom was the outstanding function of the school year."

A function may not always be defined as simply as by an algebraic expression. For example,  $\log x$  and  $\sin x$  are not. Functions may be defined for a limited number of values of the independent variable by means of a table whose entries may be obtained from experimental or observational data. Other ways of defining functions may be discussed elsewhere. In this book most of the functions will be defined by algebraic expressions.

## EXERCISES

*Find the value of each of the following functions when  $x = -2$ :*

$$1. y = 2x^2 - 3x + 1. \quad 2. y = x^3 + 2x^2. \quad 3. y = \frac{8}{2 - x}$$

*Find the value of each of the following functions when  $x = -2$  and  $y = 3$ :*

$$4. z = xy - 3x^2. \quad 5. z = x + y + 1.$$



For what values of  $x$  is each of the following functions not defined?

6.  $y = 1/x.$

7.  $y = \frac{8}{2-x}.$

8.  $y = \frac{8}{x^2-4}.$

9.  $y = \frac{1}{(x-2)(x-3)}.$

**32. Functional notation.** A given function of  $x$ , such as  $2x + 1$ , may be conveniently denoted by a symbol such as  $f(x)$ , read " $f$  of  $x$ ," or "the  $f$ -function of  $x$ ." If  $f(x) = 2x + 1$ , then  $f(2)$  stands for the value of the dependent variable, or value of the function, when  $x = 2$ . Note that  $f(x)$  does *not* mean a number  $f$  times a number  $x$ , but is simply a compact symbol for a function of  $x$ . If  $a$  is a constant, then  $f(a)$  stands for the value of the function  $f(x)$  when  $x$  has the value  $a$ , or when  $a$  is substituted for  $x$ . Hence, in the above example,  $f(2) = 5$ ,  $f(3) = 7$ ,  $f(a) = 2a + 1$ .

If two functions have to be discussed in the same problem, we may use similar symbols, such as  $F(x)$ ,  $g(x)$ ,  $D(x)$ ,  $R(x)$ , etc., to avoid confusion. Hence, if  $F(x) = x^2 + 1$  and  $f(x) = 2x + 1$ , we have  $f(2) = 5$ ,  $F(2) = 5$ ,  $f(3) = 7$ ,  $F(3) = 10$ ,  $f(-2) = -3$ ,  $F(-2) = 5$ , and so on.

A function of two variables may be similarly denoted by a symbol of the form  $f(x,y)$ . Thus, if  $f(x,y) = x^2 + 3y$ , then  $f(2,1) = 7$ ,  $f(3,2) = 15$ ,  $f(1,1) = 4$ ,  $f(-1,0) = 1$ , and so on.

If  $F(x) = \frac{1}{x-2}$ , we shall say that  $F(2)$  does not exist, or is not defined, since 2 is not in the range of  $x$ .

## EXERCISES

Given that  $f(x) = 2x + 6$ , find:

- |            |   |            |             |             |               |
|------------|---|------------|-------------|-------------|---------------|
| 1. $f(2).$ | — | 2. $f(3).$ | 3. $f(-1).$ | 4. $f(-2).$ | 5. $f(-3).$   |
| 6. $f(0).$ |   | 7. $f(n).$ | 8. $f(a).$  | 9. $f(-r).$ | 10. $f(10k).$ |

Given that  $d(x) = x^2 - 9$ , find:

- |             |              |             |              |             |
|-------------|--------------|-------------|--------------|-------------|
| 11. $d(2).$ | 12. $d(-2).$ | 13. $d(3).$ | 14. $d(-3).$ | 15. $d(r).$ |
|-------------|--------------|-------------|--------------|-------------|

Given that  $F(x) = 3x^2 - 2x + 1$ , find:

- |                        |               |                |                       |
|------------------------|---------------|----------------|-----------------------|
| 16. $F(2).$            | 17. $F(-2).$  | 18. $F(r).$    | 19. $F(\frac{1}{2}).$ |
| 20. $F(-\frac{1}{2}).$ | 21. $F(1.2).$ | 22. $F(-1.2).$ |                       |



Given that  $G(x) = \frac{2x - 3}{x - 4}$ , find:

23.  $G(5)$ .      24.  $G(\frac{1}{2})$ .      25.  $G(-\frac{1}{2})$ .      26.  $G(\frac{3}{4})$ .      27.  $G(4)$ .

Given that  $Q(x) = \frac{1}{x - 2} + \frac{1}{x - 3}$ , find:

28.  $Q(1)$ .      29.  $Q(4)$ .      30.  $Q(\frac{1}{2})$ .

**33. Applications of functions in science. Formulas.** The study of functions came to be of overwhelming importance in the 17th century, when scientists became greatly interested in the study of varying quantities. If  $y$  is a function of  $x$ , we say that the value of  $y$  depends on the value of  $x$ . Speaking loosely, we might say that  $y$  varies with  $x$ , or that a change in  $x$  induces a corresponding change in  $y$ . This statement is not strictly true, for we might well have a function which assigns the same value to  $y$  no matter what value is given to  $x$ , such as the function  $y = 3$ . Such a function is called a **constant function**. But it will do no great harm if you think of the value of  $y$  changing as the value of  $x$  changes, provided it is agreed that the change in  $y$  may be zero.

Every formula, whether taken from geometry, physics, or other subjects, expresses some variable quantity as a function of other variable quantities.

*Example 1.* The area  $A$  of a square of side  $x$  is given by the formula  $A = x^2$ .

*Example 2.* The perimeter  $P$  of a square of side  $x$  is given by the formula  $P = 4x$ .

*Example 3.* The distance  $s$ , measured in feet, through which a body falls in  $t$  seconds is given approximately by the formula  $s = 16t^2$ , air resistance being neglected.

*Example 4.* The area of a triangle is given by the formula  $A = \frac{1}{2}bh$ , where  $b$  is the base and  $h$  the altitude.

*Example 5.* The area of a trapezoid is given by the formula  $A = \frac{1}{2}h(b + b')$ , where  $h$  is the altitude and  $b$  and  $b'$  are the bases (Fig. 11).

*Example 6.* The simple interest  $I$  on  $P$  dollars at the rate  $R$  (expressed as a decimal) per year for  $T$  years is given by the formula  $I = PRT$ .

Examples 1, 2, 3 are functions of one variable, example 4 is a function of two variables, and examples 5 and 6 are functions



of three variables. It is clear that examples of functions abound in many different subjects. They may be exact, as in examples 1, 2, 4, 5, and 6, or approximate, as in example 3.

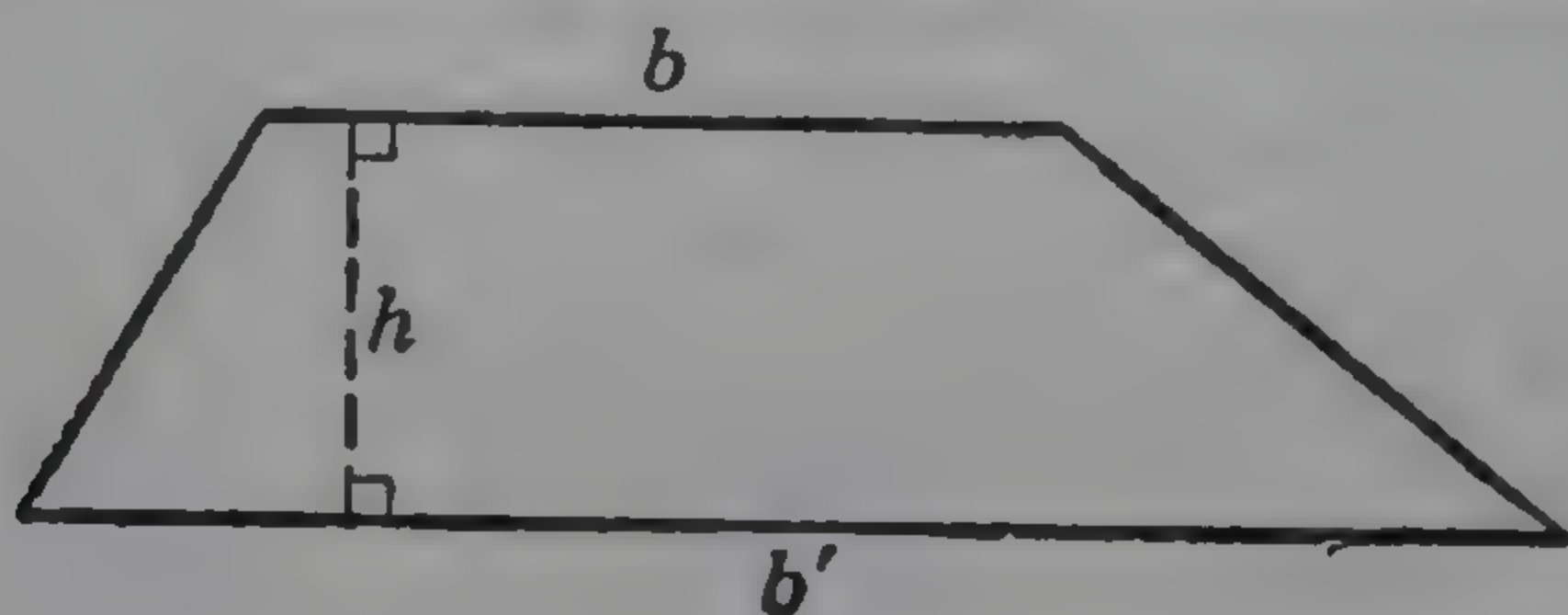


Fig. 11.

## EXERCISES

1. Using the formula of example 3, above, how far will a body fall in (a) 3 seconds? (b) 4 seconds?
2. Using the formula of example 4, above, what is the area of a triangle with base 3 feet and altitude 4 feet?
3. Using the formula of example 4, above, what is the area of a triangle with base 6 feet and altitude 4 inches?
4. Using the formula of example 5, above, what is the area of a trapezoid with altitude 6 inches and bases 3 inches and 7 inches, respectively?
5. Using the formula of example 6, above, find the simple interest on \$124.00 at the rate of 4% per year for 3 years. (*Hint*: express the rate as a decimal, that is, as 0.04.)
6. The time  $T$  measured in seconds, required for a complete oscillation of a simple pendulum of length  $k$  feet is given by the formula  $T = 2\pi \sqrt{k/g}$  where  $\pi = 3.14$  approximately and  $g = 32.2$  approximately. If  $k = 2.34$  feet, find  $T$  correct to the nearest tenth.
7. If  $F$  is the temperature in degrees Fahrenheit and  $C$  is the temperature in degrees Centigrade, then  $F = \frac{9}{5}C + 32$ . Find the Fahrenheit temperature corresponding to  $40^\circ$  Centigrade.
8. The volume  $V$  of a sphere of radius  $r$  is given by the formula  $V = \frac{4}{3}\pi r^3$ . Find the volume of a sphere of radius 3 feet, correct to the nearest tenth, using  $\pi = 3.14$  approximately.
9. The surface area  $S$  of a sphere of radius  $r$  is given by the formula  $S = 4\pi r^2$ . Find the surface area of a sphere of radius 3 feet, correct to the nearest tenth, using  $\pi = 3.14$  approximately.
34. **Rectangular coordinates.** Choose two perpendicular lines in a plane. Call one the  $x$ -axis and the other the  $y$ -axis. The point  $O$  where they intersect is called the **origin**. It is customary to consider the  $x$ -axis horizontal and the  $y$ -axis vertical. Choose



a unit of length and lay it off repeatedly on both axes. Choose a positive direction on each axis; it is customary to make the positive side of the  $x$ -axis the right side, and the positive side of the  $y$ -axis the upper side. Assign real numbers to the points of the axes in the usual way. Then to each point in the plane we may attach a pair of real numbers, called the  $x$ -coordinate (or *abscissa*) and the  $y$ -coordinate (or *ordinate*) of the point, respectively. Together these coordinates of a point  $P$  constitute

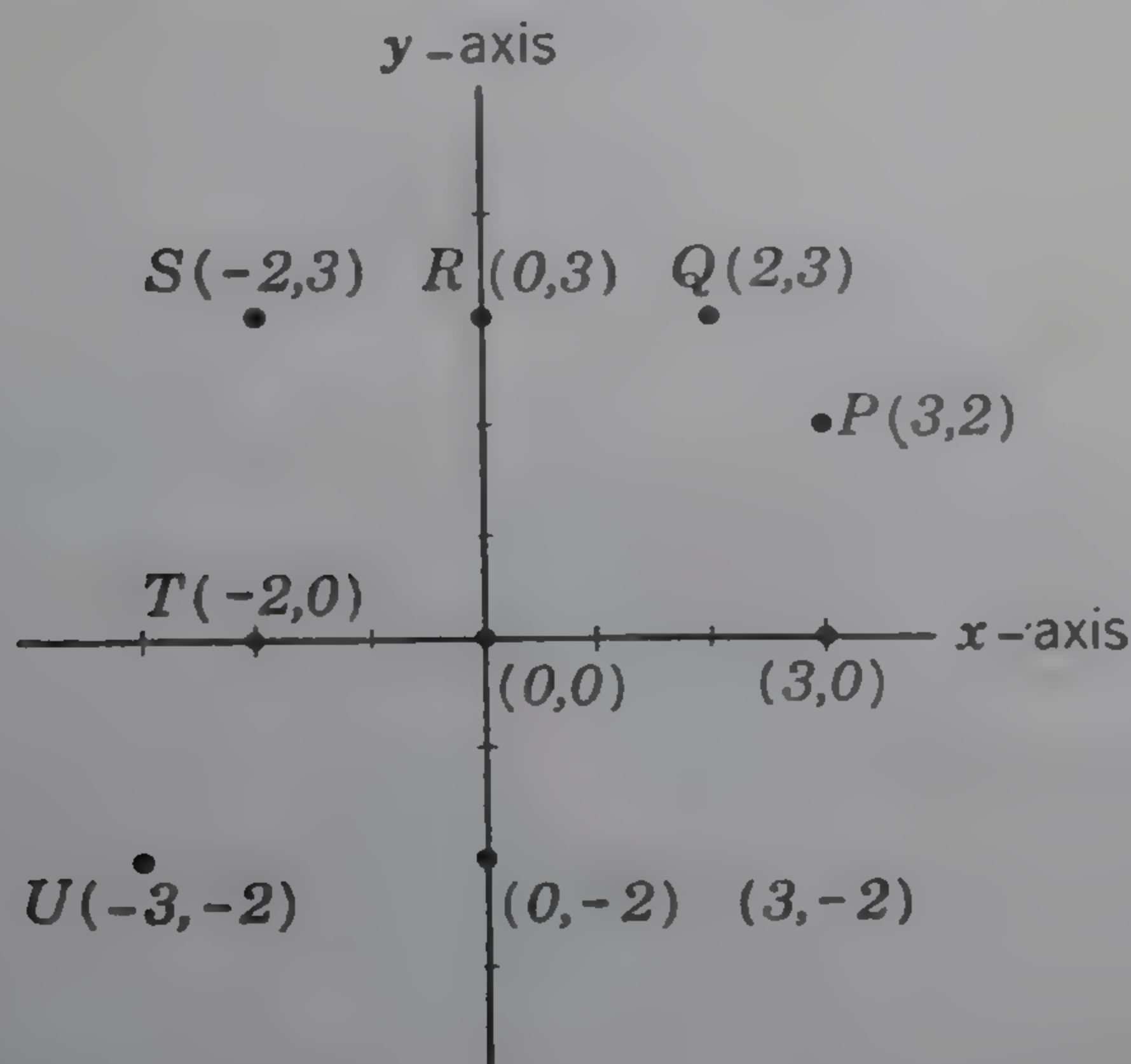


Fig. 12.

directions for getting to  $P$  from  $O$ . The  $x$ -coordinate tells us how to proceed from  $O$  along the  $x$ -axis, and then the  $y$ -coordinate tells us how to go in the direction of the  $y$ -axis. The  $x$  and  $y$  coordinates are indicated by a symbol of the form  $(x,y)$ , the  $x$ -coordinate always being given first. Thus the point  $P$  in Fig. 12 has the coordinates  $(3,2)$  while  $Q$  has the coordinates  $(2,3)$ . Similarly,  $R$  has the coordinates  $(0,3)$ ,  $S$  the coordinates  $(-2,3)$ , and so on as in Fig. 12. Note that *every point on the  $x$ -axis has  $y$ -coordinate equal to zero*. Marking a point with given coordinates is called **plotting the point**.

Use of these so-called **rectangular coordinates** was developed by the great French mathematician-philosopher René Descartes, who founded the important subject of analytic geometry\* on this basis in 1637.

\* For further indications of how these coordinates enable us to make huge strides in geometry, and for a discussion of other types of coordinates, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, Chapter IX.



## EXERCISES

1. Plot the points whose coordinates are  $(1,2)$ ,  $(2,1)$ ,  $(-1,2)$ ,  $(2,-1)$ ,  $(-2,-1)$ .
2. Plot the points whose coordinates are  $(2,0)$ ,  $(0,2)$ ,  $(-2,0)$ ,  $(0,-2)$ ,  $(0,0)$ .
3. What is the  $y$ -coordinate of any point on the  $x$ -axis?
4. What is the  $x$ -coordinate of any point on the  $y$ -axis?

35. The graphs of equations and functions. By the graph of an equation in two variables  $x$  and  $y$  we mean the set or totality of all those points, and only those points, whose coordinates satisfy the equation. *Only real numbers will be considered.* To plot the graph of an equation we may substitute various real values for one variable in the equation, calculate from the equation the corresponding value of the other variable, and tabulate the results; then we plot the corresponding points and join them with a smooth curve.

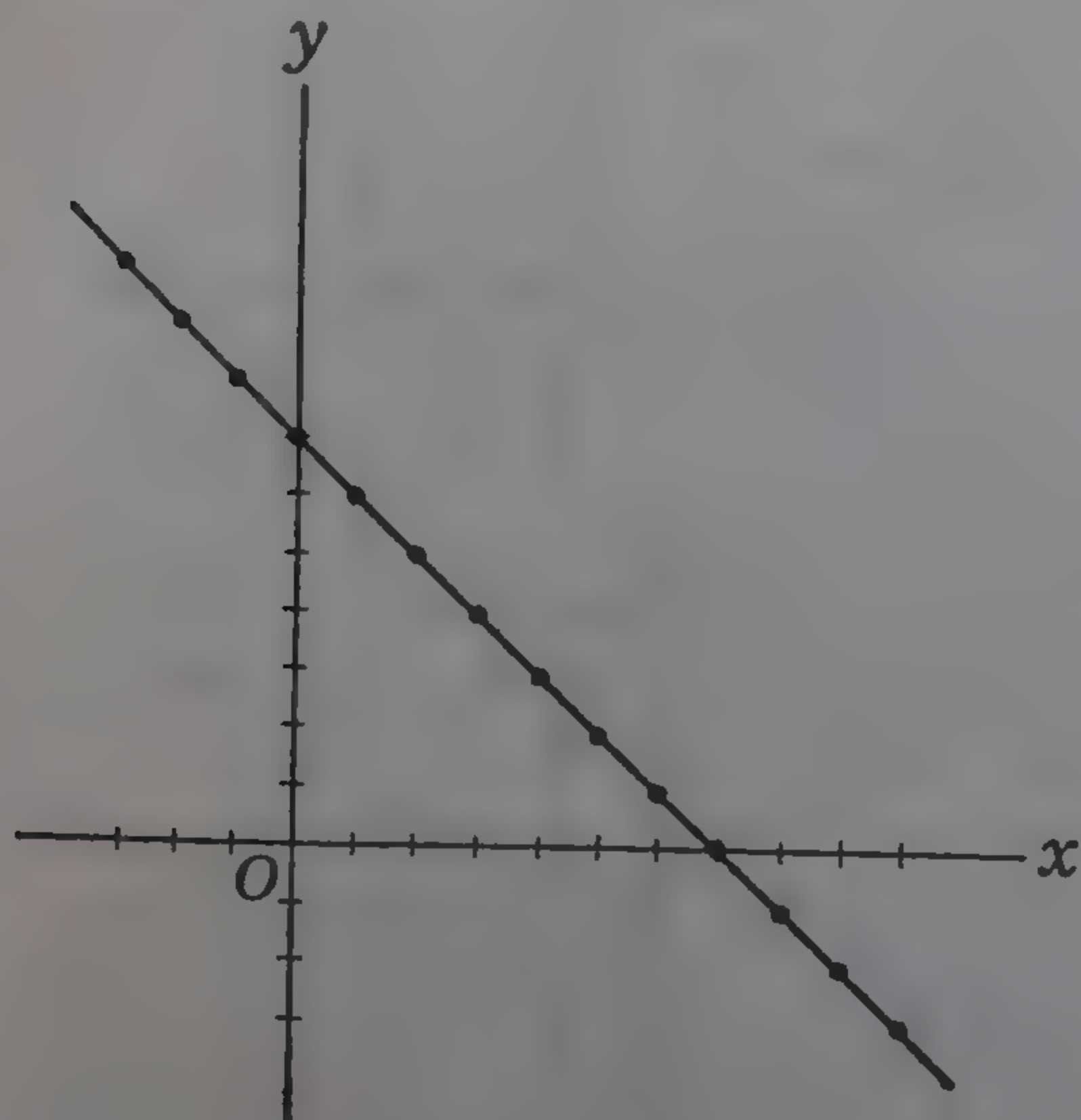


Fig. 13.

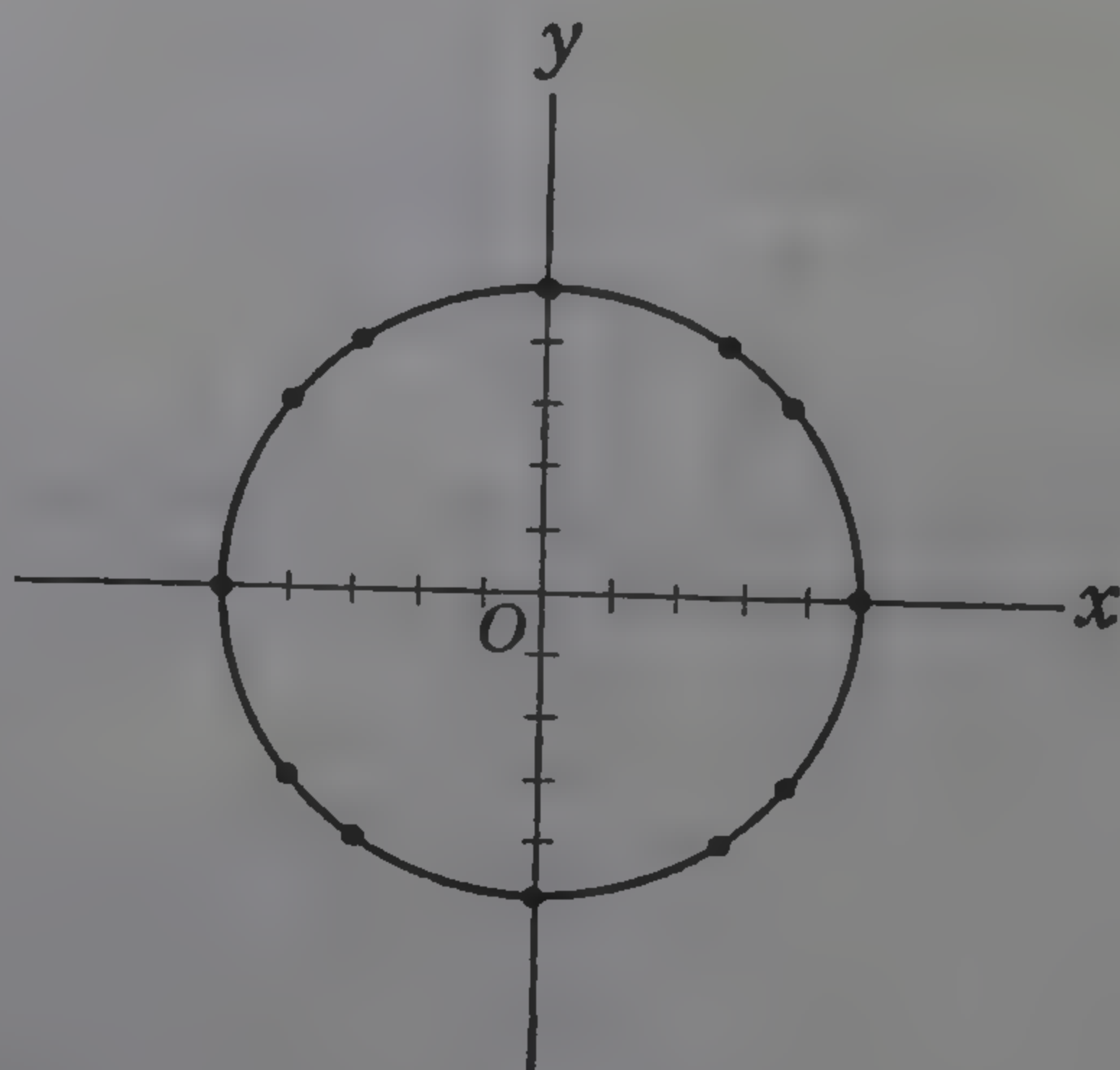


Fig. 14.

*Example 1.* Draw the graph of the equation  $x + y = 7$ . We obtain the table

|     |    |    |    |   |   |   |   |   |   |   |   |    |    |    |
|-----|----|----|----|---|---|---|---|---|---|---|---|----|----|----|
| $x$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8  | 9  | 10 |
| $y$ | 10 | 9  | 8  | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 |

Plotting these points, we get the straight line in Fig. 13.

It will be shown later that every linear equation in  $x$  and  $y$ , with real coefficients, represents a straight line. With this knowledge, it would have sufficed to plot two points, with a possible third



point as a check. In general, however, many points are necessary to get the shape of the graph.

*Example 2.* Draw the graph of the equation  $x^2 + y^2 = 25$ . We get the table

|     |         |         |         |         |         |         |
|-----|---------|---------|---------|---------|---------|---------|
| $x$ | 0       | $\pm 5$ | $\pm 3$ | $-3$    | $\pm 4$ | $-4$    |
| $y$ | $\pm 5$ | 0       | $\pm 4$ | $\pm 4$ | $\pm 3$ | $\pm 3$ |

Plotting these points, we get the circle in Fig. 14.

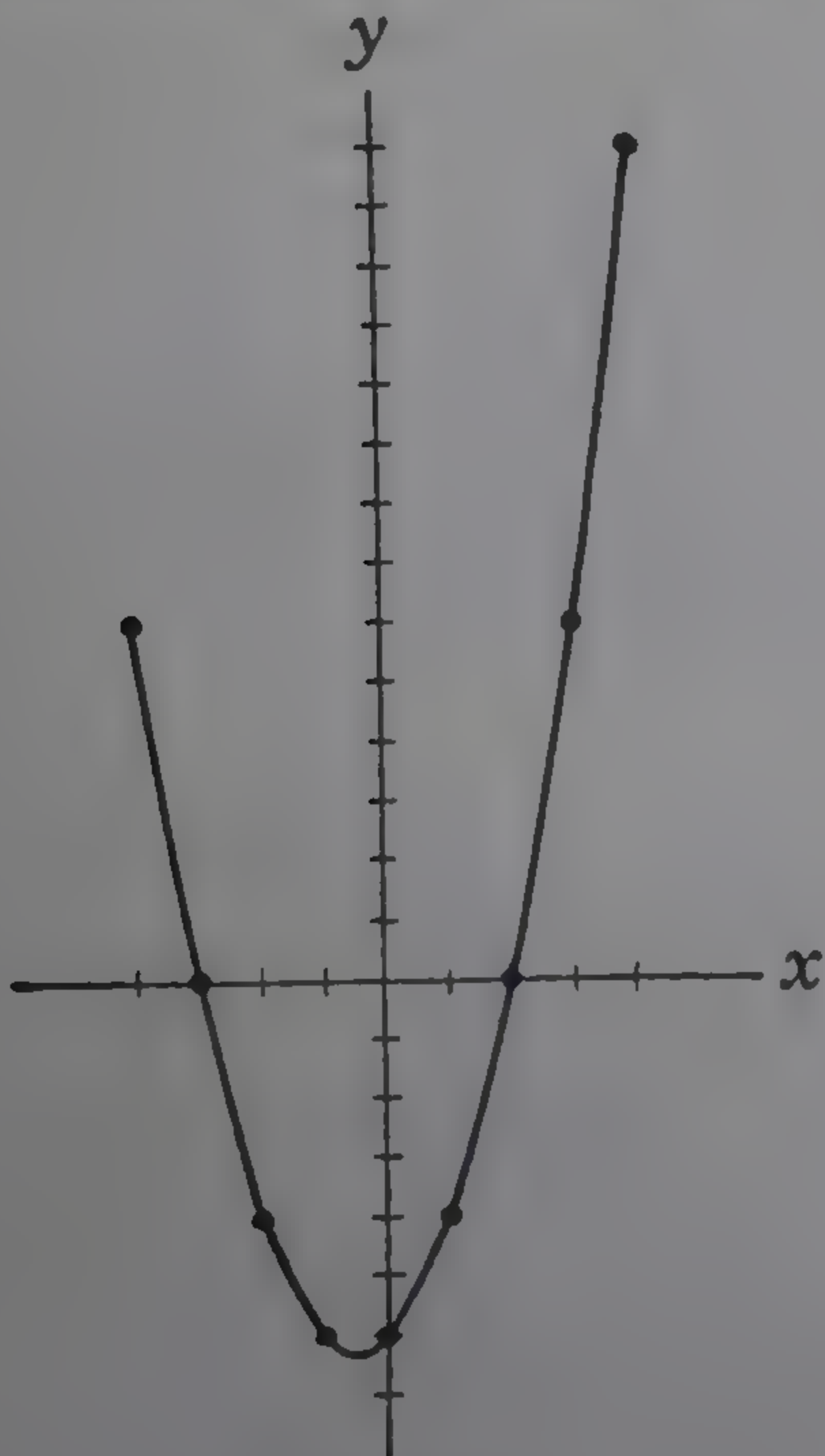


Fig. 15.

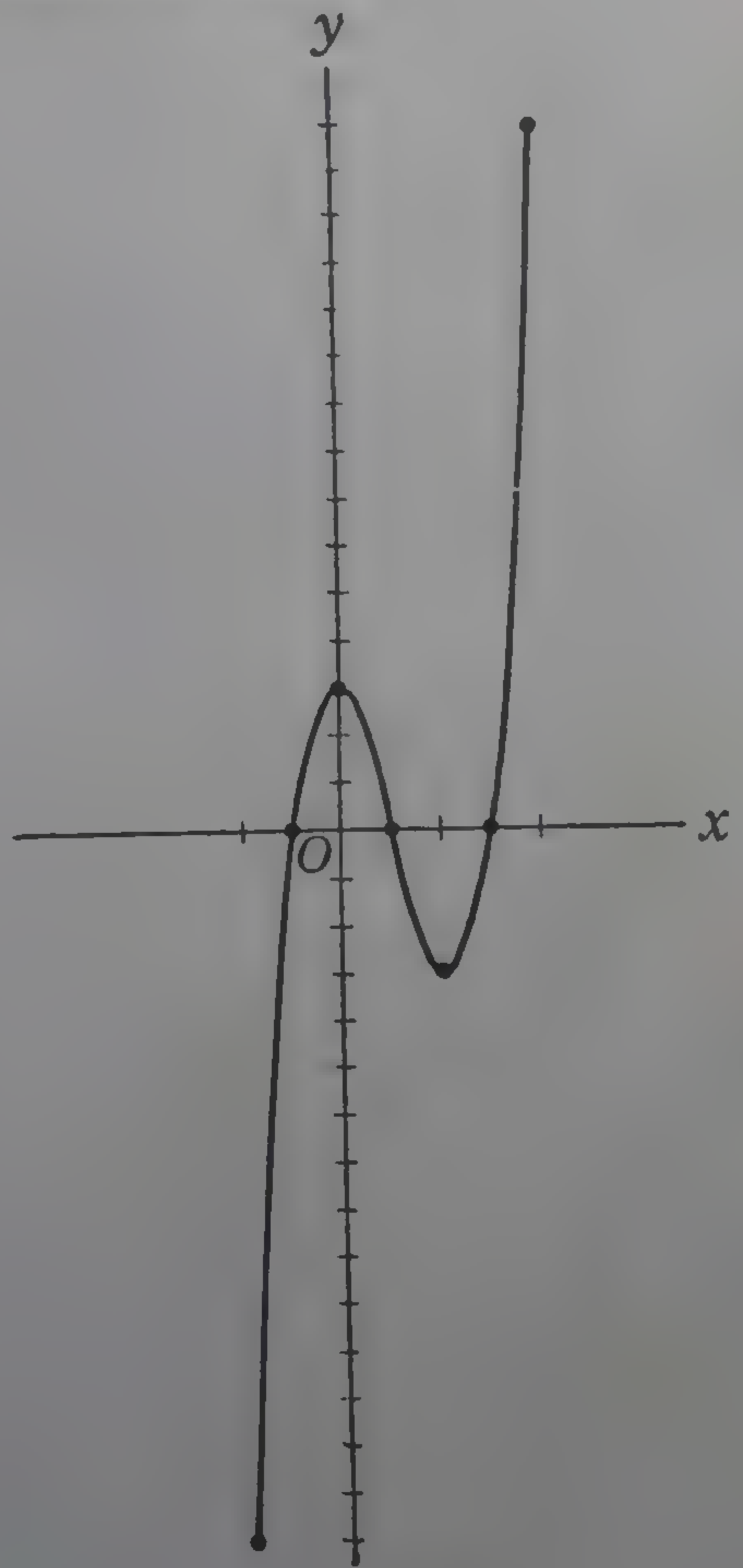


Fig. 16.

By the graph of a function  $f(x)$  we mean the graph of the equation  $y = f(x)$ .

*Example 3.* Plot the graph of the function  $x^2 + x - 6$ . From the equation  $y = x^2 + x - 6$  we get the table

|     |    |    |    |   |    |   |    |    |    |
|-----|----|----|----|---|----|---|----|----|----|
| $x$ | 0  | 1  | -1 | 2 | -2 | 3 | -3 | 4  | -4 |
| $y$ | -6 | -4 | -6 | 0 | -4 | 6 | 0  | 14 | 6  |

Plotting, we get the parabola in Fig. 15.



*Example 4.* Draw the graph of the function  $x^3 - 3x^2 - x + 3$ . From the equation  $y = x^3 - 3x^2 - x + 3$  we get the table

|     |   |   |    |    |     |   |    |
|-----|---|---|----|----|-----|---|----|
| $x$ | 0 | 1 | -1 | 2  | -2  | 3 | 4  |
| $y$ | 3 | 0 | 0  | -3 | -15 | 0 | 15 |

Plotting these points, we get the graph of Fig. 16.

Much information concerning *how* the quantity  $y$  varies as  $x$  varies can be obtained at least approximately from the graph. Thus, one might infer from the graph of Fig. 15 that  $y$  has a minimum value of something less than  $-6$  and assumes this minimum value for a value of  $x$  about midway between  $x = -1$  and  $x = 0$ .

## EXERCISES

*Plot the graph of each of the following functions and estimate from the graph for what (real) values of  $x$  the function assumes the value zero:*

1.  $2x - 6$ .    2.  $3x + 6$ .    3.  $\frac{1}{2}x + 1$ .    4.  $\frac{1}{3}x - 1$ .    5.  $x^2 - 4$ .
6.  $x^2 - 9$ .    7.  $x^2 + 1$ .    8.  $x^2 + x + 1$ .    9.  $x^2 - x - 6$ .
10.  $x^2 - 4x + 3$ .    11.  $x^3 - 2x^2 - 5x + 6$ .    12.  $x^3 + 2x^2 - 5x - 6$ .

*Plot the graphs of each of the following equations:*

13.  $2x + 4y = 5$ .    14.  $3x - 2y = 1$ .    15.  $x^2 + y^2 = 100$ .
16.  $4x^2 + 9y^2 = 36$ .    17.  $x^2 - y^2 = 1$ .    18.  $4x^2 - 9y^2 = 36$ .
19.  $xy = 1$ .    20.  $y(x - 2) = 1$ .

**36. Continuous functions.** In plotting the graph of a function  $f(x)$  we join the points actually plotted by a smooth curve when

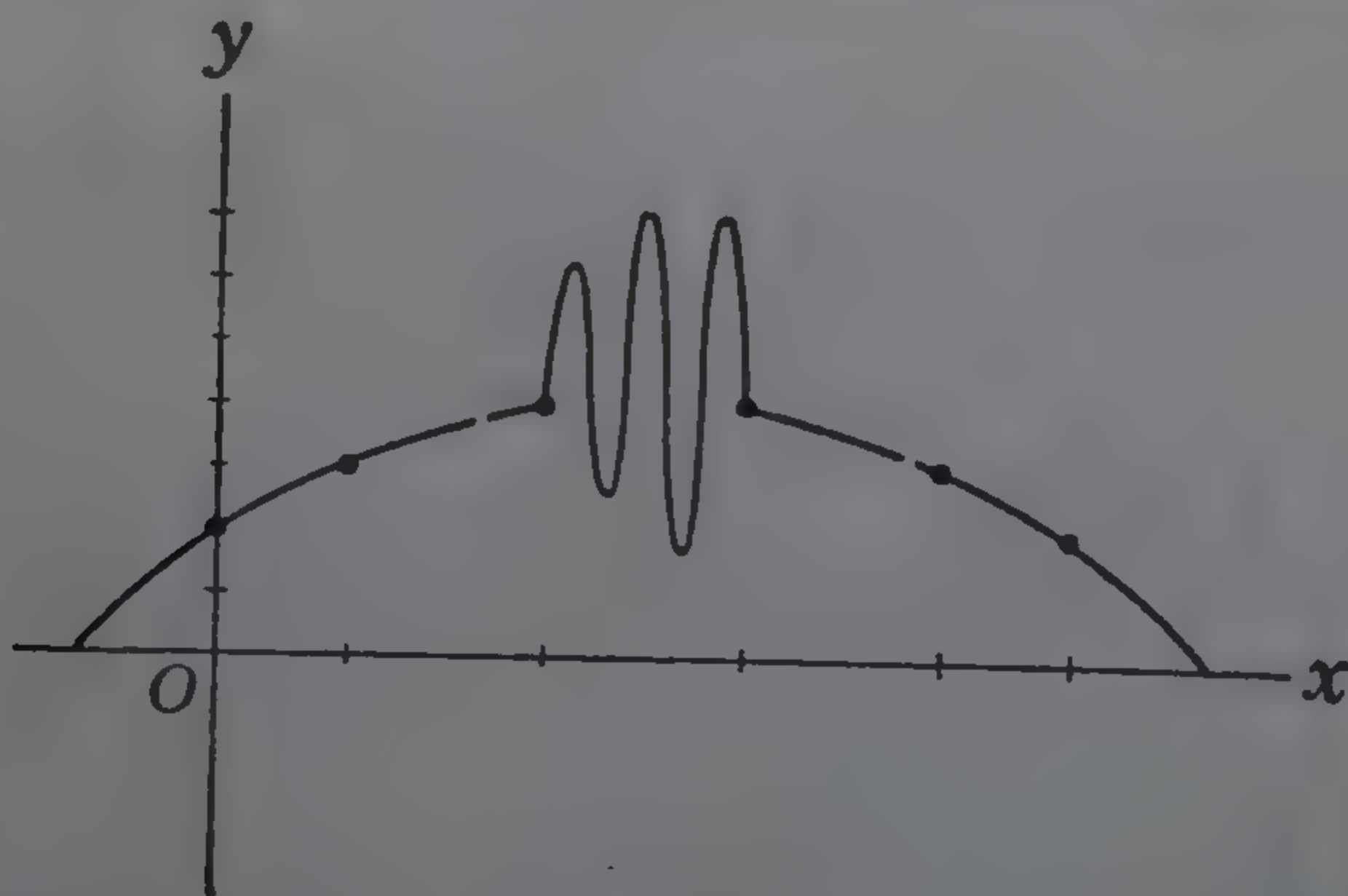


Fig. 17.

we have enough points to judge the shape of the curve. This involves the assumption, among others, that the curve does not behave wildly between the plotted points, as in Fig. 17. This



assumption, although rash at this stage of the game, can be justified for polynomial functions by advanced analysis, beyond the scope of this book. By a **continuous function** is meant, speaking very roughly,\* one whose graph can be drawn with one continuous stroke of the pencil, that is, without taking pencil from paper. It can be shown that *polynomials in  $x$  are continuous at all points of their graphs*, and that *rational functions in  $x$ , expressed as  $\frac{f(x)}{d(x)}$  where  $f(x)$  and  $d(x)$  are polynomials, are continuous except at the roots of the denominator polynomial  $d(x)$ .*

**37. Graphical solution of equations.** Let  $f(x)$  be any function of  $x$ . If the value of  $f(x)$  is zero when the number  $r$  is substituted

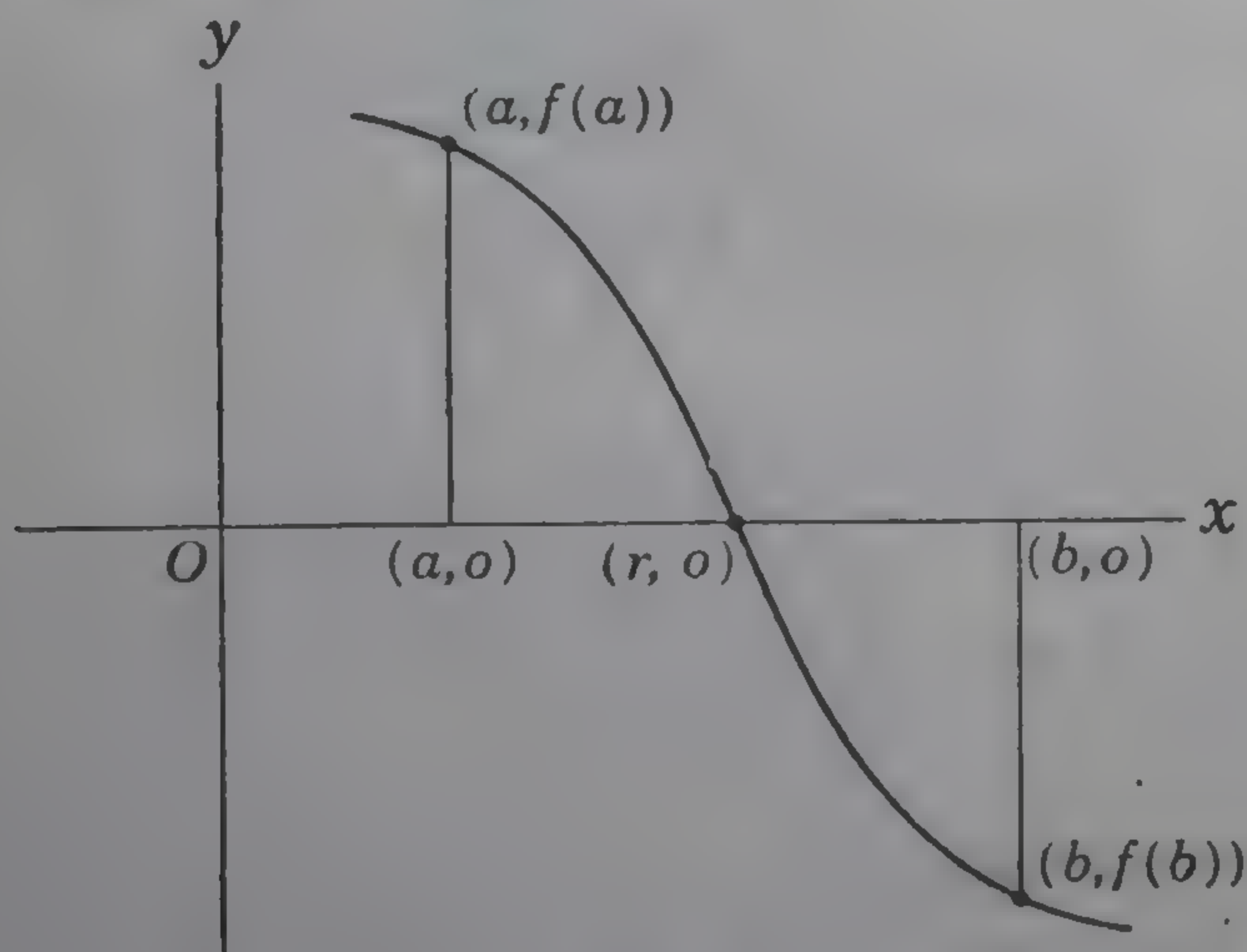


Fig. 18.

for  $x$ , that is, if  $f(r) = 0$ , then  $r$  is a **root** of the equation  $f(x) = 0$ . If the root  $r$  is a real number, then the graph of the function  $y = f(x)$  must have a point in common with the  $x$ -axis at  $x = r$  since  $y = 0$  at  $x = r$  (Fig. 18). Therefore *the real roots of an equation  $f(x) = 0$  are the  $x$ -coordinates of the points of intersection of the graph of  $y = f(x)$  with the  $x$ -axis*. Consequently, the real roots of an equation may be estimated from the graph. Imaginary roots cannot be found this way, since the  $x$ - and  $y$ -coordinates on our graph are real numbers.

An important property of all continuous functions, and hence, in particular, of all polynomials, is the following:

\* This is highly inaccurate, but will suffice for present purposes. For a more precise discussion of continuity, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, p. 296.



If  $f(x)$  is continuous, and if  $f(a)$  and  $f(b)$ , (where  $a < b$ ), have opposite signs, then there is at least one real root  $r$  of the equation  $f(x) = 0$  between  $a$  and  $b$  (that is,  $a < r < b$ ).

This principle is intuitively clear from Fig. 18, but is too difficult to prove here.

*Example 1.* The real roots of  $x^2 + x - 6 = 0$  are 2 and  $-3$ . See Fig. 15.

*Example 2.* The real roots of  $x^3 - 3x^2 - x + 3 = 0$  are  $\pm 1$ , and 3. See Fig. 16.

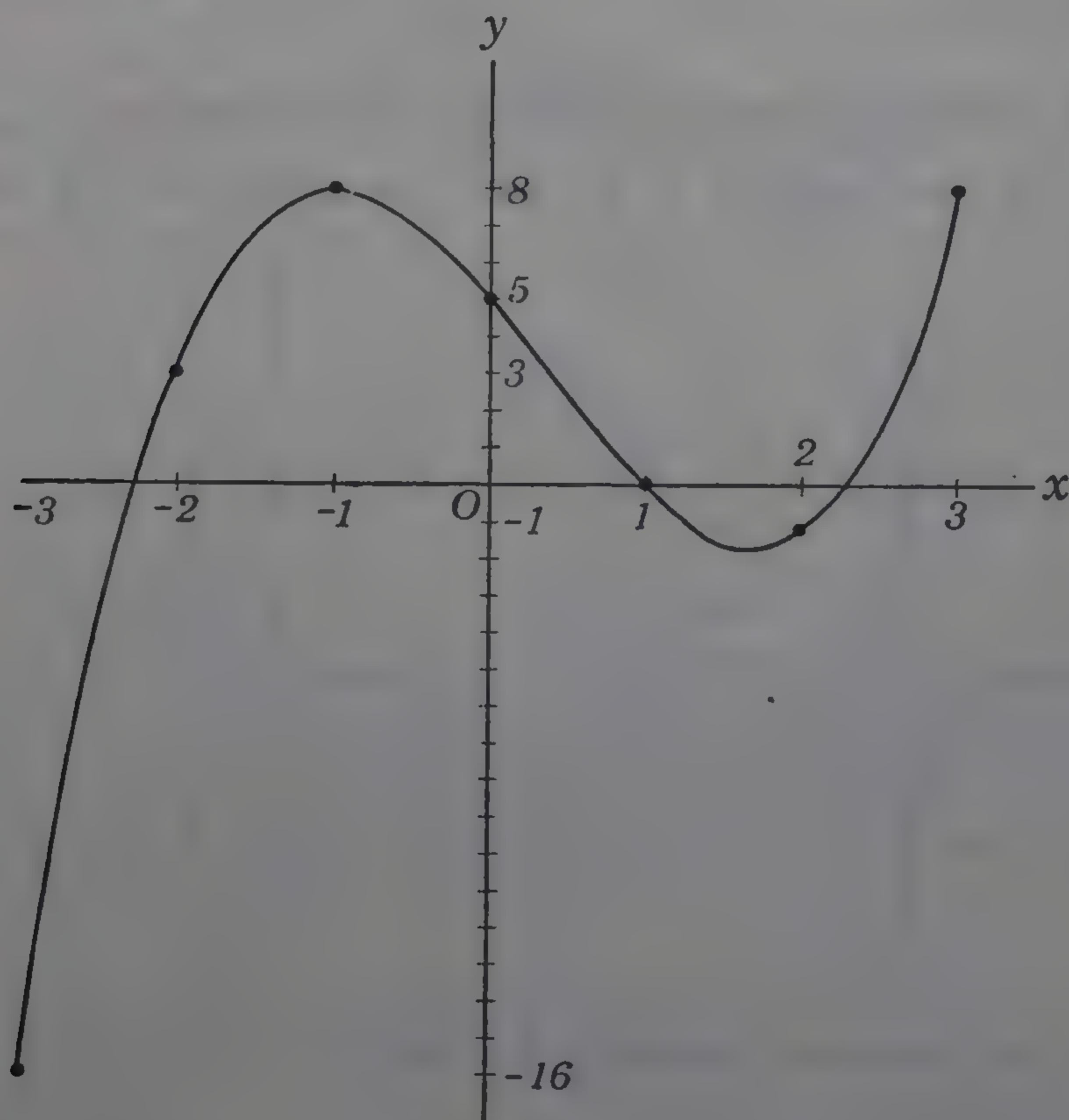


Fig. 19.

*Example 3.* Estimate graphically the real roots of  $x^3 - x^2 - 5x + 5 = 0$ . Plotting the function  $y = x^3 - x^2 - 5x + 5$ , we find the table

| $x$ | 0 | 1 | 2  | 3 | -1 | -2 | -3  |
|-----|---|---|----|---|----|----|-----|
| $y$ | 5 | 0 | -1 | 8 | 8  | 3  | -16 |

and the graph of Fig. 19. Hence the roots are 1, something between 2 and 3, and something between  $-2$  and  $-3$ .

This method will be elaborated further in Chapter XV.



## EXERCISES

Estimate from the graph the real roots of each of the following equations:

1.  $2x - 8 = 0$ .    2.  $3x + 6 = 0$ .    3.  $\frac{1}{2}x + 1 = 0$ .    4.  $\frac{1}{3}x - 1 = 0$ .
5.  $x^2 - 4 = 0$ .    6.  $x^2 - 9 = 0$ .    7.  $x^2 - 4x + 3 = 0$ .
8.  $x^2 - 5x + 4 = 0$ .    9.  $x^2 - 2x - 3 = 0$ .    10.  $x^2 + 2x - 3 = 0$
11.  $x^3 - 4x^2 + x + 6 = 0$ .    12.  $x^3 - 7x - 6 = 0$ .
13.  $6x^2 - 5x - 4 = 0$ .    14.  $6x^2 - 7x - 3 = 0$ .
15.  $x^2 - 2 = 0$ .    16.  $x^2 - 3 = 0$ .

\*38. **Infinites of a rational function.** The function  $y = \frac{1}{x}$  is not defined for the value  $x = 0$ . But the graph of the function

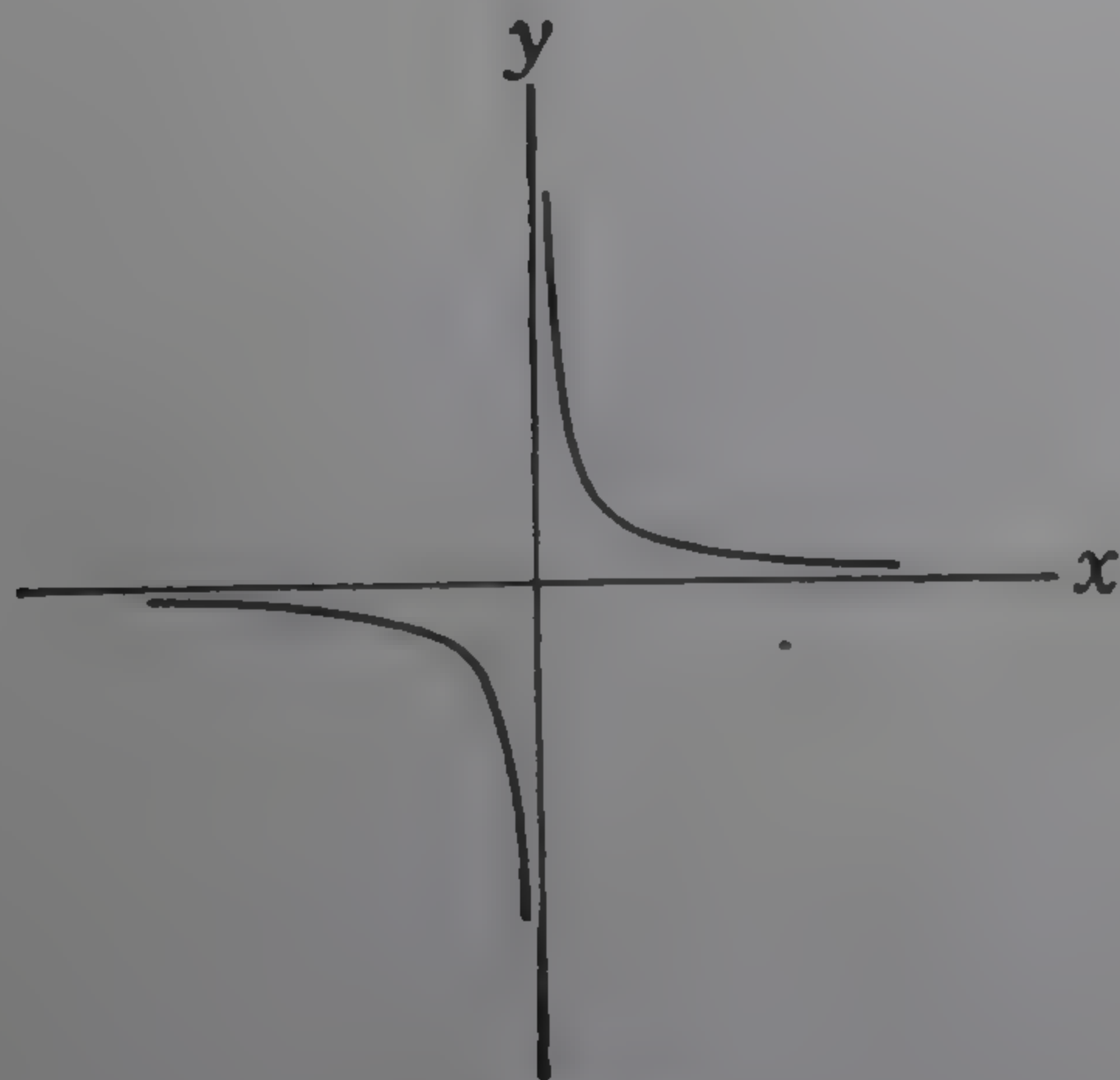


Fig. 20.

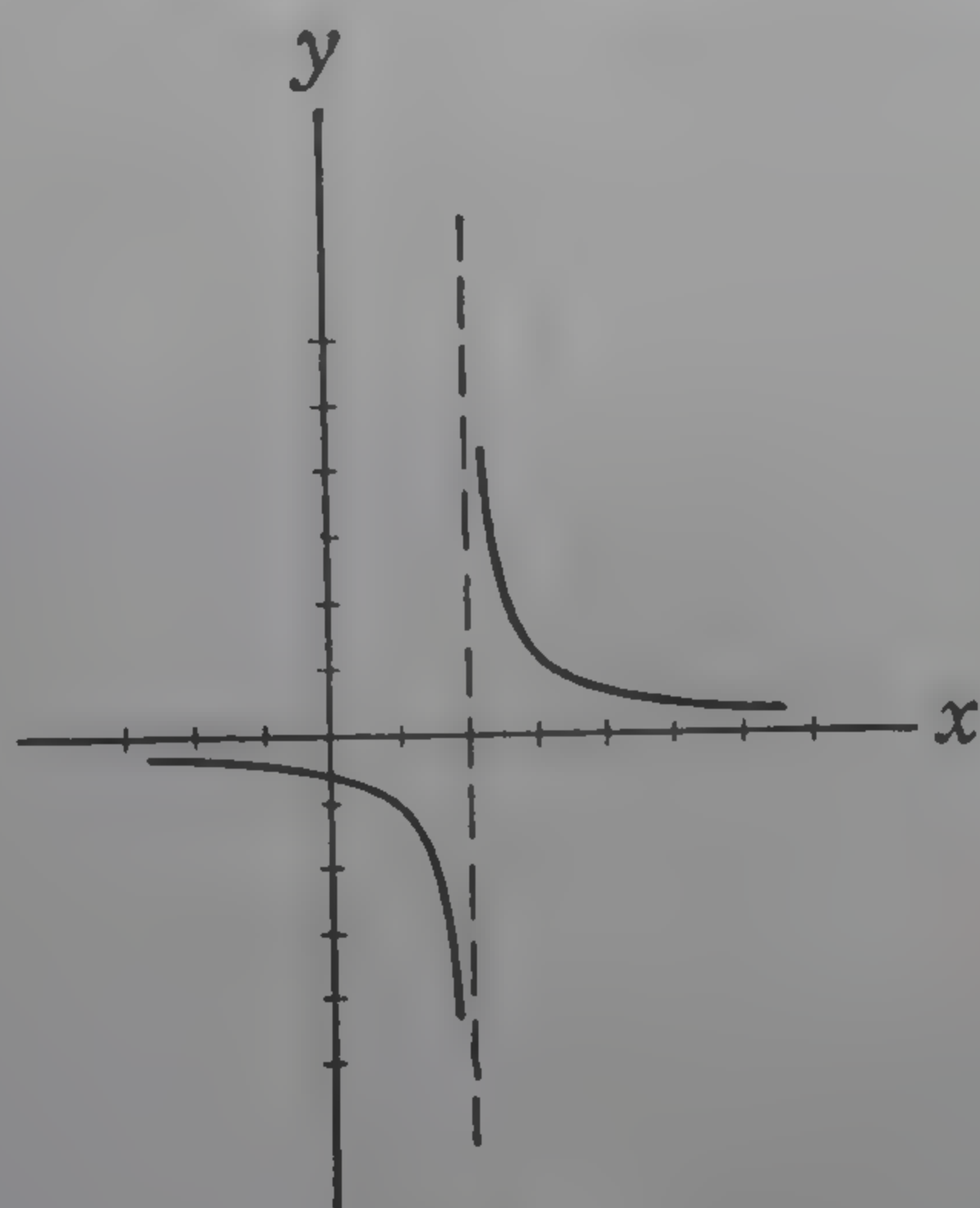


Fig. 21.

exhibits an important property for values of  $x$  near zero. Note the following table:

| $x$ | $\pm 1$ | $\pm 0.1$ | $\pm 0.01$ | $\pm 0.001$ | $\pm 0.0001$ | $\dots$ | 0           |
|-----|---------|-----------|------------|-------------|--------------|---------|-------------|
| $y$ | $\pm 1$ | $\pm 10$  | $\pm 100$  | $\pm 1000$  | $\pm 10000$  | $\dots$ | not defined |

It is clear that as  $x$  gets nearer to 0,  $y$  increases in absolute value without limit. In technical language we say that  $y$  becomes infinite as  $x$  approaches 0. In symbols,  $y \rightarrow \infty$  as  $x \rightarrow 0$ , read

" $y$  becomes infinite as  $x$  approaches 0," or  $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$ , read "the

\* This section may be omitted on first reading without disturbing the continuity of the book.



limit of  $1/x$  as  $x$  approaches 0 is infinity."\* The graph (Fig. 20) indicates what this means.

Similarly  $y = \frac{1}{x-2}$  becomes infinite at  $x = 2$  (Fig. 21). It can be shown that a rational function  $y = \frac{f(x)}{d(x)}$ , where  $f(x)$  and  $d(x)$  are polynomials, becomes infinite at every root of  $d(x)$  which is not also a root of  $f(x)$ . A value of  $x$  for which a function  $y = F(x)$  becomes infinite may be called an **infinity** or **infinite discontinuity**† of the function  $F(x)$ .

## EXERCISES

*Plot the graph of each of the following functions and find its infinities:*

1.  $y = \frac{2}{x}$
2.  $y = \frac{1}{2x}$
3.  $y = \frac{1}{x-3}$
4.  $y = \frac{1}{2x-3}$
5.  $y = \frac{1}{3x+2}$
6.  $y = \frac{1}{3x-4}$
7.  $y = \frac{1}{x^2-4}$
8.  $y = \frac{1}{9-x^2}$
9.  $y = \frac{1}{(x-1)(x-3)}$
10.  $y = \frac{1}{x(x-2)}$
11.  $y = \frac{x}{x^2-2x}$
12.  $y = \frac{x-2}{x^2-2x}$
13.  $y = \frac{x^2-2x}{x-2}$

\* It is misleading to write  $1/0 = \infty$ , although this is often done, because it seems to suggest that we get something by dividing 1 by 0. We have seen in section 11 that division by zero is excluded. When the symbol  $1/0 = \infty$  is encountered, it should be understood as an abbreviation for the situation described in the text.

† For a further discussion of this and other types of discontinuities see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941.



## CHAPTER V

### Elementary Operations with Polynomials

**39. Integers and polynomials. Hindu-Arabic notation.** It has already been remarked (section 26) that integers and polynomials (or rational integral expressions) have many similar properties. There are deep-lying reasons for their similarity which are beyond the scope of this book. But even the elementary operations of addition, subtraction, multiplication, and division are much alike for integers and polynomials. These similarities may be traced to the fact that an integer, written in the usual way, is nothing but a particular value of a polynomial in  $x$ , namely the value for  $x = 10$ . Thus 2345 is the value of the polynomial  $2x^3 + 3x^2 + 4x + 5$  when  $x = 10$ , or  $2 \cdot 10^3 + 3 \cdot 10^2 + 4 \cdot 10 + 5$ . Similarly 5078 is the value of the polynomial  $5x^3 + 7x + 8$  when  $x = 10$ ; and  $-407$  is the value of the polynomial  $-4x^2 - 7$  when  $x = 10$ .

Our way of writing integers is called the Hindu-Arabic notation because it was invented by the Hindus about 500 A.D. and brought into Europe by the Arabs when they overran Spain in the 8th century. It represents a great advance over the Roman numerals, as one can see by multiplying 1626 by 62 in the usual way and then trying to multiply the same numbers MDCXXVI by LXII, using only Roman numerals. In practice, the ancients used the abacus, counting-boards with pebbles, and other devices. (The word "calculate" comes from the Latin "calculus" meaning "pebble.") Despite its obvious superiority over previous methods, the Hindu-Arabic notation was not widely used in Europe until the 15th century, after a long struggle against the repressive forces of tradition and resistance to change. In fact the new notation was adopted by the enlightened merchants long before the "learned" authorities became reconciled to it. The importance of the Hindu-Arabic notation for the operation of our complicated modern industrial and commercial civilization is indicated by the fact that it made it possible for the general public, instead of experts only, to learn arithmetic.



The Hindu-Arabic notation is called a *decimal* notation because each digit in a number represents the coefficient of a power of 10. It is called a *positional*\* notation because the position of each digit tells us the power of 10 of which it is the coefficient. Thus, in school, the student learned to speak of the units column, the tens column, the hundreds column, etc.

**40. Addition and subtraction of polynomials.** In discussing the elementary operations of addition, subtraction, multiplication, and division of polynomials, *it will be understood that all polynomials will first be written in standard form.*† This is a convenient practice, although not absolutely essential.

*To add two polynomials, we write like terms under each other and then add by columns, as in the following example.*

*Example 1.* To add  $2x^2 + 5x + 8$  and  $3x^2 + 6x + 5$  we write

$$\begin{array}{r} 2x^2 + 5x + 8 \\ 3x^2 + 6x + 5 \\ \hline 5x^2 + 11x + 13. \end{array}$$

Hence the sum is  $5x^2 + 11x + 13$ .

*Justification of the procedure.* The sum  $(2x^2 + 5x + 8) + (3x^2 + 6x + 5)$  can be written as  $(2x^2 + 3x^2) + (5x + 6x) + (8 + 5)$  by using the associative and commutative laws for addition. This is accomplished in practice by writing like terms under each other in columns. Adding up by columns amounts to using the distributive law which enables us to write  $2x^2 + 3x^2 = (2 + 3)x^2 = 5x^2$ , and  $5x + 6x = (5 + 6)x = 11x$ . Thus the sum is  $5x^2 + 11x + 13$ .

*Comparison with adding integers.* To add 258 and 365, we write

$$\begin{array}{r} 258 (= 2 \cdot 10^2 + 5 \cdot 10 + 8) \\ 365 (= 3 \cdot 10^2 + 6 \cdot 10 + 5) \\ \hline 623 (= 5 \cdot 10^2 + 11 \cdot 10 + 13). \end{array}$$

The sum 623 is indeed the value of  $5x^2 + 11x + 13$  when  $x = 10$ , or  $5 \cdot 10^2 + 11 \cdot 10 + 13$ , but must be written 623 because we allow only coefficients from 0 to 9 inclusive. Therefore we use the trick of "carrying." That is, we write  $5 \cdot 10^2 + 11 \cdot 10 + 13 =$

\* For a more detailed discussion of this system and the possibility of using similar positional systems based on powers of numbers other than ten, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, Chapter VII.

† See section 26 for the definition of standard form of a polynomial.



$$5 \cdot 10^2 + (10 + 1)10 + (10 + 3) = 5 \cdot 10^2 + 1 \cdot 10^2 + 1 \cdot 10 + 1 \cdot 10 + 3 = 6 \cdot 10^2 + 2 \cdot 10 + 3.$$

To subtract a first polynomial from a second, we write the first under the second, placing like terms under each other, change the signs of the terms of the lower polynomial, and add.

*Example 2.* To subtract  $2x^2 - 3x + 5$  from  $5x^2 - 2x - 3$ , we write

$$\begin{array}{r} 5x^2 - 2x - 3 \\ - \quad + \quad - \\ \oplus 2x^2 \ominus 3x \oplus 5 \\ \hline 3x^2 + x - 8. \end{array}$$

*Justification of the procedure.* By the rules for removing parentheses (section 25),  $(5x^2 - 2x - 3) - (2x^2 - 3x + 5) = 5x^2 - 2x - 3 - 2x^2 + 3x - 5$ . The rest of the process is merely addition.

*Checking.* A convenient way to check is to substitute  $x = 10$ , say, in the polynomials involved in the process, and perform the corresponding arithmetical operations. Thus, to check example 1, we do what was done above in the paragraph marked "Comparison with adding integers." To check example 2, we similarly substitute  $x = 10$  and verify that

$$\begin{array}{rcl} 5 \cdot 10^2 - 2 \cdot 10 - 3 & = & 477 \\ -(2 \cdot 10^2 - 3 \cdot 10 + 5) & = & -175 \\ \hline 3 \cdot 10^2 + 1 \cdot 10 - 8 & = & 302. \end{array}$$

Such a check by substituting a particular value for  $x$  cannot, of course, prove that the relation among the polynomials involved is an identity, but it will usually catch any error. Any number may be substituted as a check, but it is advisable to avoid 0 and  $\pm 1$  because they are more likely to fail to reveal common errors, such as wrong exponents, since all powers of 0 are 0 and all powers of 1 are 1.

Subtraction of course may be checked by applying the definition of subtraction (section 5) and adding the result to the polynomial in the second line, obtaining the polynomial in the first line. Thus example 2 can be checked as follows:

$$\begin{array}{r} 3x^2 + x - 8 \\ 2x^2 - 3x + 5 \\ \hline 5x^2 - 2x - 3. \end{array}$$

The procedures for polynomials in more than one variable are similar.



# EXERCISES

Calculate, and check by substitution:

1. Add  $2x^2 - 3x + 5$  and  $3x^2 + x - 2$ .
2. Add  $6x^2 - 4x - 2$  and  $x^2 - 3x + 6$ .
3. Add  $4x^2 - 3x^2y + 6y^2$  and  $2x^2 + 2xy^2 - 2y^2$ .
4. Add  $2x^2 + 3xy - 2y^2$  and  $x^2 - 5xy + 5y^2$ .
5. Subtract  $x^2 - 2x + 4$  from  $3x^2 - 5x + 1$ .
6. Subtract  $5x^2 + 3x - 4$  from  $2x^2 - x + 2$ .
7. Subtract  $x^2 - 2xy + 3y^2$  from  $3x^2 - xy - 2y^2$ .
8. Subtract  $2x^2 + 3xy - 4y^2$  from  $x^2 + xy + y^2$ .
9. Subtract  $x^2 - 3x + 4$  from the sum of  $2x^2 + 5x - 3$  and  $3x^2 - 8x + 2$ .
10. Subtract the sum of  $2x^2 - 5x + 3$  and  $-3x^2 + 2x - 5$  from the sum of  $x^2 + 3x + 7$  and  $x^2 - 2x - 5$ .
11. Subtract the sum of  $x^3 + 4x^2y - 2y^3$  and  $2x^3 - 5x^2y + 5y^3$  from zero.
12. Subtract the sum of  $2x^3 - 3xy^2 + 4y^3$  and  $x^3 + 4xy^3 - 2y^3$  from zero.

**41. Multiplication of polynomials.** To multiply two monomials, we multiply the coefficients and combine the variable factors according to the rules for exponents (section 23).

*Example 1.* The product of  $2x^3$  and  $3x^2$  is  $6x^5$ .

*Justification of the procedure.* The product  $(2x^3)(3x^2)$  may be written as  $(2 \cdot 3)(x^3 \cdot x^2)$  by the associative and commutative laws for multiplication. The rest is obvious.

To multiply a first polynomial by a second polynomial, we multiply each term of the first polynomial by each term of the second and take the algebraic sum of these products. The work may be arranged as in the following example.

*Example 2.* To multiply  $2x^2 - 3x + 4$  by  $5x - 6$ , we write

$$\begin{array}{r}
 2x^2 - 3x + 4 \\
 \phantom{2x^2 - 3x + 4} 5x - 6 \\
 \hline
 = 12x^3 + 18x^2 - 24x \\
 10x^2 - 15x^2 + 20x \\
 \hline
 10x^3 - 27x^2 + 38x - 24
 \end{array}$$

*Justification of the procedure.* By the distributive law we may write



$$\begin{aligned}
 (2x^2 - 3x + 4)(5x - 6) \\
 &= (2x^2 - 3x + 4)5x + (2x^2 - 3x + 4)(-6) \\
 &= 10x^3 - 15x^2 + 20x - 12x^2 + 18x - 24.
 \end{aligned}$$

The rest is obvious.

*Comparison with multiplication of integers.* The comparison is clear from the following example in which the right side is obtained from the left by setting  $x = 10$ .

|                   |                            |                            |
|-------------------|----------------------------|----------------------------|
| <i>Example 3.</i> | $2x^2 + 5x + 3$            | $253$                      |
|                   | $4x + 2$                   | $42$                       |
|                   | <hr style="width: 100%;"/> | <hr style="width: 100%;"/> |
|                   | $4x^2 + 10x + 6$           | $506$                      |
|                   | $8x^3 + 20x^2 + 12x$       | $1012$                     |
|                   | <hr style="width: 100%;"/> | <hr style="width: 100%;"/> |
|                   | $8x^3 + 24x^2 + 22x + 6$   | $10626$                    |

*Checking.* To check the multiplication of polynomials in  $x$ , we may substitute  $x = 10$ , say, and perform the corresponding arithmetical process. We should obtain for the answer the value of the product polynomial when  $x = 10$ . Thus the right side of example 3 is a check for the left side.

The rules for polynomials in more than one variable are similar.

## EXERCISES

*Multiply, and check by substitution:*

- |                                              |                                     |                          |
|----------------------------------------------|-------------------------------------|--------------------------|
| 1. $2x(5y + 3z)$ .                           | 2. $3x(2y - 4w)$ .                  | 3. $-2y(3x - 5z)$ .      |
| 4. $-3z(-2x + 4y)$ .                         | 5. $2x(x^2 - 3x + 4)$ .             | 6. $3x(2x^2 + 2x - 4)$ . |
| 7. $3x(2x^2 + xy - 2y^2)$ .                  | 8. $2y(3x^2 - 2xy + 5y^2)$ .        |                          |
| 9. $-3x^2(5x^3 - 2y)$ .                      | 10. $5x^2y(2x^3 + 3y^2)$ .          |                          |
| 11. $(x - 2)(x - 5)$ .                       | 12. $(x - 3)(x - 2)$ .              |                          |
| 13. $(x - 3)(x + 4)$ .                       | 14. $(x + 2)(x + 4)$ .              |                          |
| 15. $(2x - 1)(3x - 2)$ .                     | 16. $(2x + 3)(4x - 5)$ .            |                          |
| ✓ 17. $(x - 3)(x^2 + x + 5)$ .               | 18. $(x + 2)(x^2 - 3x + 4)$ .       |                          |
| 19. $(2x - 3)(3x^2 - 2x + 4)$ .              | 20. $(3x - 2)(2x^2 + 5x - 3)$ .     |                          |
| 21. $(x + 2y)(x^2 - xy + y^2)$ .             | 22. $(2x - 3y)(x^2 - 2xy + 3y^2)$ . |                          |
| ✓ 23. $(x + y)(x^2 - xy + y^2)$ .            | 24. $(x - y)(x^2 + xy + y^2)$ .     |                          |
| 25. $(x + y)(x + y)$ .                       | 26. $(x + y)(x - y)$ .              |                          |
| 27. $(ax^o + b)(ax^o - b)$ .                 | 28. $(ax^o + b)(ax^o + b)$ .        |                          |
| ✓ 29. $(ax^o - b)(ax^o - b)$ .               | 30. $(x - y)(x - y)$ .              |                          |
| 31. $(2x^2 - 2x + 3)(3x^2 + x - 2)$ .        |                                     |                          |
| 32. $(3x^2 + 2xy - y^2)(2x^2 - xy + 3y^2)$ . |                                     |                          |

**42. Division by monomials.** To divide a quantity  $a$  by a quantity  $b$  means to find a quantity  $c$  such that  $bc = a$ . To divide



a monomial by a monomial, we divide their coefficients and treat the variable factors according to the laws for exponents.

*Example 1.*  $6x^5/3x^2 = 2x^3$ .

To check, we multiply  $3x^2 \cdot 2x^3 = 6x^5$ .

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial and take the algebraic sum of the results.

*Example 2.*  $(6x^5 + 15x^4 + 12x^3 + 9x^2) \div 3x^2 = 2x^3 + 5x^2 + 4x + 3$ .

To check, we multiply  $3x^2(2x^3 + 5x^2 + 4x + 3) = 6x^5 + 15x^4 + 12x^3 + 9x^2$ , or else check by substitution.

The justification of this process is obvious, since division has been defined as the inverse of multiplication. However, the quotient may not always be a polynomial or rational integral function, although it is necessarily a rational function; just as the quotient of two integers may fail to be an integer although it is always a rational number. For example,  $4x/2x^3 = 2/x^2$  and  $5 \div 3 = 5/3$ . Another concept of division for both integers and polynomials will be introduced in the next section.

## EXERCISES

*Divide and check:*

- |                                                   |                                                     |
|---------------------------------------------------|-----------------------------------------------------|
| 1. $10x^6/2x^2$ .                                 | 2. $18x^8/3x^4$ .                                   |
| 3. $16x^6y^{10} \div 8x^2y^2$ .                   | 4. $28x^3y^9 \div 7xy^3$ .                          |
| 5. $(21x^3y^2 + 18x^4y^5) \div (3xy^2)$ .         | 6. $(15x^{12} - 25x^6) \div 5x^3$ .                 |
| 7. $\frac{12x^8 - 8x^6 + 4x^2}{-4x^2}$ .          | 8. $\frac{4x^4 - 16x^3 + 8x^2 - 4x}{4x}$ .          |
| 9. $\frac{15x^3y^4 - 12x^2y^2 - 9xy^2}{-3xy^2}$ . | 10. $\frac{18x^3y^5 - 27x^2y^3 + 15xy^3}{-3xy^3}$ . |

**43. The division algorithm.\*** It is a mildly unpleasant fact that the word "division" is used in two different senses in connection with integers, and in the same two senses in connection with polynomials. In the sense already taken up, to divide an integer  $f$  by an integer  $d$  ( $\neq 0$ ) means to find a number  $q$  such that  $f = dq$ . In this sense the quotient  $q = f/d$  may not always exist in the system of integers but always does in the system of rational numbers. Thus  $7 \div 2 = 7/2$ . In this sense, division is the

\* The word "algorithm" means "a process of computation" and is derived from the name of al-Khowarizmi, an Arab of the 9th century, who wrote a book on mathematics.

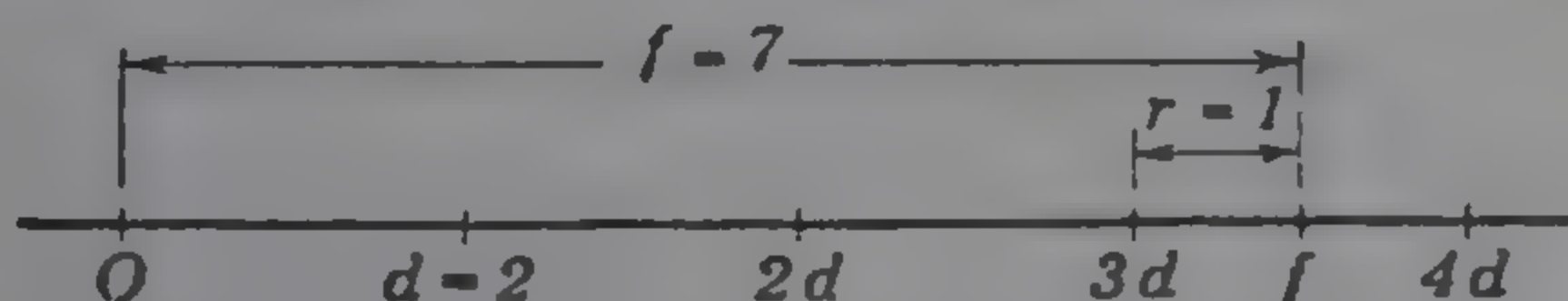


inverse of multiplication; thus  $2 \cdot \frac{7}{2} = 7$ . But there is another sense given to the word *division*, in which division of integers always can be performed within the system of integers, namely, *division with a remainder*. This may be defined as follows:

**DEFINITION 1.** To **divide** a positive integer  $f$ , the dividend, by a positive integer  $d$ , the divisor, ( $d \leq f$ ), means to find a positive integer  $q$ , the quotient, and a non-negative integer  $r$ , the remainder, such that (a)  $f = dq + r$ , and (b)  $0 \leq r < d$ .

*Example 1.* Dividing 7 by 2, we obtain the quotient 3 and the remainder 1, so that  $7 = 2 \cdot 3 + 1$  and  $1 < 2$ .

Note that also  $7 = 2 \cdot 2 + 3$ , but we would not say that in dividing 7 by 2 we get the quotient 2 and the remainder 3, because the remainder is required to be less than the divisor.



$$\begin{aligned} f &= dq + r \\ 7 &= 2 \cdot 3 + 1 \end{aligned}$$

Fig. 22.

This sense of “division” is also a reversal of multiplication. Multiplication by a positive integer is repeated addition, and this kind of division amounts to repeated subtraction as far as it can be done. Thus we can subtract three 2’s from 7 and have 1 left over. In general  $f = dq + r$  is equivalent to  $f - dq = r$  and we subtract as many  $d$ ’s ( $q$  of them) as we can until a remainder less than  $d$  is left.

It is intuitively clear from Fig. 22 that  $q$  and  $r$  always exist and are uniquely determined; this could be proved but the proof will be omitted.

The process of long division is really repeated subtraction, as can be seen from the following example.

$$\begin{aligned} \text{Example 2.} \quad & \begin{array}{rcl} 23 & = & q = 20 + 3 \\ d = 12 \overline{) 281} & = & f \\ 24(0) & = & 20 \cdot d \\ \hline 41 & = & f - 20 \cdot d \\ 36 & = & 3d \\ \hline 5 & = & r = f - 23d. \end{array} \end{aligned}$$



Hence  $281 = 12 \cdot 23 + 5$ . Thus when 281 is divided by 12, the quotient is 23 and the remainder is 5.

Note that  $f = dq + r$  may be written also as  $\frac{f}{d} = q + \frac{r}{d}$ . The expression  $q + \frac{r}{d}$  is the "quotient" in the earlier sense of division, since if we multiply it by  $d$  we get  $f$ . The former sense of division is sometimes called *exact division*. If  $r = 0$ , then  $d$  is a **factor** of  $f$ , or  $f$  is **exactly divisible** by  $d$ , and the two senses of division coincide in this case. Fortunately, confusion is unlikely to arise because of this double use of the same word.

As usual, polynomials are similar to integers. As the inverse of multiplication, *exact division* of a polynomial  $F(x)$  by a polynomial  $D(x)$  would mean finding a function  $Q(x)$  such that  $F(x) = D(x) \cdot Q(x)$ . But exact division of polynomials is not always possible within the system of polynomials, although it is within the system of rational functions, since  $Q(x) = F(x)/D(x)$  is always a rational function. Thus

$$(2x^3 + 5x^2 + 7x + 4) \div (x^2 + x + 1) = Q(x)$$

can be written as a rational function

$$\frac{2x^3 + 5x^2 + 7x + 4}{x^2 + x + 1}$$

but is not a polynomial, as will be seen below. As in the case of integers, we define division of polynomials within the system of polynomials as follows:

**DEFINITION 2.** To divide a polynomial  $F(x)$ , the dividend, by a polynomial  $D(x)$ , the divisor, means to find a polynomial  $Q(x)$ , the quotient, and a polynomial  $R(x)$ , the remainder, such that (a)  $F(x) = D(x) \cdot Q(x) + R(x)$ , and (b) the degree of  $R(x)$  is less than the degree of  $D(x)$ .

This is also repeated subtraction, since (a) is equivalent to  $F(x) - D(x) \cdot Q(x) = R(x)$  and we subtract as "great" a multiple of  $D(x)$  from  $F(x)$  as we can, so that the remainder has lower degree than  $D(x)$ . The work may be arranged similarly to long division of integers, as in the following example.

**Example 3.** To divide  $2x^3 + 5x^2 + 7x + 4$  by  $x^2 + x + 1$ , we write



$$\begin{array}{rcl}
& 2x + 3 & = Q(x) \\
D(x) = x^2 + x + 1 \overline{) 2x^3 + 5x^2 + 7x + 4} & = F(x) \\
& \underline{2x^3 + 2x^2 + 2x} & = 2x \cdot D(x) \\
& 3x^2 + 5x + 4 & = F(x) - 2x \cdot D(x) \\
& \underline{3x^2 + 3x + 3} & = 3 \cdot D(x) \\
& 2x + 1 & = R(x) = F(x) \\
& & - (2x + 3) \cdot D(x).
\end{array}$$

To check division, we may apply definition 2 and verify that  $D(x) \cdot Q(x) + R(x) = F(x)$ . Thus in example 2 we find by multiplication and addition that

$$(x^2 + x + 1)(2x + 3) + (2x + 1) = 2x^3 + 5x^2 + 7x + 4.$$

Or we may check by letting  $x = 10$ , say, and performing the corresponding arithmetical division. Thus, in example 3 we get

$$\begin{array}{r}
23 = q = Q(10) \\
D(10) = d = 111 \overline{) 2574} = f = F(10) \\
\begin{array}{r}
222 \\
354 \\
333 \\
\hline
21
\end{array} \\
21 = r = R(10).
\end{array}$$

Note that in example 3 we do not say after the first subtraction that the quotient is  $2x$  and the remainder  $3x^2 + 5x + 4$ , because the remainder is required to have lower degree than the divisor so that further subtraction is impossible. As for integers, we may write  $F(x) = D(x) \cdot Q(x) + R(x)$  as

$$\frac{F(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}.$$

The function  $Q(x) + \frac{R(x)}{D(x)}$  would yield  $F(x)$  if multiplied by  $D(x)$ .

If  $R(x) = 0$ , then  $D(x)$  is a **factor** of  $F(x)$  or  $F(x)$  is **exactly divisible** by  $D(x)$ , and the two senses of division coincide in this case.

It can be proved that the quotient polynomial  $Q(x)$  and remainder polynomial  $R(x)$  always exist and are uniquely determined as long as the divisor is not zero. The process of determining them, that is, the division process, is called the **division algorithm**. Recall that it is intended that all polynomials are to be written in standard form before beginning the division process. The procedure for polynomials in more than 1 variable is similar.



## EXERCISES

*Divide and check.*

1.  $(x^2 - 7x + 10) \div (x - 2)$ .
2.  $(x^2 - x - 12) \div (x + 3)$ .
3.  $(2x^2 + 4 - 9x) \div (x - 4)$ .
4.  $(3x^2 + 7x - 6) \div (3 + x)$ .
5.  $(2x^3 - 3x^2 + 5x - 2) \div (2x - 1)$ .
6.  $(6x^3 + 11x^2 - x + 2) \div (x + 2)$ .
7.  $(x^3 - 11x + 6) \div (x - 3)$ .
8.  $(x^3 + 5x^2 - 12) \div (x + 2)$ .
9.  $(3x^3 - 6x - x^2 + 2) \div (x^2 - 2)$ .
10.  $(6x^3 - 4x^2 + 9x - 6) \div (2x^2 + 3)$ .
11.  $(2x^4 + 2x^3 + 4x - 8) \div (x^2 + x - 2)$ .
12.  $(6x^4 - 3x^3 + 7x^2 - 2x + 2) \div (2x^2 - x + 1)$ .
13.  $(x^3 + 8y^3) \div (x + 2y)$ .
14.  $(x^3 - 27y^3) \div (x - 3y)$ .
15.  $(2x^3 - 7x^2y + 12xy^2 - 9y^3) \div (2x - 3y)$ .
16.  $(x^3 + x^2y - xy^2 + 2y^3) \div (x + 2y)$ .
17.  $(x^2 - 5x + 10) \div (x - 2)$ .
18.  $(x^2 + x - 6) \div (x - 3)$ .
19.  $(2x^3 + x^2 - 5x - 3) \div (x + 2)$ .
20.  $(x^3 + 6x^2 + 7x - 3) \div (x + 3)$ .
21.  $(2x^3 - 7x^2 - 16x - 1) \div (x - 5)$ .
22.  $(x^3 + 2x^2 + 3x + 4) \div (x + 5)$ .
23.  $(x^4 - 5x^2 + 2x + 5) \div (x^2 - 2)$ .
24.  $(x^4 + x^2 + 3x + 1) \div (x^2 + 3)$ .
25.  $(2x^3 - 3x^2 + 8x + 3) \div (x^2 - x + 2)$ .
26.  $(x^3 - 6x + 5) \div (x^2 + 3x - 2)$ .
27.  $(x^4 - x^2 - 10) \div (x^2 - 4)$ .
28.  $(x^4 + 2x^2 - 12) \div (x^2 - 3)$ .
29. Show that  $(x + y)$  is a factor of  $(x^3 + y^3)$  and find another factor.
30. Show that  $(x + y)$  is a factor of  $(x^5 + y^5)$  and find another factor.
31. Show that  $(x - y)$  is a factor of  $(x^3 - y^3)$  and find another factor.
32. Show that  $(x - y)$  is a factor of  $(x^5 - y^5)$  and find another factor.
33. Show that  $(x + y)$  is a factor of  $(x^4 - y^4)$  and find another factor.
34. Show that  $(x + y)$  is a factor of  $(x^6 - y^6)$  and find another factor.
35. Show that  $(x^2 + 2x + 4)$  is a factor of  $(x^4 + 2x^3 - 8x - 16)$  and find another factor.
36. Show that  $x^2 + x + 1$  is a factor of  $(x^4 - x^3 - x^2 - 5x - 3)$  and find another factor.



## CHAPTER VI

### Factoring of Polynomials

**44. Prime factors.** A polynomial  $D(x)$  is said to be a factor of a polynomial  $F(x)$  if there exists a polynomial  $Q(x)$  such that  $F(x) = D(x) \cdot Q(x)$ ; that is, if  $F(x)$  is exactly divisible by  $D(x)$ ; or if the remainder is zero when  $F(x)$  is divided by  $D(x)$ .

Thus, if two polynomials are multiplied together, either one is said to be a factor of the product.

To factor a polynomial means to express it as a product of other polynomials.

For example,  $x^2 - 1 = (x + 1)(x - 1)$ , as we can verify by multiplication. But although  $x - 1 = (\sqrt{x} + 1)(\sqrt{x} - 1)$ , this is not considered factoring, since the quantities in parentheses are not polynomials, that is, not rational integral functions.

In multiplication, we are given two factors and asked for the product. In (exact) division, we are given the product and one factor and asked for the other factor. In both situations we have a routine procedure for accomplishing our task. In factoring, we are given the product and asked for the factors from which it was compounded. Factoring, or unmultiplying, is consequently more difficult than the other two tasks, just as unscrambling eggs is more difficult than scrambling them, and chemical analysis of a compound is harder than compounding a substance from its components. In fact, we shall have to lean on our experience with multiplication in order to recognize factors, and shall develop no systematic procedure for factoring (except for one theorem to be taken up in Chapter XV).

In fact, the question whether a polynomial has factors other than plus and minus itself and  $\pm 1$  is not a simple question and depends on what kind of coefficients are to be permitted in the factors. Thus by multiplication we can verify that  $(x + \sqrt{-1})(x - \sqrt{-1}) = x^2 + 1$ . Hence  $x^2 + 1$  has these linear factors if imaginary numbers are permitted in the factors. Similarly  $x^2 - 2$  has the factors  $(x + \sqrt{2})(x - \sqrt{2})$  if irrational



numbers are permitted in the factors. Finally  $x + 1$  has the factors  $2(\frac{1}{2}x + \frac{1}{2})$ , etc., if fractions are allowed in the factors. But neither  $x^2 + 1$ ,  $x^2 - 2$ , nor  $x + 1$  has any factors at all except plus and minus itself and  $\pm 1$  if we insist on integral coefficients in the factors.

In the present chapter,\* we shall usually consider only polynomials with integral coefficients and shall require factors to be polynomials with integral coefficients only.

A polynomial may be called **prime** if it has no factors other than plus and minus itself and  $\pm 1$ . To factor completely a polynomial means to express it as a product of its prime factors. Under the circumstances mentioned these are uniquely determined just as for integers (compare section 6), except for trivial changes of sign.†

We shall not have any systematic procedure for determining whether or not a polynomial is prime. It is not hard to see that  $x^2 + 1$ ,  $x^2 - 2$ ,  $x + 1$ ,  $x^2 + y^2$ ,  $x + y$ ,  $x^2 + xy + y^2$ ,  $x^2 - xy + y^2$ , etc., are prime. If we permitted coefficients other than integers, our concept of prime polynomial would be altered, but our procedures would not.

*Factoring always should be checked by multiplication.*

**45. Special products.** Since factoring will be done largely from experience with multiplication, it is advisable to become familiar with the following types of products which are of frequent occurrence. Each can be verified by multiplication.

$$(1) \ a(u + v + w + \dots) = au + av + aw + \dots$$

$$(2) \ (a + b)(a - b) = a^2 - b^2.$$

$$(3) \ (a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2.$$

$$(4) \ (a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2.$$

$$(5) \ (x + b)(x + d) = x^2 + (b + d)x + bd.$$

$$(6) \ (ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

It will be found convenient to memorize formulas (2), (3), and (4). In the case of (5) and (6) one can think mentally of the usual form of the work of multiplication. Thus in the product

$$\begin{array}{r} ax + b \\ \times \quad cx + d \\ \hline \end{array}$$

\* In Chapter XV we shall permit any complex coefficients.  
 † Thus  $2 + 2 = (-1)(-1) + (-2)(-2)$ .



it is seen that the term  $(ad + bc)x$  may be called the sum of the "cross-products." Formula (5) is merely the special case of (6) in which  $a = c = 1$ .

*Note.* It is important to realize that in formulas such as those listed above, the letters  $a$ ,  $b$ , etc., unlike your instructor, will stand for anything (more precisely, for any quantity). For example,

$$(2x + 3y)(2x - 3y) = 4x^2 - 9y^2$$

is of the type (2) where  $a$  stands for  $2x$  and  $b$  for  $3y$ . Much of the superiority of algebra over arithmetic is due to the generality which is due to the fact that letters often may be used to represent any quantity whatever.

## EXERCISES

Multiply each of the following by inspection:

- |                                                |                                      |                         |
|------------------------------------------------|--------------------------------------|-------------------------|
| 1. $3a(2x + 5y)$ .                             | 2. $-2a(x - 3y)$ .                   |                         |
| 3. $-2a^2b(3a^3 - ab)$ .                       | 4. $2b(x - 3y + 2z)$ .               |                         |
| 5. $(x + 3y)(x - 3y)$ .                        | 6. $(2x + 5y)(2x - 5y)$ .            |                         |
| 7. $(1 - 3x^2)(1 + 3x^2)$ .                    | 8. $(3 - 5xy)(3 + 5xy)$ .            |                         |
| 9. $(3x + 2)^2$ .                              | 10. $(2x + 5y)^2$ .                  | 11. $(5x - 3y)^2$ .     |
| 12. $(2ax + b)^2$ .                            | 13. $(3cx - ay)^2$ .                 | 14. $(x^2 - 3y^2)^2$ .  |
| 15. $(4x^2 - 3y^2)^2$ .                        | 16. $(5x^3 + y)^2$ .                 | 17. $(x + 2)(x + 5)$ .  |
| 18. $(x + 2)(x + 6)$ .                         | 19. $(x - 2)(x + 7)$ .               | 20. $(x + 2)(x - 7)$ .  |
| 21. $(u - 3)(u - 4)$ .                         | 22. $(y - 5)(y - 3)$ .               | 23. $(2x - 1)(x + 4)$ . |
| 24. $(3x - 2)(4x + 5)$ .                       | 25. $(6x - y)(2x + 3y)$ .            |                         |
| 26. $(x + 2y)(3x - 2y)$ .                      | 27. $(2x + 5y)(3x - y)$ .            |                         |
| 28. $(4x - 3)(2x + 5)$ .                       | 29. $(2x^2 - 5)(3x^2 + 2)$ .         |                         |
| 30. $(x^2 - 3y)(5x^2 + 2y)$ .                  | 31. $(4x^2 - 3y^2)(x^2 + 2y^2)$ .    |                         |
| 32. $(2x^2 - 3y^2)(4x^2 + 5y^2)$ .             | 33. $[(x + y) - 2z][(x + y) + 2z]$ . |                         |
| 34. $[(x + 2y) - 3][(x + 2y) + 3]$ .           | 35. $[(x + 3) + 2y]^2$ .             |                         |
| 36. $[(2x + 1) - 3y]^2$ .                      | 37. $(x + y + z)(x + y - z)$ .       |                         |
| 38. $(x + 2y + 3z)(x + 2y - 3z)$ .             | 39. $(x - y - 3)^2$ .                |                         |
| 40. $(2x - 3y - 4)^2$ .                        |                                      |                         |
| 41. $[(x + y) + (u - v)][(x + y) - (u - v)]$ . |                                      |                         |
| 42. $[(x - y) - (u + v)][(x - y) + (u + v)]$ . |                                      |                         |
| 43. $(x^a + y^b)^2$ .                          | 44. $(x^a - y^b)^2$ .                |                         |
| 45. $(x^a + y^b)(x^a - y^b)$ .                 | 46. $(ax^b + cy^d)^2$ .              |                         |

46. Simple types of factoring. The examples of factoring taken up in this section are done merely by recognizing the type of product which fits the given expression. To check the factor-



ing, we multiply out the factors and see if we get back the original expression.

*Case 1. Monomial common factor.* Since  $a(b - c + d - e) = ab - ac + ad - ae$ , we may take out the greatest possible common factor from an algebraic sum of terms.

*Example 1.*  $2x^3y^2 - 6x^2y^3 + 4x^2y^2 = 2x^2y^2(x - 3y + 2)$ .

*Case 2. Difference of two squares.* Since  $(a + b)(a - b) = a^2 - b^2$ , the difference of the squares of two quantities may be expressed as the product of the sum of the two quantities by their difference.

*Example 2.*  $4x^2 - 9y^2 = (2x)^2 - (3y)^2 = (2x + 3y)(2x - 3y)$ .

*Case 3. Trinomial perfect squares.* Since  $(a + b)^2 = a^2 + 2ab + b^2$  and  $(a - b)^2 = a^2 - 2ab + b^2$ , we may factor these right members by setting them equal to the left members.

*Example 3.*  $4x^2 + 12x + 9 = (2x)^2 + 2(2x) \cdot 3 + 3^2 = (2x + 3)^2$ .

*Example 4.*  $9x^2 - 30xy + 25y^2 = (3x)^2 - 2 \cdot (3x)(5y) + (5y)^2 = (3x - 5y)^2$ .

*Case 4. Sum and difference of two cubes.* By multiplication we find that  $(a + b)(a^2 - ab + b^2) = a^3 + b^3$  and  $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ . Hence we may factor the right members by setting them equal to the left members.

*Example 5.*  $8x^3 + 27y^3 = (2x)^3 + (3y)^3 = (2x + 3y)(4x^2 - 6xy + 9y^2)$ .

*Example 6.*  $8x^3 - 27y^3 = (2x)^3 - (3y)^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$ .

*Case 5. Combination of different types.* It is advisable to take out the greatest possible common factor first, and then apply whatever types of factoring seem appropriate.

*Example 7.*

$$\begin{aligned} 2x^4y^2 - 2x^3y^2 &= 2x^3y^2(x^4 - 1) = 2x^3y^2(x^2 + 1)(x^2 - 1) \\ &= 2x^3y^2(x^2 + 1)(x + 1)(x - 1). \end{aligned}$$

It is important to note that the letters  $a$ ,  $b$ , etc., in the typical formulas above may stand for any quantities. See the note of section 45.

## EXERCISES

Factor each of the following completely, and check:

1.  $2ax - 6a^2$ .

2.  $6x^2y + 10xy^2$ .

3.  $4x^3y - 6xy^2 + 8xy^3$ .

4.  $10x^3y^2 - 14x^2y^2 - 6x^2y^3$ .



5.  $16x^2 - 9y^2$ .
6.  $25x^2 - 4$ .
7.  $9 - 64x^2y^2$ .
8.  $1 - 36a^2x^2$ .
9.  $144u^2 - 25v^2$ .
10.  $49m^2 - 121n^2$ .
11.  $4x^2 + 12xy + 9y^2$ .
12.  $25x^2 + 20x + 4$ .
13.  $16x^2 - 24xy + 9y^2$ .
14.  $9x^2 - 30x + 25$ .
15.  $64x^2y^2 - 16xy + 1$ .
16.  $1 - 6x + 9x^2$ .
17.  $4a^2x^2 + 4abx + b^2$ .
18.  $a^2x^2 + 2abxy + b^2y^2$ .
19.  $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}$ .
20.  $121x^2 - 88x + 16$ .
21.  $9x^3 - 4xy^2$ .
22.  $8x^2 - 50y^2$ .
23.  $32x^4 - 18x^2y^2$ .
24.  $144x^2y^2 - 64z^2$ .
25.  $x^4 - 16$ .
26.  $16x^4 - 81y^4$ .
27.  $27x^3 + 8y^3$ .
28.  $8p^3q^3 + 1$ .
29.  $64x^3 - y^3$ .
30.  $1 - 27y^3$ .
31.  $x^6 - 64y^6$ .
32.  $729x^6 - y^6$ .
33.  $2a(x + y) + b(x + y)$ .
34.  $3(x - 2y) - 5a(x - 2y)$ .
35.  $(x + y)^2 - 9$ .
36.  $(2x - 3y)^2 - 16z^2$ .
37.  $25x^2 - (y - z)^2$ .
38.  $4x^2 - (2y - 3z)^2$ .
39.  $(x - 2y)^2 - 6(x - 2y) + 9$ .
40.  $(2x - y)^2 + 4(2x - y) + 4$ .
41.  $(x - 2y)^2 - (3x + y)^2$ .
42.  $(2x - 3y)^2 - (x - y)^2$ .
43.  $2x^2y^4 - 16x^5y$ .
44.  $192x^6y - 3x^3y^4$ .
45.  $(x^2 + 6x + 9) - (y - 1)^2$ .
46.  $(4x^2 - 4x + 1) - (3y + 2)^2$ .
47.  $(x + 2y)^3 + 8y^3$ .
48.  $(2x - y)^3 - 8y^3$ .
49.  $(x + 2)(x - 3y)^2 - (x + 2)(2x - y)^2$ .
50.  $(2x + y)(2x - y)^2 - (2x + y)(x - 3y)^2$ .
51.  $5x^5y - 80xy^5$ .
52.  $2x^3y^2 - 12x^2y^2 + 18xy^2$ .
53.  $3x^5y + 24x^2y^4$ .
54.  $2x^5y^2 - 162xy^2$ .

**47. Factoring by grouping.** Sometimes various terms may be grouped together in such a way that one of the preceding types then applies.

*Example 1.*  $ax + bx - ay - by = (a + b)x - (a + b)y$ . Now  $(a + b)$  is a common factor, so that the original expression may be factored into  $(a + b)(x - y)$ .

*Example 2.*  $x^2 + 4x + 4 - y^2 + 6yz - 9z^2 = (x^2 + 4x + 4) - (y^2 - 6yz + 9z^2) = (x + 2)^2 - (y - 3z)^2 = [(x + 2) + (y - 3z)][(x + 2) - (y - 3z)] = [x + 2 + y - 3z][x + 2 - y + 3z]$ .

It is sometimes possible to add and subtract the same quantity so that the preceding method may then be applied.

*Example 3.* To factor  $x^4 + x^2y^2 + y^4$  we add and subtract



$x^2y^2$ , obtaining  $x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2 + y^2)^2 - x^2y^2 = (x^2 + y^2 + xy)(x^2 + y^2 - xy)$ .

## EXERCISES

Factor each of the following, and check:

1.  $ax + 3x - ay - 3y$ .
2.  $2cx + 2cy + dx + dy$ .
3.  $x^2 - 3x - xy + 3y$ .
4.  $x^3 + 3x^2 - 2xy - 6y$ .
5.  $2x^3 + 3x^2 - 4x - 6$ .
6.  $x^3 + 4x^2 + 3x + 12$ .
7.  $x^2 - 4y^2 + x - 2y$ .
8.  $x^2 - y^2 - 3x + 3y$ .
9.  $x^2 + 6x + 9 - y^2$ .
10.  $3x^2 - 3y^2 + x + y$ .
11.  $3x + y - 6x^2 - 2xy$ .
12.  $3x^3 + 3x^2 - x - 1$ .
13.  $x^2 + 4xy + 4y^2 - u^2 - 2u - 1$ .
14.  $x^3 - x^2y + 3xy^2 - 3y^3$ .
15.  $x^4 - 3x^2y^2 + y^4$ .
16.  $x^4 - 7x^2y^2 + y^4$ .
17.  $x^4 - 6x^2y^2 + y^4$ .
18.  $x^4 + x^2 + 1$ .
19.  $x^4 + 2x^2 + 9$ .
20.  $x^4 + 3x^2 + 4$ .
21.  $x^4 + 4$ .
22.  $x^4 + 64$ .
23.  $x^4 - 7x^2y^2 + 9y^4$ .
24.  $64x^4 + y^4$ .
25.  $a(x - y) + b(y - x)$ .
26.  $3a(2x - y) - 5(y - 2x)$ .

48. Quadratic trinomials. If a quadratic trinomial is not prime, it may be factored by trial, as follows.

*Example 1.* To factor  $x^2 - 5x + 6$  we realize that, if it can be factored, it must have factors of the form  $(x + b)(x + d)$ . Since  $(x + b)(x + d) = x^2 + (b + d)x + bd$ , we must find integers  $b$  and  $d$  whose product is 6 and whose sum is  $-5$ . Trying various pairs of factors of 6, we arrive at  $(x - 2)(x - 3)$ .

*Example 2.* To factor  $4x^2 - 11x + 6$  we realize that, if it can be factored, it must have factors of the form  $(ax + b)(cx + d)$ . Since  $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$ , we must find integral values of  $a$ ,  $b$ ,  $c$ , and  $d$  such that  $ac = 4$ ,  $bd = 6$ , and  $ad + bc = -11$ . After trying various possibilities, we find the factors  $(4x - 3)(x - 2)$ .

It is important to check by multiplying out the factors, paying careful attention to the sum of the cross-products.

*Note.* Examples of this type may be done also by the method of grouping. Thus,  $x^2 - 5x + 6$  may be factored by writing  $x^2 - 5x + 6 = x^2 - 3x - 2x + 6 = x(x - 3) - 2(x - 3) = (x - 2)(x - 3)$ . Similarly, example 2 can be done by observing that  $4x^2 - 11x + 6 = 4x^2 - 3x - 8x + 6 = (4x - 3)x - (4x - 3)2 = (4x - 3)(x - 2)$ .



## EXERCISES

Factor each of the following, and check:

1.  $x^2 - 7x + 12$ .
2.  $x^2 + 7x + 6$ .
3.  $x^2 - x - 12$ .
4.  $x^2 - 4x - 12$ .
5.  $x^2 + 7x - 18$ .
6.  $x^2 - 17x - 18$ .
7.  $x^2 + 9x - 36$ .
8.  $k^2 - 16k - 36$ .
9.  $m^2 - 10m - 24$ .
10.  $p^2 + 14p + 24$ .
11.  $x^2 - 22xy - 48y^2$ .
12.  $x^2 + 2xy - 48y^2$ .
13.  $x^2 - 8xy - 48y^2$ .
14.  $x^2 - 16y + 48y^2$ .
15.  $6x^2 + 7x - 5$ .
16.  $4x^2 + 8x - 5$ .
17.  $6x^2 - 19x + 3$ .
18.  $6x^2 + 5x - 4$ .
19.  $8x^2 - 22x + 15$ .
20.  $2x^2 + 7x + 3$ .
21.  $21x^2 + 5xy - 4y^2$ .
22.  $3x^2 - 19xy + 20y^2$ .
23.  $12x^2 - 8xy - 15y^2$ .
24.  $x^2 - 3xy - 28y^2$ .
25.  $(x + 2)^2 + 5(x + 2) + 6$ .
26.  $(x + 3)^2 - 7(x + 3) + 12$ .
27.  $(x + y)^2 + 7(x + y) - 18$ .
28.  $(x + y)^2 + 9(x + y) - 36$ .
29.  $(2x - y)^2 - 10(2x - y) - 24$ .
30.  $(3x + 4)^2 - 8(3x + 4) - 48$ .
31.  $(x^2 - 6x + 9) - 5(x - 3) + 6$ .
32.  $(x^2 - 10xy + 25y^2) - 7(x - 5y) + 12$ .
33.  $12x^3y^2 + 2x^2y^3 - 4xy^4$ .
34.  $2x^3 + 4x^2 - 30x$ .
35.  $2x^4y - 10x^3y^2 - 12x^2y^3$ .
36.  $6x^3y - 27x^2y^2 + 12xy^3$ .
37.  $x^4 + 7x^2 + 12$ .
38.  $x^4 - 8x^2 + 12$ .
39.  $x^6 + 7x^3 - 8$ .
40.  $x^6 - 3x^3 - 40$ .
41.  $x^4 - 8x^2y^2 - 9y^4$ .
42.  $4x^4 - 6x^2y^2 - 18y^4$ .

\*49. Sum and difference of two like powers. The following theorems can be proved by methods taken up later; see exercise 21, section 101, and exercise 19, section 142.

THEOREM A. If  $n$  is any positive integer,  $x^n - y^n$  has the factor  $x - y$ .

THEOREM B. If  $n$  is an even positive integer,  $x^n - y^n$  has the factor  $x + y$ .

THEOREM C. If  $n$  is an odd positive integer,  $x^n + y^n$  has the factor  $x + y$ .

THEOREM D. If  $n$  is an even positive integer, neither  $x + y$  nor  $x - y$  is a factor of  $x^n + y^n$ .

Another factor, in each case, may be found by division.

\* This section may be omitted without disturbing the continuity of the chapter.



*Example 1.*  $x^5 + 32y^5$  has the factor  $x + 2y$  by theorem C. Dividing  $x^5 + 32y^5$  by  $x + 2y$ , we find that  $(x^5 + 32y^5) = (x + 2y)(x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4)$ .

*Example 2.*  $64x^6 - y^6$  has the factor  $2x + y$ , by theorem B, and the factor  $2x - y$  by theorem A. Hence, dividing by  $(2x + y)(2x - y) = 4x^2 - y^2$  we obtain the factors  $(2x + y)(2x - y)(16x^4 + 4x^2y^2 + y^4)$ . The last factor can be factored further as follows:  $16x^4 + 4x^2y^2 + y^4 = 16x^4 + 8x^2y^2 + y^4 - 4x^2y^2 = (4x^2 + y^2)^2 - 4x^2y^2 = (4x^2 + y^2 - 2xy)(4x^2 + y^2 + 2xy)$ . Hence  $64x^6 - y^6 = (2x + y)(2x - y)(4x^2 + y^2 - 2xy)(4x^2 + y^2 + 2xy)$ .

This example could be done also by writing  $64x^6 - y^6 = (8x^3 + y^3)(8x^3 - y^3)$  and factoring each of these factors by theorems A and C.

Some expressions of the form  $x^n + y^n$  with even  $n$  can be factored, as in the following examples.

*Example 3.*  $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$ .

*Example 4.*  $x^6 + 4y^6 = x^6 + 4x^2y^2 + 4y^6 - 4x^2y^2 = (x^2 + 2y^2)^3 - 4x^2y^2 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$ .

## EXERCISES

Factor each of the following, and check:

- |                   |                    |                  |
|-------------------|--------------------|------------------|
| 1. $x^3 + y^3$    | 2. $x^3 - y^3$     | 3. $x^5 - 32y^5$ |
| 4. $32x^6 + y^6$  | 5. $16x^4 - y^4$   | 6. $x^5 + 32$    |
| 7. $x^6y^4 + 243$ | 8. $x^3 - 128$     | 9. $x^6 + 64$    |
| 10. $x^6 + 64y^6$ | 11. $x^6 - 64y^6$  | 12. $x^7 + y^7$  |
| 13. $x^4 - y^4$   | 14. $x^4 - 256y^4$ |                  |

**50. H.C.F. and L.C.M.** The highest common factor (H.C.F.) of several polynomials (with integral coefficients) is the polynomial of highest degree, with the largest coefficients in absolute value, which is a factor of all the given polynomials.

To find the H.C.F. of several polynomials, we may first express each of them as a product of its prime factors. Then the H.C.F. is the product of all the different prime factors common to all the given polynomials, each taken the least number of times it occurs in any of the given polynomials. See section 6.

*Example 1.* To find the H.C.F. of  $6x^3y^2 - 6xy^4$  and  $10x^3y + 20x^2y^2 + 10xy^3$  we write  $6x^3y^2 - 6xy^4 = 6xy^2(x^2 - y^2) = 6xy(x + y)(x - y)$  and  $10x^3y + 20x^2y^2 + 10xy^3 = 10xy(x^2 + 2xy + y^2) = 10xy(x + y)^2$ . Hence the H.C.F. is  $2xy(x + y)$ .



The lowest common multiple (L.C.M.) of several polynomials (with integral coefficients) is the polynomial of lowest degree, with the smallest coefficients in absolute value, of which each of the given polynomials is a factor.

To find the L.C.M. of several polynomials we may first express each of them as a product of its prime factors. Then the L.C.M. is the product of all the different prime factors occurring in the given polynomials, each taken the greatest number of times it occurs in any of the given polynomials.

*Example 2.* The L.C.M. of the two polynomials in example 1 is  $30xy^2(x+y)^2(x-y)$ .

The H.C.F. and L.C.M. are unique except for a factor  $\pm 1$ . The definitions and procedures here described are analogous to those for integers discussed in section 6. As proved there for integers, we may prove the following theorem:

*If  $L$  is the L.C.M. of two polynomials  $A$  and  $B$  and if  $H$  is their H.C.F., then  $LH = \pm AB$ , or  $L = \pm AB/H$ . This may be used as a check; thus in examples 1 and 2 above we find that  $LH = AB = 60x^2y^3(x+y)^3(x-y)$ .*

## EXERCISES

*Find the H.C.F. and the L.C.M. of each of the following sets of polynomials, leaving answers in factored form:*

1.  $6x^2y, 4xy^2$ .
2.  $6x^3y^2, 3xy^3$ .
3.  $9xy^3, 6x^2y^2, 2x^3y$ .
4.  $8x^4y^3, 12x^2y^3, 4x^3y^2$ .
5.  $2x - 6, 3x - 9$ .
6.  $4x + 8, 5x + 10$ .
7.  $2x - 4, x^2 - 4$ .
8.  $4x - 12, 2x^2 - 18$ .
9.  $x^2 - y^2, x^2 + 2xy + y^2$ .
10.  $x^2 - 9, x^2 - 6x + 9$ .
11.  $x^2 - 4, x^2 - 5x + 6$ .
12.  $x^2 - x - 2, x^2 + x - 6$ .
13.  $x^2 - 9, x^3 - 27, x^2 - x - 6$ .
14.  $x^3 + 8, x^2 - 4, x^2 - 4x + 4$ .
15.  $2x^2 - 12x + 18, 4x^2 - 36, 2x^2 + 4x - 30$ .
16.  $6x^2 - 4x, 9x^2 - 4, 9x^2 - 12x + 4$ .
17.  $8x^3 - 2xy^2, 12x^3 + 2x^2y - 4xy^2, 6x^2 + 4xy$ .
18.  $3x^2 - 6xy, x^2 - 4y^2, 2x^2 - 8xy + 8y^2$ .
19.  $3x - 6, 4 - x^2, x^2 - 4x + 4$ .
20.  $6 - 2x, x^2 - 9, x^2 - x - 6$ .
21.  $x^3 - y^3, x^2 - y^2, x^2 - 2xy + y^2$ .
22.  $x^3 + y^3, x^2 - y^2, x^2 - xy - 2y^2$ .
23.  $x^4 - y^4, x^2 + y^2, x^2 + xy - 2y^2$ .
24.  $x^4 - y^4, x^2 - y^2, x^2 + 2xy - 3y^2$ .
25.  $9 - 4x^2, 2x^2 + x - 6, 6x^2 - 9x$ .
26.  $8x^3 - 27y^3, 4x^2 - 9y^2, 6xy - 4x^2$ .



## CHAPTER VII

### Elementary Operations with Fractional Expressions

Since a fractional expression or (fractional) rational function becomes a number for any allowable value of the variables, all definitions concerning and operations with fractional expressions are similar to those for numerical fractions. Therefore the student should review carefully sections 7 to 11 inclusive before proceeding with this chapter. Fractional expressions are also referred to as *algebraic fractions* or simply as *fractions*.

**51. Reduction to lowest terms.** In Chapter I it was pointed out that a *non-zero common factor* may be either introduced into or cancelled from both numerator and denominator of a fraction; in symbols,

$$1) \quad \frac{ac}{bc} = \frac{a}{b}, \text{ provided } c \neq 0.$$

The same statement therefore may be made for fractional expressions. When all common factors except  $\pm 1$  have been cancelled from numerator and denominator of a fractional expression, it is said to be reduced to lowest terms or expressed in simplest form. Hence to reduce a fractional expression to lowest terms, express numerator and denominator as products of their prime factors and cancel common factors.

**Example 1.**  $\frac{2x^3y^2z}{6x^2yz^3} = \frac{xy}{3z^2}$

**Example 2.**  $\frac{x^2 - 7x + 12}{x^2 - 5x + 6} = \frac{(x-4)(x-3)}{(x-2)(x-3)} = \frac{x-4}{x-2}$

Note that the expression obtained by reducing to lowest terms is not always quite the same function as that with which we start. Thus in example 1, the reduced fractional expression is defined, and has the value 0 when  $x = y = 0, z \neq 0$ , while the original expression is undefined for such values of the variable. Similarly in example 2, the reduced fractional expression is defined and has the value  $-1$  for  $x = 3$ , while the original expression is undefined for  $x = 3$ . This is, however, a subtle difference, which usually may be overlooked without fear by the beginning student.



Recall that  $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$  while  $\frac{-a}{-b} = \frac{a}{b}$  (see (4), section 10). It follows that, if we think of three signs attached to every fraction, one before the numerator, one before the denominator, and one before the entire fraction, as  $+\frac{+a}{+b}$ , any two of the three signs may be changed without altering the value of the fraction.

*Example 3.*  $\frac{x-y}{2(y-x)} = -\frac{x-y}{2(x-y)} = -\frac{1}{2}$ , since  $-(y-x) = x-y$ .

## EXERCISES

*Reduce to lowest terms, leaving the answer in factored form:*

1.  $\frac{12x}{15x}$
2.  $\frac{10y}{12y}$
3.  $\frac{15x^2y}{18xy^2}$
4.  $\frac{18x^2y}{22xy}$
5.  $\frac{21ax^2y^3}{24a^2xy}$
6.  $\frac{16xy^2}{20xy^5}$
7.  $\frac{7(a+b)}{8a(a+b)}$
8.  $\frac{4a^2(x+2y)}{8a(x+2y)}$
9.  $\frac{12x^2y(x-y)}{9xy(x-y)}$
10.  $\frac{15x^2y(2x-y)}{12xy^2(2x-y)}$
11.  $\frac{3x+3xy}{4+4y}$
12.  $\frac{4x^2y-2xy}{6x-3}$
13.  $\frac{x^2-y^2}{3x+3y}$
14.  $\frac{2x^2-2y^2}{3x-3y}$
15.  $\frac{x^2-6x+9}{x^2-9}$
16.  $\frac{x^2-64y^2}{x^2+16xy+64y^2}$
17.  $\frac{4x^2-12x}{x^2-x-6}$
18.  $\frac{x^2+x-12}{2x^2+8x}$
19.  $\frac{2x^2-7x+3}{2x^2-13x+6}$
20.  $\frac{6x^2-7x-3}{12x^2-16x-3}$
21.  $\frac{x^2-y^2}{x^3-y^3}$
22.  $\frac{x^3+8y^3}{x^2-4y^2}$
23.  $\frac{2y-2x}{x^2-y^2}$
24.  $\frac{x^2-4y^2}{6xy-3x^2}$
25.  $\frac{x^2-2xy+y^2}{xy-x^2}$
26.  $\frac{x^2-7x+12}{8-2x}$
27.  $\frac{27x^3-1}{1-9x^2}$
28.  $\frac{x^4-y^4}{(x^2+y^2)(y-x)}$
29.  $\frac{6x^2-35+x}{15x^2+14-41x}$
30.  $\frac{-x^2+x+6}{5x-6-x^2}$
31.  $\frac{2x^2-8y^2}{24y^3-3x^3}$
32.  $\frac{15-13x+2x^2}{2x^2-16x+30}$

**52. Multiplication and division of fractional expressions.** By definition 4 (section 7), the product of two fractions is the fraction whose numerator is the product of the two given numerators and whose denominator is the product of the two given denominators; in symbols,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Hence to multiply two fractional expressions we have only to follow this procedure, and then, in order to put the answer in simplest form, reduce to lowest terms as in the preceding section.



$$\text{Example 1. } \frac{2}{3} \cdot \frac{9}{4} = \frac{2 \cdot 3 \cdot 3}{3 \cdot 2 \cdot 2} = \frac{3}{2}.$$

$$\text{Example 2. } \frac{2x^2y^3z}{3xy^2z^3} \cdot \frac{9xyz}{4x^2yz} = \frac{3y}{2z^2}.$$

$$\begin{aligned} \text{Example 3. } \frac{x^2 - 5x + 6}{x^2 - 6x + 9} \cdot \frac{x(x - 3)}{x^2 - 4x + 4} \\ = \frac{(x - 2)(x - 3)x(x - 3)}{(x - 3)(x - 3)(x - 2)(x - 2)} = \frac{x}{x - 2}. \end{aligned}$$

By theorem 2 (section 7) the quotient of two fractions may be obtained by inverting the divisor and multiplying.

$$\text{Example 4. } \frac{2}{3} \div \frac{4}{9} = \frac{2}{3} \cdot \frac{9}{4} = \frac{3}{2}.$$

$$\text{Example 5. } \frac{3x^2y}{4z^3} \div \frac{6x^2y^2}{2z^2} = \frac{3x^2y}{4z^3} \cdot \frac{2z^2}{6x^2y^2} = \frac{1}{4yz}.$$

$$\begin{aligned} \text{Example 6. } \frac{x^2 - 5x + 6}{x^2 - x - 6} \div \frac{4 - x^2}{x^2 + 4x + 4} \\ = \frac{(x - 2)(x - 3)}{(x - 3)(x + 2)} \cdot \frac{(x + 2)(x + 2)}{-(x^2 - 4)} \\ = - \frac{(x - 2)(x - 3)(x + 2)(x + 2)}{(x - 3)(x + 2)(x + 2)(x - 2)} = -1. \end{aligned}$$

## EXERCISES

Perform the indicated operations and simplify:

$$1. \frac{3}{5} \cdot \frac{15}{12} \quad 2. \frac{1}{4} \cdot \frac{12}{5} \quad 3. 4 \div \frac{2}{3} \quad 4. \frac{3}{10} \div 6 \quad 5. \frac{4xy}{3z} \cdot \frac{9z^2}{8x^2y^3}$$

$$6. (3 + 6x) \cdot \frac{y}{3} \quad 7. \frac{ab + ac}{2a^2y} \div \frac{3b + 3c}{4ay^3} \quad 8. \frac{2x - 6}{5} \div \frac{3x - 9}{4}$$

$$9. \frac{x^2 - 4}{2x} \div (x + 2).$$

$$10. \frac{2x^2 - 18}{3x} \div (x + 3).$$

$$11. \frac{x - 2}{x^2 - 5x + 6} \cdot (x - 3).$$

$$12. \frac{2x^2y}{x + 5} \cdot \frac{3x^2 + 15x}{6x^3y^2}.$$

$$13. \frac{15a^3b^2c}{2xyz^3} \div \frac{5a^2b^3c^2}{6x^2yz^2}.$$

$$14. \frac{6x^3yz^4}{35a^2b^3c} \div \frac{30xy^3z^2}{14a^3bc^3}.$$

$$15. \frac{x + y}{x + 2y} \cdot \frac{x^2 - 4y^2}{x^2 - y^2}.$$

$$16. \frac{4 - x^2}{x^3 + 3x^2} \cdot \frac{3x^2}{x^3 - 5x + 6}.$$

$$17. \frac{x^2 + x - 6}{x^2 - x - 12} \div \frac{x^2 - x - 2}{x^2 - 3x - 4}.$$

$$18. \frac{x^2 + 5xy + 6y^2}{x^2 - y^2} \div \frac{x^2 - 9y^2}{x^2 - 2xy - 3y^2}.$$

$$19. - \frac{x^2 - xy - 6y^2}{4y^2 - x^2} \div \frac{x^2 - 8xy + 15y^2}{x^3 - 2x^2y}.$$



$$20. \frac{x^2 - 2x - 3}{x^2 - 9} \div \frac{x^2 + 5x}{x^2 + 8x + 15}.$$

$$21. \frac{1 - 4x^2}{2x^2 - 7x + 3} \cdot \frac{2x - x^2}{6x^2 + 13x + 5} \div \frac{2x^2 - 4x}{3x^2 - 4x - 15}.$$

$$22. \frac{x^2 - xy - 20y^2}{x^2 + 2xy - 8y^2} \cdot \frac{x^2 + 5xy}{x^2 - 25y^2} \div \frac{x^2 - xy - 2y^2}{x + y}.$$

$$23. \frac{x^3 - y^3}{9x^2 - 4y^2} \cdot \frac{3x^2 + xy - 2y^2}{x^2 - y^2} \div (x^2 + xy + y^2).$$

$$24. \frac{x^2 + 7x + 10}{x^2 - 5x + 6} \cdot \frac{x^2 + x - 6}{x^2 - 25} \div \frac{x^2 + 5x + 6}{x^2 - 3x - 10}.$$

$$25. \frac{x^2 - 4x + 4}{x^2 - 7x + 12} \cdot \frac{x^2 - 6x + 9}{x^2 - 4} \div \frac{x^2 - 5x + 6}{x^2 - 2x - 8}.$$

$$26. \frac{4x^2 - 4x + 1}{3x^2 + 4x - 4} \cdot \frac{x^2 - 4}{3x^2 + 13x - 10} \div \frac{2x^2 - 5x + 2}{9x^2 - 12x + 4}.$$

$$27. (a) \frac{2x^2 - 3x - 2}{x^2 - 4} \div \frac{2x^2 - 3x - 2}{x^2 - 3x - 10}; (b) \frac{\frac{2x^2 - 3x - 2}{x^2 - 4}}{\frac{x^2 - 4}{x^2 - 3x - 10}}.$$

$$28. (a) \frac{2x^2 - xy - y^2}{x^2 - y^2} \div \frac{2x^2 - xy - y^2}{2x^2 + 3xy + y^2}; (b) \frac{\frac{2x^2 - xy - y^2}{x^2 - y^2}}{\frac{2x^2 + 3xy + y^2}{2x^2 - xy - y^2}}.$$

$$29. \frac{\frac{6x^2 + xy - 2y^2}{4x^2 - 7xy - 2y^2}}{\frac{4x^2 - y^2}{x^2 - 4y^2}}.$$

$$30. \frac{\frac{2ax^2 + 2bx}{3ax^2 - 6abx + 3ab^2}}{\frac{ax^2 + abx + bx + b^2}{x^2 - b^2}}.$$

**53. Addition and subtraction of fractional expressions.** By section 8, to add or subtract two or more fractions having the same denominator, we form the fraction whose numerator is the algebraic sum of the given numerators and whose denominator is the common denominator.

$$\text{Example 1. } \frac{2}{6} + \frac{3}{6} = \frac{2 + 3}{6} = \frac{5}{6}.$$

$$\text{Example 2. } \frac{2}{6} - \frac{3}{6} = \frac{2 - 3}{6} = -\frac{1}{6}.$$

$$\begin{aligned} \text{Example 3. } & \frac{3}{x+3} - \frac{3x-5}{x+3} + \frac{5x-2}{x+3} \\ &= \frac{3 - (3x - 5) + (5x - 2)}{x + 3} = \frac{3 - 3x + 5 + 5x - 2}{x + 3} \\ &= \frac{2x + 6}{x + 3} = \frac{2(x + 3)}{x + 3} = 2. \end{aligned}$$



To add or subtract two or more fractions having different denominators, we express the given fractions as equal fractions having a common denominator, by multiplying their numerators and denominators by suitable factors, and then proceed as above.

$$\text{Example 4. } \frac{1}{3} + \frac{1}{2} = \frac{1 \cdot 2}{3 \cdot 2} + \frac{3 \cdot 1}{3 \cdot 2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}.$$

$$\text{Example 5. } \frac{1}{2} - \frac{1}{3} = \frac{1 \cdot 3}{2 \cdot 3} - \frac{2 \cdot 1}{2 \cdot 3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}.$$

$$\begin{aligned} \text{Example 6. } & \frac{x}{x-2} - \frac{2}{x+2} + \frac{-8}{x^2-4} \\ &= \frac{x}{x-2} - \frac{2}{x+2} + \frac{-8}{(x+2)(x-2)} \\ &= \frac{x(x+2)}{(x-2)(x+2)} - \frac{2(x-2)}{(x-2)(x+2)} + \frac{-8}{(x+2)(x-2)} \\ &= \frac{x(x+2) - 2(x-2) - 8}{(x+2)(x-2)} = \frac{x^2 - 4}{(x+2)(x-2)} \\ &= \frac{(x+2)(x-2)}{(x+2)(x-2)} = 1. \end{aligned}$$

By the **least common denominator** (L.C.D.) of several fractions is meant the L.C.M. of their denominators. It is usually desirable, although not essential, to use the L.C.D. in obtaining the algebraic sum of fractional expressions. In any case answers should be reduced to lowest terms.

## EXERCISES

Perform the indicated operations and simplify:

1.  $\frac{4}{15} + \frac{7}{15} - \frac{8}{15}$
2.  $\frac{5}{12} - \frac{7}{12} + \frac{4}{12}$
3.  $\frac{1}{2} + \frac{3}{4} - \frac{5}{8}$
4.  $\frac{3}{8} - \frac{1}{4} + \frac{5}{6}$
5.  $\frac{1}{2} - \frac{5x}{6} + \frac{2x}{3}$
6.  $\frac{5x}{4} + \frac{x}{8} - \frac{3x}{2}$
7.  $\frac{1}{a} + \frac{2}{3a} + \frac{a}{3}$
8.  $\frac{2a}{3} + \frac{5}{6a} - \frac{3}{2a}$
9.  $\frac{1}{a} - \frac{1}{b} + \frac{1}{c}$
10.  $\frac{3}{x} + \frac{2}{y} - \frac{1}{z}$
11.  $\frac{4}{3x} + \frac{3}{2x^2} - \frac{1}{6x^2}$
12.  $\frac{2}{3x} + \frac{x}{3} - \frac{1}{2x^2}$
13.  $\frac{x-1}{2} + \frac{3x}{4} - \frac{2x-3}{2x}$
14.  $\frac{2x-1}{3x^2} + \frac{3}{2x} - \frac{x-3}{2x^2}$
15.  $\frac{x}{x-2} + \frac{2}{x-5} - \frac{1}{x-10}$
16.  $\frac{1}{x+1} - \frac{1}{x-2} + \frac{1}{x+1}$
17.  $\frac{x-2}{2x-1} + \frac{x}{x-3} - \frac{1}{2x-1}$
18.  $\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3}$



19.  $\frac{x-y}{x+y} - \frac{x+y}{x-y}$

20.  $\frac{2x-y}{x-2y} - \frac{x+2y}{2x-y}$

21.  $\frac{1}{(x+a)^2} + \frac{1}{x^2-a^2}$

22.  $\frac{2}{x+2} + \frac{3}{x-2} - \frac{5x}{x^2-4}$

23.  $\frac{1}{x-3} - \frac{1}{x^2-5x+6}$

24.  $\frac{1}{x-4} - \frac{1}{x^2-7x+12}$

25.  $\frac{3}{x^2-x-2} - \frac{2}{x^2-1} + \frac{1}{x^2-3x+2}$

26.  $\frac{1}{x^2-4} - \frac{1}{x^2+2x-8} + \frac{1}{x^2+6x+8}$

27.  $\frac{1}{x^2-15x+54} - \frac{4}{36-x^2}$

28.  $\frac{2x}{x^2-4} + \frac{1}{2-x} + \frac{1}{x+2}$

29.  $\frac{1}{x^2-3xy+2y^2} - \frac{1}{x^2-4xy+3y^2} + \frac{1}{x^2-5xy+6y^2}$

30.  $\frac{x-2y}{x+2y} + \frac{8xy}{x^2-4y^2} - \frac{x+2y}{2y-x}$

31.  $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{2}{x^2-4x+3}$

32.  $\frac{4a^2}{x^2-a^2} + \frac{x+a}{a-x} - \frac{a-x}{x+a}$

33.  $\frac{x-2}{x^2-2x-24} + \frac{x+4}{x^2+8x+16} - \frac{x-4}{x^2-2x-8}$

34.  $\frac{x+y}{(y-1)(y-x)} + \frac{y+1}{(x-1)(y-1)} + \frac{x+1}{(x-y)(x-1)}$

**54. Complex fractions.** By a complex fraction is meant a fraction whose numerator or denominator or both contains one or more fractions.

*Example 1.*  $\frac{2 + \frac{1}{3}}{1 + \frac{2}{3}}$  is a complex fraction.

To evaluate a complex fraction, we may perform the indicated operations in the numerator and in the denominator, and then divide the resulting numerator by the resulting denominator.

Thus in example 1, we may write

$$\frac{2 + \frac{1}{3}}{1 + \frac{2}{3}} = \frac{7}{3} \div \frac{5}{3} = \frac{7}{5}$$



Note that, to avoid ambiguity, we make some fraction lines longer or heavier than others. Thus

$$\frac{\frac{a}{\bar{b}}}{c} = \frac{\left(\frac{a}{\bar{b}}\right)}{\left(\frac{c}{\bar{1}}\right)} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc},$$

while

$$\frac{\frac{a}{\bar{b}}}{\frac{c}{\bar{c}}} = \frac{\left(\frac{a}{\bar{1}}\right)}{\left(\frac{b}{\bar{c}}\right)} = \frac{a}{1} \cdot \frac{c}{b} = \frac{ac}{b}.$$

Example 2. 
$$\frac{\frac{1}{x} - \frac{1}{y}}{\frac{1}{x} + \frac{1}{y}} = \frac{\frac{y-x}{xy}}{\frac{y+x}{xy}} = \frac{y-x}{xy} \cdot \frac{xy}{y+x} = \frac{y-x}{y+x}.$$

Another procedure, sometimes expedient, is to multiply both the entire numerator and denominator of the given complex fraction by a suitable quantity. Thus in example 2 we may write

$$\frac{\frac{1}{x} - \frac{1}{y}}{\frac{1}{x} + \frac{1}{y}} \cdot \frac{xy}{xy} = \frac{y-x}{y+x}.$$

This amounts to multiplying the given complex fraction by 1.

The algebraic sum of one or more integral rational expressions (or polynomials) and one or more fractional rational expressions is often called a **mixed expression**.\*

Example 3.  $x + 3 + \frac{2}{x-1}$  is a mixed expression.

To evaluate a mixed expression, we may write the integral part over the denominator 1, and proceed in the usual way. Thus in example 3 we write

$$\begin{aligned} \frac{x+3}{1} + \frac{2}{x-1} &= \frac{(x+3)(x-1)}{x-1} + \frac{2}{x-1} \\ &= \frac{x^2 + 2x - 1}{x-1}. \end{aligned}$$

\* A mixed expression should be distinguished from the look on a student's face when he sees one; the latter is more properly called a "mixed up" or "confused" expression.



## EXERCISES

Reduce each of the following to a simple fraction in lowest terms:

$$1. \frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$$

$$2. \frac{\frac{4}{5} - \frac{1}{2}}{\frac{3}{4} + \frac{2}{5}}$$

$$3. \frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

$$4. \frac{\frac{1}{x} + \frac{1}{y}}{\frac{2}{x} + \frac{2}{y}}$$

$$5. \frac{4 - \frac{4}{5}}{2 + \frac{2}{5}}$$

$$6. \frac{6 + \frac{4}{3}}{3 + \frac{1}{3}}$$

$$7. \frac{\frac{x^2}{x-y} - x}{\frac{y^2}{x-y} + y}$$

$$8. \frac{x - \frac{x}{y}}{1 - \frac{1}{y}}$$

$$9. \frac{x + \frac{3}{3}}{1 + \frac{3}{x}}$$

$$10. \frac{\frac{3}{3} - x}{\frac{3}{x} - 1}$$

$$11. \frac{1 - \frac{x}{y}}{\frac{y}{x} - 1}$$

$$12. \frac{\frac{x^2}{y^2} - 1}{\frac{x}{y} - 1}$$

$$13. \frac{\frac{\frac{4}{y-x} - \frac{2}{x^2-y^2}}{1} - \frac{1}{y-x} - \frac{1}{y+x}}{\frac{x}{x+y} - \frac{y-x}{x}}$$

$$14. \frac{\frac{\frac{3}{x+2} + \frac{2}{x-1}}{3} - \frac{2}{x+2} - \frac{2}{x-1}}{\frac{3}{x^2-9} + \frac{2}{3-x}}$$

$$15. \frac{\frac{\frac{x}{x+y} - \frac{y-x}{x}}{x} + \frac{y-x}{x}}{3 + \frac{2}{x-3}}$$

$$16. \frac{\frac{3}{x^2-9} + \frac{2}{3-x}}{3 + \frac{2}{x-3}}$$

$$17. \left( \frac{x+2}{x-2} - \frac{x-2}{x+2} \right) \div \left( \frac{x}{x-2} - \frac{x}{x+2} \right)$$

$$18. \frac{\frac{\frac{x^2}{y^2} - 1}{\frac{x^2}{y^2} - \frac{2x}{y} + 1}}{\frac{x^2}{y^2} - \frac{2x}{y} + 1}$$

$$19. \frac{x + 5 + \frac{6}{x}}{x + 1 - \frac{6}{x}}$$

$$20. \frac{2x + 3 - \frac{2}{x}}{2x - 9 + \frac{4}{x}}$$

$$21. \frac{\frac{x}{2} + \frac{x+2}{x-2}}{\frac{x}{2} - \frac{x-2}{x+2}}$$

$$22. \frac{x + \frac{y^2}{x-2y}}{\frac{2y^2}{x - \frac{x+y}{x+y}}}$$

$$23. \frac{\frac{x}{y}}{z} \div \frac{\frac{x}{y}}{\frac{y}{z}}$$

$$24. \frac{x + \frac{y^3}{x^2}}{1 + \frac{y^2}{x^2} - \frac{y}{x}}$$

$$25. \frac{x^2 - \frac{y^3}{x}}{x + \frac{y^2}{x} + y}$$



26. Find the value of  $\frac{x-y}{x+y}$  in terms of  $a$  if  $x = \frac{a}{1+a}$  and  $y = \frac{1-a}{a}$ .

27. Find the value of  $\frac{x-y}{x+y}$  in terms of  $a$  and  $b$  if  $x = \frac{a}{a+b}$  and  $y = \frac{b-a}{a}$ .

$$28. 1 + \frac{1}{a + \frac{1}{a}}.$$

$$29. 1 + \frac{2}{a + \frac{3}{a + \frac{4}{a}}}.$$

$$30. 1 - \frac{2}{a + \frac{3}{a - \frac{4}{a}}}.$$



## CHAPTER VIII

### Linear Equations and Linear Functions

**55. Solution of linear equations in one variable.** A root of an equation in  $x$  is a number which satisfies the equation, or converts it into a true statement when substituted for the variable  $x$ . To solve an equation means to find all its roots.

*Note.* Not every equation has a root. Thus  $x = x + 1$  and  $\frac{1}{x} = 0$  have no roots at all.

A linear equation in  $x$  is one that can be written in the standard form\*  $ax + b = 0$ ,  $a \neq 0$ , where  $a$  and  $b$  are constants.

The technique of solving an equation is to manipulate it so that the variable or unknown stands alone on one side of the equation. In the case of a linear equation, this can always be done with the help of axiom XIV, section 17. That is, if equals are (a) added to, (b) subtracted from, (c) multiplied by, (d) divided by equals, the results are equal. (See section 29.)

*Example 1.* Solve

$$(1) \qquad 3x - 1 = x + 13.$$

We reason as follows. If equation (1) has a root, then this root will also satisfy

$$(2) \qquad 2x = 14$$

which is obtained by transposing, or by adding one to both sides and subtracting  $x$  from both sides. But this root will also satisfy the equation obtained by dividing both sides of (2) by 2. Thus

$$(3) \qquad x = 7.$$

We have thus proved that if  $x$  satisfies (1), then  $x$  must be 7. This means that 7 is the only possible candidate for the position of root. But does it prove that 7 is a root of (1)? This requires the proposition "if  $x = 7$ , then it satisfies (1)." But this is the converse of what we proved, and it is well known that a correct proposition may have an incorrect converse. For example, "if  $x$

\* See section 30.



is a donkey, then  $x$  is a mammal" is true; but the converse "if  $x$  is a mammal, then  $x$  is a donkey" is false, as every reader can testify personally. Therefore we must prove the converse. This can be done either by verifying that each step of the reasoning above is reversible (see sections 28 and 29), which is the case in example 1, or, more simply, by substituting 7 for  $x$  in the original equation (1). Thus,  $3 \cdot 7 - 1 \stackrel{?}{=} 7 + 13$ , or  $20 = 20$ . Hence 7 is a root and certainly the only root.

The process of substituting the candidate or candidates for the title of root in the original equation is often called *checking* the solution. This unfortunately suggests that it is a superfluous operation designed only to allay the insecurity of the student. *We have seen that it is an essential part of the work*, unless all steps used are known to be reversible. The student should check every solution.

*Example 2.* Solve  $\frac{5x}{2} - 3 = \frac{x}{3} + 10$ . Multiplying both sides by the L.C.D. 6, we obtain  $15x - 18 = 2x + 60$ . Transposing, we get  $13x = 78$ . Dividing both sides by 13, we get  $x = 6$ . Hence 6 is the only possible root. Checking, we find that 6 is a root, since  $\frac{5 \cdot 6}{2} - 3 = \frac{6}{3} + 10$ , or  $12 = 12$ .

## EXERCISES

*Solve and check:*

1.  $x + 3 = 7$ .
2.  $x - 3 = 7$ .
3.  $3x = 12$ .
4.  $\frac{x}{3} = 4$ .
5.  $3x + 2 = x + 6$ .
6.  $5x - 3 = 9 - x$ .
7.  $3x + 5 = 2(x + 7)$ .
8.  $5(x - 2) = 2x + 5$ .
9.  $3 - 2x = 3(5 - 2x)$ .
10.  $2 + 3(x + 2) = 4x - 1$ .
11.  $2x + 4 = 3(-2x + 5) - 27$ .
12.  $3(x + 4) = 2(2 - 3x) + 6x - 1$ .
13.  $2y - 15 = 5y - (y + 5)$ .
14.  $6z + 4(z - 1) = 3(z + 3) - 20$ .
15.  $2(x + 3) = 4x + 1$ .
16.  $5(x - 2) = x + 2(x + 1)$ .
17.  $\frac{x}{2} + \frac{3}{4} = 2\left(x + \frac{1}{4}\right)$ .
18.  $\frac{x}{3} + \frac{5}{6} = 3\left(x + \frac{1}{9}\right)$ .
19.  $2(x - [2x + 1] - 3) = \frac{x}{2} + 1$ .
20.  $x + (2x - [3x - 4]) = \frac{x}{3} + 2$ .
21.  $\frac{3x - 1}{2} + \frac{x - 4}{4} = \frac{2x + 1}{2}$ .
22.  $\frac{x}{3} + \frac{2x - 1}{6} = x - 5$ .



23.  $\frac{x}{2} - \frac{3x}{4} + 3 = 2x - \frac{1}{8}$ .

24.  $\frac{5x}{6} - \frac{1}{2} = \frac{3}{2} - \frac{x}{6}$ .

25.  $\frac{x-2}{2} = \frac{x}{5} + \frac{1}{2}$ .

26.  $\frac{1}{3} + \frac{x+2}{2} = \frac{2x}{3} - \frac{1}{6}$ .

27.  $x - \frac{5}{12} = \frac{1}{3} \left( x + \frac{3}{2} \right)$ .

28.  $\frac{3x-1}{2} - \frac{2x}{3} = 3 - \frac{x-2}{2}$ .

29.  $3x - 0.2 = 0.25 - 6x$ .

30.  $4.08 - 6x = 5.2 - 4x$ .

56. Equations leading to linear equations. Frequently an equation, not linear itself, leads to a linear equation.

*Example 1.* Solve  $\frac{1}{6} + \frac{1}{10} = \frac{1}{x}$ . Multiplying both members by the L.C.D.  $30x$ , we get  $5x + 3x = 30$ , or  $8x = 30$ , or  $x = \frac{30}{8} = \frac{15}{4}$ . This proves that  $\frac{15}{4}$  is the only possible root. To show that it is a root, we check by substitution:  $\frac{1}{6} + \frac{1}{10} = \frac{1}{15/4}$  or  $\frac{8}{30} = \frac{4}{15}$ .

*Example 2.* Solve  $\frac{2x}{x-2} = \frac{4}{x-2}$ . Multiplying both sides by the L.C.D.  $x-2$  we get  $2x = 4$ , or  $x = 2$ . This proves that if  $x$  satisfied the original equation, then  $x$  could only be 2. But checking by substitution reveals that 2 does not satisfy the original equation, since division by zero is excluded. Hence the given equation has no root at all.

Recall that multiplying both sides by a quantity involving unknowns is not always a reversible step and may lead to non-equivalent equations as in example 2. Compare section 29. Therefore *checking is an essential part of the solution in such a case and not merely a superfluous operation designed to reassure the student.*

## EXERCISES

*Solve and check:*

1.  $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 2$ .

2.  $\frac{5}{x} + 1 = \frac{6}{x}$ .

3.  $\frac{2x-1}{2x} = \frac{3}{x} + \frac{1}{2}$ .

4.  $\frac{2}{x} - \frac{12+x}{9x} = \frac{4}{9}$ .

5.  $\frac{1}{x-1} = \frac{2}{x-2}$ .

6.  $\frac{3}{x-2} = \frac{2}{x-3}$ .



7.  $\frac{2x}{x-5} = \frac{10}{x-5}$ .

8.  $\frac{6x}{x-3} = \frac{18}{x-3}$ .

9.  $x(x-3) + 4 = x^2 - 2(x+4)$ .

10.  $(x-1)(x-2) = (x+3)(x-4) - 3(x-1)$ .

11.  $\frac{x+1}{x-2} = \frac{x-3}{x-4}$ .

12.  $\frac{2x-1}{x-3} = \frac{2x+4}{x+4}$ .

13.  $\frac{3}{2x+3} + \frac{6}{6x+9} = \frac{5}{9}$ .

14.  $\frac{1}{x^2-4} = \frac{2}{2x^2-3x+4}$ .

15.  $(x-2)^2 = (x-3)^2$ .

16.  $(x+2)^2 = (x-5)^2$ .

17.  $3(x-2)^2 = (x-2)(x+2) + 2(x+3)^2$ .

18.  $(2x-3)^2 - (x-1)(x-3) = 3(x+2)^2$ .

19.  $\frac{2}{x} + \frac{x+2}{x(x-2)} = \frac{4}{x(x-2)}$ .

20.  $\frac{2}{x-1} - \frac{1}{2(x-1)} + \frac{1}{2(x-3)} = \frac{1}{(x-1)(x-3)}$ .

21.  $\frac{2}{2x^2-7x+3} = \frac{5}{5x^2-17x+6}$ .

22.  $\frac{5}{x} - \frac{2}{x+1} = \frac{3}{x-1}$ .

23.  $\frac{7}{x-1} = \frac{9}{x-2} - \frac{2}{x-3}$ .

24.  $\frac{x+2}{x+3} + \frac{x+3}{x+2} = \frac{x^2-9}{x^2+5x+6}$ .

25.  $\frac{3}{x+1} - \frac{3}{x+2} + \frac{5}{2x+2} = 0$ .

26.  $\frac{4y-3}{y+2} = \frac{7y-2}{y-5} - 3$ .

27.  $\frac{4}{x-1} + \frac{3}{2(x-1)} - \frac{3}{2(x+1)} = \frac{3}{x^2-1}$ .

28.  $\frac{4}{(x-2)(x+3)} + \frac{4}{5(x-2)} = \frac{1}{x-2} + \frac{4}{5(x+3)}$ .

**57. Literal equations and formulas.** The operations of adding, subtracting, multiplying, or dividing equals with equals may be used to solve linear equations with literal\* coefficients.

*Example 1.* Solve  $ax - b = cx + d$  for  $x$ .

*Solution.* Adding  $b$  to both sides of the equation and subtracting  $cx$  from both sides, we obtain  $ax - cx = b + d$ , or

$$(a - c)x = b + d.$$

Provided  $a - c \neq 0$ , or  $a \neq c$ , we may divide both sides by  $a - c$ , obtaining

\* That is, coefficients which are unspecified quantities.



$$x = \frac{b + d}{a - c}.$$

The reader should check this solution by substituting in the original equation.

This sort of manipulation will arise frequently in the student's scientific studies when it is desired to rearrange formulas connecting two or more quantities so as to express a certain one of them in terms of the others.

*Example 2.* Let  $F$  be the number of degrees Fahrenheit and  $C$  the number of degrees Centigrade. Then  $F = \frac{9}{5}C + 32$ . Solve this formula for  $C$  in terms of  $F$ .

*Solution.* Subtracting 32 from both sides, we have

$$\frac{9}{5}C = F - 32.$$

Multiplying both sides by  $\frac{5}{9}$ , we have

$$C = \frac{5}{9}(F - 32).$$

## EXERCISES

*Solve the following equations for  $x$ , stating any conditions which become necessary to avoid division by zero.*

1.  $2ax - 3ab = 7ab.$

3.  $3a - 2x = x - 6b.$

5.  $3ax - 2b = 7b + 6x.$

7.  $2(ax - 3b) = b(x - 5).$

9.  $\frac{x + 5}{3a} = \frac{2}{3} + a$ , where  $a \neq 0$ .

11.  $2ax + b^2 = 4a^2 + bx.$

13.  $\frac{x - b}{a} = \frac{x - a}{b}.$

15.  $\frac{1}{a} - \frac{1}{x} = \frac{1}{x} - \frac{1}{b}.$

16.  $(x + a)^2 - (x - b)^2 = 2a(a + b).$

17.  $\frac{1}{x - a} + \frac{1}{x - b} = 0.$

2.  $3x - c = x + 5c.$

4.  $6ax + 3ab = 3ax + 6ab.$

6.  $ax + c = d + bx.$

8.  $3ax - 4a^2 = 12a^2 - 5ax.$

10.  $\frac{3a - x}{4} = \frac{x - 3a}{3}.$

12.  $4ax + b^2 = 16a^2 + bx.$

14.  $a(a - x) = b(x - b) - 2ab.$

*Solve each of the following for  $y$ :*

19.  $2x + 3y = 4.$

20.  $3x - 2y = 6.$

21.  $2x + 5y - 2 = 3x - 2y + 5.$

22.  $5x - 2y + 1 = 3x + 3y + 11.$



23.  $Ax + By = C.$

24.  $\frac{x}{a} + \frac{y}{b} = 1.$

Solve each of the following formulas for the indicated letter, all letters being understood to represent positive quantities:

25.  $pv = c$  for  $p$ .

27.  $A = \frac{1}{2}bh$  for  $h$ .

29.  $P = 2l + 2w$  for  $w$ .

31.  $C = \frac{5}{9}(F - 32)$  for  $F$ .

33.  $V = \frac{1}{3}\pi r^2 h$  for  $h$ .

35.  $F = G \frac{m_1 m_2}{r^2}$  for  $m_1$ .

37.  $S = \frac{n}{2} [2a + (n - 1)d]$  for  $d$ .

39.  $nE = I(R + nr)$  for  $r$ .

41.  $d_1 w_1 = d_2 w_2$  for  $w_2$ .

43.  $c = \frac{P - p}{pt}$  for  $t$ .

45.  $p = p_1 - c(h - h_1)$  for  $h$ .

46.  $T = T_1 \left( 1 - \frac{n - 1}{n} \cdot \frac{h}{h_0} \right)$  for  $h$ .

47.  $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$  for  $C$ .

48.  $V = \frac{q}{\epsilon_0} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$  for  $r_2$ .

26.  $I = PRT$  for  $T$ .

28.  $A + B + C = 180$  for  $C$ .

30.  $d = \frac{1}{2}gt^2$  for  $g$ .

32.  $A = \frac{1}{2}h(B + b)$  for  $B$ .

34.  $A = P(1 + rt)$  for  $r$ .

36.  $S = \frac{a - rl}{1 - r}$  for  $r$ .

38.  $\frac{V_1}{V_2} = \frac{P_2}{P_1}$  for  $P_1$ .

40.  $I = \frac{E}{r + \frac{R}{n}}$  for  $n$ .

42.  $S = 2\pi r(r + h)$  for  $h$ .

44.  $\frac{v^2}{2g} + \frac{p}{c} = H$  for  $c$ .

58. Graph\* of a linear function. Consider a linear function  $y = ax + b$  ( $a \neq 0$ ) with real coefficients. It can be proved† that the graph of a linear function in  $x$  with real coefficients is an oblique straight line, that is, one which is parallel to neither axis. Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be any two points on its graph; that is,

$$y_1 = ax_1 + b$$

$$y_2 = ax_2 + b.$$

\* The student should review sections 34 and 35.

† For proof see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, section 80, or textbooks on analytic geometry.



Then  $y_2 - y_1 = a(x_2 - x_1)$ , or\*

$$(1) \quad \frac{y_2 - y_1}{x_2 - x_1} = a.$$

The constant  $a$  is called the **slope** of the line, since it represents geometrically the ratio of vertical rise over horizontal progress toward the right (Fig. 23), or the vertical rise per unit of horizontal progress toward the right.

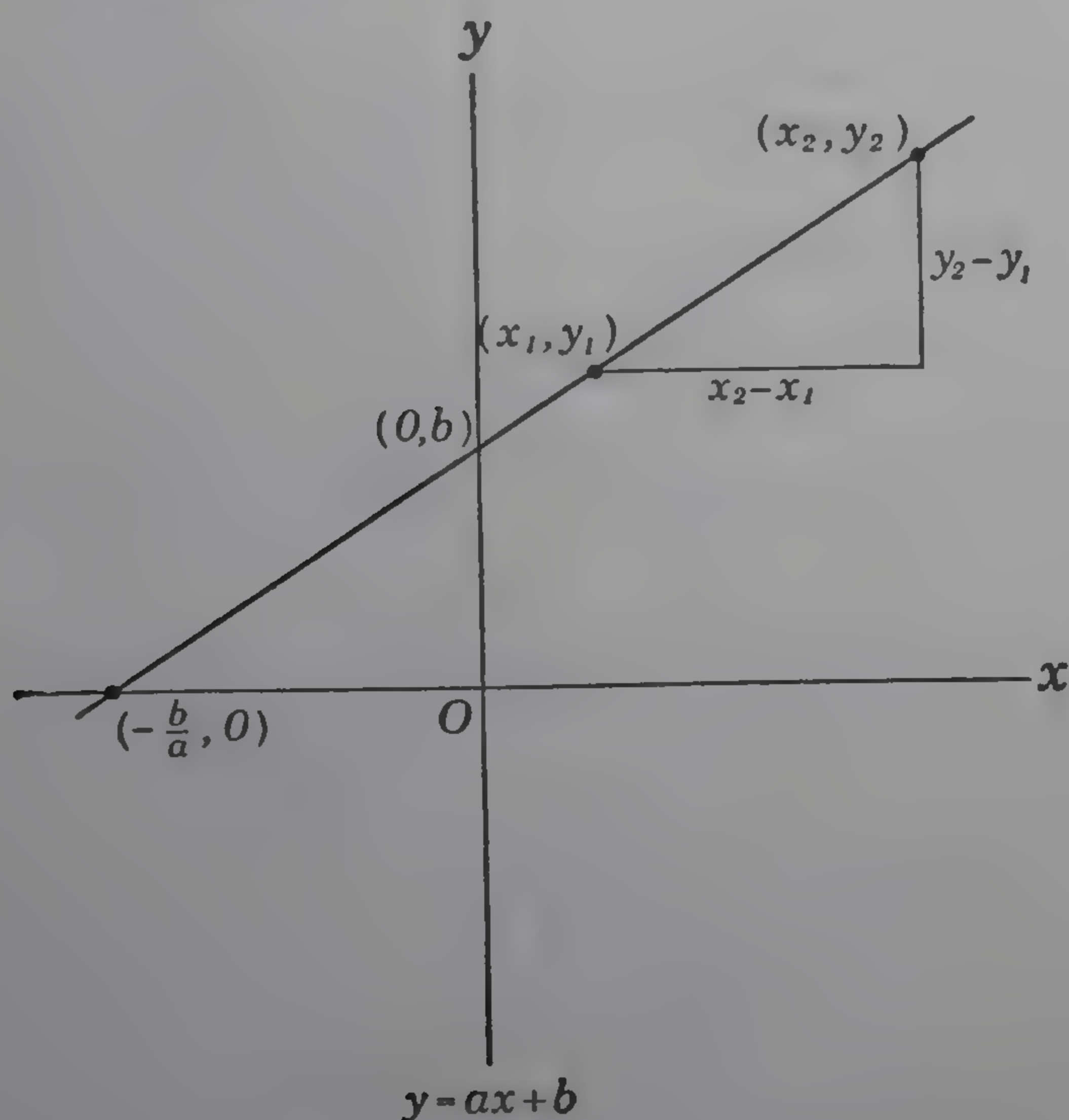


Fig. 23.

*Example 1.* The graph (Fig. 24) of the function  $y = 2x + 4$  rises 2 units per unit of horizontal progress toward the right.

*Example 2.* The graph (Fig. 25) of the function  $y = -\frac{1}{2}x + 4$  sinks 1 unit (rises  $-1$  unit) for every 2 units of horizontal progress toward the right.

These examples indicate the fact that a line with positive slope rises as it goes to the right, while a line with negative slope sinks as it goes to the right.

The constant  $b$  is called the  **$y$ -intercept** of the line  $y = ax + b$ , since, by substitution, it is clear that  $(0, b)$  are the coordinates

\* Since a linear function is single-valued, two different points on its graph must have different  $x$ -coordinates. Thus  $x_1 \neq x_2$  or  $x_2 - x_1 \neq 0$ ; hence division by  $x_2 - x_1$  is permissible.



of the point where the graph intersects the  $y$ -axis. See Figs. 23, 24, 25.

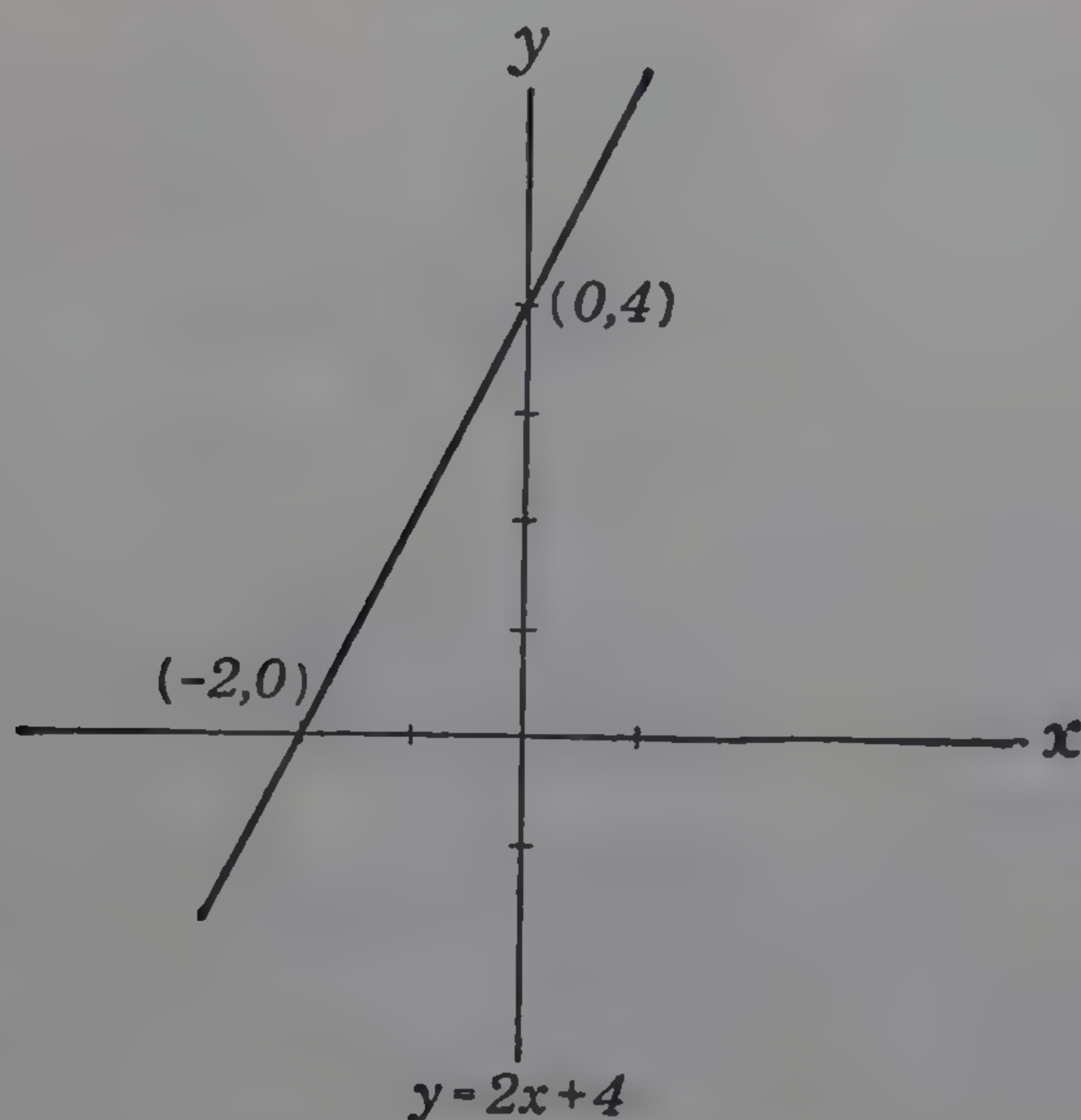


Fig. 24.

Let two oblique lines have the equations  $y = a_1x + b_1$  and  $y = a_2x + b_2$  respectively. If  $a_1 = a_2$  and  $b_1 = b_2$ , then the two lines are clearly the same. If  $a_1 = a_2$  and  $b_1 \neq b_2$ , the two lines are

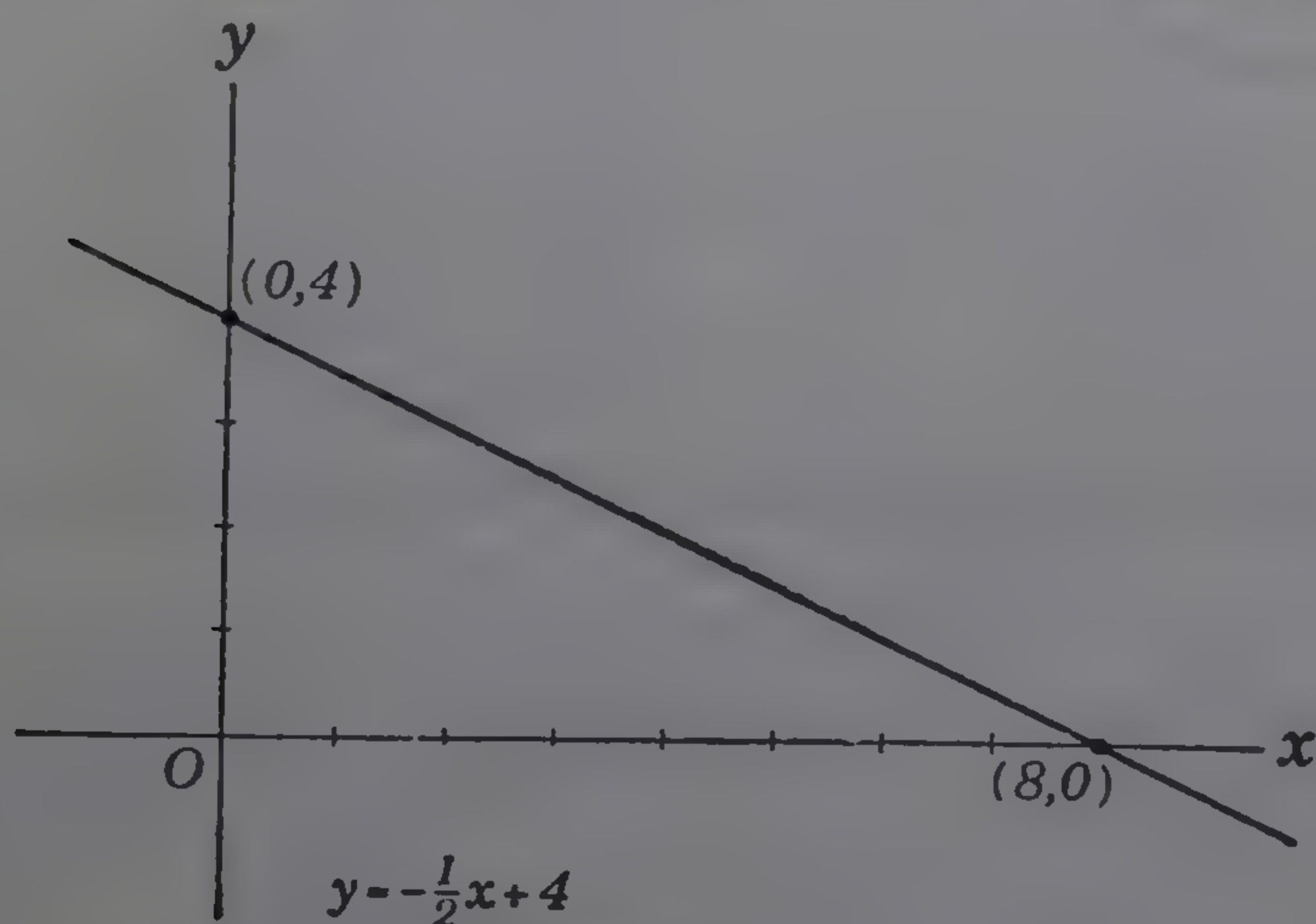


Fig. 25.

*parallel.* The proof is left to the reader. (*Hint:* consider the geometrical meaning of slope as vertical rise over horizontal progress toward the right.)

The solution or root of the equation  $ax + b = 0$  is  $x = -b/a$ ,



which is the  $x$ -coordinate of the point at which the graph of  $y = ax + b$  intersects the  $x$ -axis. See Figs. 23, 24, 25. This point is called the  **$x$ -intercept** of the line.

*Example 3.* Solve the equation  $2x + 4 = 0$  graphically.

*Solution.* From the graph (Fig. 24), the solution is  $x = -2$ , since the line crosses the  $x$ -axis at  $x = -2$ .

*Example 4.* The solution of the equation  $-\frac{1}{2}x + 4 = 0$  is seen similarly to be  $x = 8$  (Fig. 25).

It can be seen from the graph that equation (1) above means that the linear function  $y = ax + b$  indicates that the variable  $y$  has a constant rate of change  $a$  with respect to  $x$ . That is, the change in  $y$  induced by a change in  $x$  is always the same constant  $a$  times the change in  $x$ .

Since a straight line is determined by two points, plotting two points suffices to draw the graph of a linear function. A third point is desirable as a check; if the three points do not lie on a straight line, an error has been made.

## EXERCISES

*Plot the graph of each of the following functions:*

- |                             |                             |
|-----------------------------|-----------------------------|
| 1. $y = 2x + 1.$            | 2. $y = 2x - 1.$            |
| 3. $y = \frac{1}{2}x + 1.$  | 4. $y = \frac{1}{2}x - 1.$  |
| 5. $y = -2x + 1.$           | 6. $y = -2x - 1.$           |
| 7. $y = -\frac{1}{2}x + 1.$ | 8. $y = -\frac{1}{2}x - 1.$ |
| 9. $y = x.$                 | 10. $y = -x.$               |
| 11. $y = 2x.$               | 12. $y = -\frac{1}{2}x.$    |

*Solve each of the following equations algebraically and check by plotting the graph:*

- |                             |                                       |
|-----------------------------|---------------------------------------|
| 13. $2x - 6 = 0.$           | 14. $3x - 6 = 0.$                     |
| 15. $\frac{1}{2}x + 3 = 0.$ | 16. $\frac{1}{3}x - 2 = 0.$           |
| 17. $5 - 2x = 0.$           | 18. $1 - 3x = 0.$                     |
| 19. $3 - \frac{1}{2}x = 0.$ | 20. $\frac{1}{2} - \frac{1}{4}x = 0.$ |
21. The line  $y = 2x + b$  passes through the point (3,4). Find  $b$ .  
 22. The line  $y = ax + 2$  passes through the point (2,6). Find  $a$ .  
 23. Find the slope of the line passing through (1,2) and (3,8).  
 24. Find the slope of the line passing through (-1,2) and (1,4).



Write an equation of the line satisfying each of the following conditions:

25. Passing through (3,2) with slope 4.
26. Passing through (2,3) with slope 1.
27. Passing through  $(-1, -3)$  with slope  $\frac{1}{2}$ .
28. Passing through  $(-3, 3)$  with slope  $-1$ .
29. Passing through (1,2) and (3,8).
30. Passing through  $(-1, 2)$  and  $(2, -1)$ .
31. Passing through (0,4) and parallel to  $y = 2x - 3$ .
32. Passing through  $(0, -3)$  and parallel to  $y = \frac{1}{2}x + 1$ .
33. Passing through  $(-2, 0)$  and parallel to  $y = -\frac{1}{3}x + 1$ .
34. Passing through (3,0) and parallel to  $y = \frac{1}{2}x - 1$ .
35. Having  $x$ -intercept 3 and  $y$ -intercept  $-6$ .
36. Having  $x$ -intercept  $-3$  and  $y$ -intercept 6.
37. Show that the points (1,2), (3,8), and (4,11) are on the same straight line.
38. Show that the points  $(-1, -2)$ , (1,1), and (3,4) are on the same straight line.

59. Systems of two linear equations in two unknowns. The general linear equation in  $x$  and  $y$  may be written in the form  $Ax + By = C$  where  $A$ ,  $B$ , and  $C$  are constants and where  $A$  and  $B$  are not both zero. A solution of such an equation is a pair of values, one for  $x$  and one for  $y$ , which satisfies the equation. In general, a single equation in more than one unknown may have infinitely many solutions. For example,  $x + y = 7$  has the solutions  $(x = 1, y = 6)$ ,  $(x = 2, y = 5)$ ,  $(x = 3, y = 4)$ ,  $(x = -1, y = 8)$ , etc. No matter what value we choose for  $x$ , that value together with the value  $y = 7 - x$  will constitute a solution. Therefore there is little point in asking one to "solve" such an equation. But if we have two such equations, one may ask whether they have a common solution, and, if so, what it is. A common solution of two or more equations in two or more unknowns is called a solution of the system of equations. Thus the system of equations

$$\begin{array}{l} (1) \\ (2) \end{array} \quad \left. \begin{array}{l} x + y = 7 \\ x - y = 1 \end{array} \right\}$$

has the solution  $(x = 4, y = 3)$  and no other solution.



The technique of solving a system of two linear equations in two unknowns is to reduce it to one equation in one unknown by eliminating one of the unknowns. This may be done by substitution, as follows.

*Example 1.* Solve the system of equations (1) and (2). We reason as follows. If  $x$  and  $y$  satisfy (1), then they also satisfy  $y = 7 - x$ . Substituting this in (2), we have  $x - (7 - x) = 1$ , or  $x - 7 + x = 1$ , or  $2x = 8$ , or  $x = 4$ . Then  $y = 7 - x = 7 - 4 = 3$ . This proves that if  $(x, y)$  satisfy the system, then  $x$  can only be 4 and  $y$  can only be 3. The converse is also true, since all steps taken are reversible. Nevertheless, it is desirable to check the converse by substitution in both equations. Thus  $4 + 3 = 7$  and  $4 - 3 = 1$ . This illustrates the first method for solving systems of two linear equations in two unknowns  $x$  and  $y$ .

*Method of elimination by substitution.* (a) Solve one equation for one unknown in terms of the other. (b) Substitute this expression in the second equation, thus obtaining an equation in the other unknown only. (c) Solve this equation for the other unknown. (d) Substitute this value in the expression obtained in step (a), thus obtaining the value for the first unknown. (e) Check by substitution in both given equations.

*Example 2.* Solve the system

$$\begin{array}{l} (3) \qquad 2x + 3y = 13 \\ (4) \qquad 5x - 2y = 4. \end{array} \quad \left. \vphantom{\begin{array}{l} (3) \\ (4) \end{array}} \right\}$$

*Solution.* (a) From (3), we obtain  $y = \frac{13 - 2x}{3}$ . (b) Substituting this in (4), we obtain  $5x - 2\left(\frac{13 - 2x}{3}\right) = 4$ . (c) Then  $15x - 26 + 4x = 12$ , or  $19x = 38$ , and  $x = 2$ . (d)  $y = \frac{13 - 4}{3} = 3$ . (e) Checking,  $2 \cdot 2 + 3 \cdot 3 = 13$  and  $5 \cdot 2 - 2 \cdot 3 = 4$ .

A second method, often more convenient, will be illustrated by using it to solve example 2.

We obtain equations equivalent to (3) and (4) by multiplying them by any constant except 0. Let us multiply both sides of (3) by 2 and both sides of (4) by 3, obtaining

$$\begin{array}{l} 4x + 6y = 26 \\ 15x - 6y = 12. \end{array} \quad \left. \vphantom{\begin{array}{l} 4x + 6y = 26 \\ 15x - 6y = 12. \end{array}} \right\}$$

and

Since if equals are added to equals the results are equal, we may add the left members and right members of these equations and



obtain the equation

$$19x = 38.$$

Hence  $x = 2$ .

Similarly, multiplying (3) and (4) by 5 and  $-2$  respectively, we obtain

$$\begin{array}{r} 10x + 15y = 65 \\ -20x + 4y = -8 \end{array}$$

Adding these equations, we get

$$19y = 57.$$

Hence  $y = 3$ . The check is as above.

This illustrates the second method for solving systems of two linear equations in two unknowns.

**Method of elimination by multiplying and adding.** (a) Multiply both equations by constants chosen so that the coefficients of one unknown become negatives of each other. (b) Add the resulting equations, thus eliminating one unknown. (c) Solve the resulting equation for the other unknown. (d) Repeat the process with the roles of the unknowns interchanged, solving for the first unknown. (e) Check by substituting in both equations.

Another method of solving systems of linear equations will be taken up in Chapter XVI.

## EXERCISES

Solve by either method, unless otherwise directed, and check.

1.  $\begin{cases} 2x + 3y = 4 \\ 3x + y = 11 \end{cases}$
2.  $\begin{cases} 2x - 3y = 1 \\ 3y + y = 7 \end{cases}$
3.  $\begin{cases} 2x + 3y = 4 \\ x + 2y = 1 \end{cases}$
4.  $\begin{cases} 4x + 3y = 1 \\ y + 2z = 1 \end{cases}$
5.  $\begin{cases} 3x + 2y = 5 \\ x + 2y = 2 \end{cases}$
6.  $\begin{cases} 3y - y = 2 \\ 5x + y = 2 \end{cases}$
7.  $\begin{cases} 2x - 3y = 13 \\ 5x + 2y = 6 \end{cases}$
8.  $\begin{cases} 2x - 3y = 13 \\ 5x + 2y = 6 \end{cases}$
9.  $\begin{cases} 2x - 3y = 13 \\ 5x + 2y = 6 \end{cases}$
10.  $\begin{cases} 10 - 3y = 5 \\ 3x + 2y = 11 \end{cases}$
11.  $\begin{cases} x + 2y = 1 \\ 4x - 3y + 2 = 11 \end{cases}$
12.  $\begin{cases} 2x = 5y - 2 \\ 3x = 6y + 7 \end{cases}$
13.  $\begin{cases} \frac{x}{2} - \frac{y}{3} = \frac{1}{6} \\ \frac{x}{3} - \frac{y}{2} = \frac{1}{6} \end{cases}$
14.  $\begin{cases} \frac{x}{2} + \frac{y}{3} = \frac{1}{6} \\ \frac{x}{3} + \frac{y}{2} = \frac{1}{6} \end{cases}$
15.  $\begin{cases} x + \frac{y}{2} = \frac{1}{2} \\ \frac{x}{2} + \frac{y}{3} = \frac{1}{6} \end{cases}$



$$16. \begin{cases} \frac{x}{2} + \frac{2y}{5} = 1, \\ \frac{2x}{3} - \frac{y}{3} + 3 = 0. \end{cases}$$

$$17. \begin{cases} \frac{x}{3} - 1 = y, \\ \frac{2x}{5} + \frac{4}{5} = 3y + 2. \end{cases}$$

$$18. \begin{cases} \frac{x}{2} + \frac{y}{3} = 2, \\ \frac{x}{3} - \frac{y}{2} + 3 = 0. \end{cases}$$

$$19. \begin{cases} \frac{x-2}{2} + \frac{y+3}{3} = 7, \\ \frac{x+3}{3} - \frac{y+2}{2} = -4. \end{cases}$$

$$20. \begin{cases} \frac{x+y}{2} - \frac{x-y}{2} = 2, \\ \frac{x-y}{4} + \frac{x+y}{2} = 5. \end{cases}$$

$$21. \begin{cases} \frac{x+3}{3} + \frac{y-3}{3} = 2, \\ \frac{x-2}{2} - \frac{y-4}{4} = -6. \end{cases}$$

$$22. \begin{cases} \frac{2x+3}{2} + \frac{3y+2}{3} = 2, \\ \frac{2x-3}{2} - \frac{3y-4}{3} = -1. \end{cases}$$

$$23. \begin{cases} x + y = 5a, \\ x - y = a. \end{cases}$$

$$24. \begin{cases} 2x + 5y = a, \\ 3x - 2y = -8a. \end{cases}$$

$$25. \begin{cases} 3x + y = 5a, \\ 2x - 3y = 11 - 4a. \end{cases}$$

$$26. \begin{cases} 2x + y = 9a, \\ 7x - 2y = 4a. \end{cases}$$

$$27. \begin{cases} x - 3y = 7b, \\ 2x + y = 7a. \end{cases}$$

$$28. \begin{cases} x + y = 2a, \\ x - y = 2b. \end{cases}$$

$$29. \begin{cases} \frac{x}{4a} + \frac{y}{3a} = \frac{1}{12}, (a \neq 0), \\ \frac{x}{3b} + \frac{y}{5b} = \frac{1}{15}, (b \neq 0). \end{cases}$$

$$30. \begin{cases} 5x + y = 4a, \\ 7x - y = 2a. \end{cases}$$

$$31. \begin{cases} ax + by = a^2 + b^2, \\ bx - ay = a^2 + b^2, (a^2 + b^2 \neq 0). \end{cases}$$

$$32. \begin{cases} ax + by = a^2 - b^2, \\ bx + ay = a^2 - b^2, (a^2 - b^2 \neq 0). \end{cases}$$

$$33. \begin{cases} a_1x + b_1y = k_1, \\ a_2x + b_2y = k_2, (a_1b_2 - a_2b_1 \neq 0). \end{cases}$$

$$34. \begin{cases} ax - by = c, \\ dx + ey = f, (ae + bd \neq 0). \end{cases}$$

60. Graphical interpretation. Inconsistent and dependent equations. It can be proved\* that every linear equation

$$(1) \quad ax + by = c$$

with real coefficients  $a$ ,  $b$ , and  $c$ , not both  $a$  and  $b$  equal to zero, has a straight line for its graph; and conversely every straight line in the plane has such a linear equation. If  $a = 0$ ,  $b \neq 0$ , we may

\* For a proof, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, or textbooks on analytic geometry.



write the equation (1) as  $y = \frac{c}{b}$ , which clearly represents a horizontal straight line through the point  $(0, \frac{c}{b})$ . If  $b = 0$ ,  $a \neq 0$ , then equation (1) becomes  $x = \frac{c}{a}$ , which clearly represents a vertical line through the point  $(\frac{c}{a}, 0)$ . If neither  $a$  nor  $b$  is zero, then equation (1) may be written as  $y = -\frac{a}{b}x + \frac{c}{b}$ , which represents an oblique straight line with slope  $-\frac{a}{b}$  and  $y$ -intercept  $\frac{c}{b}$ . To plot the graph of a linear equation, therefore, requires plotting only two points; a third point may be desirable as a check.

Consider two linear equations, as

$$(2) \quad x + y = 7,$$

$$(3) \quad x - y = 1.$$

Every point in the plane whose coordinates satisfy equation (2) lies on the graph of equation (2) (Fig. 26). Every point whose coordinates satisfy (3) lies on the graph of equation (3). Hence a *common solution of the system of equations (2) and (3) must be the coordinates of a point of intersection of these two lines, and conversely*. Usually two linear equations have one and only one common solution, but not always. Two equations which have a unique common solution are often called **simultaneous**.

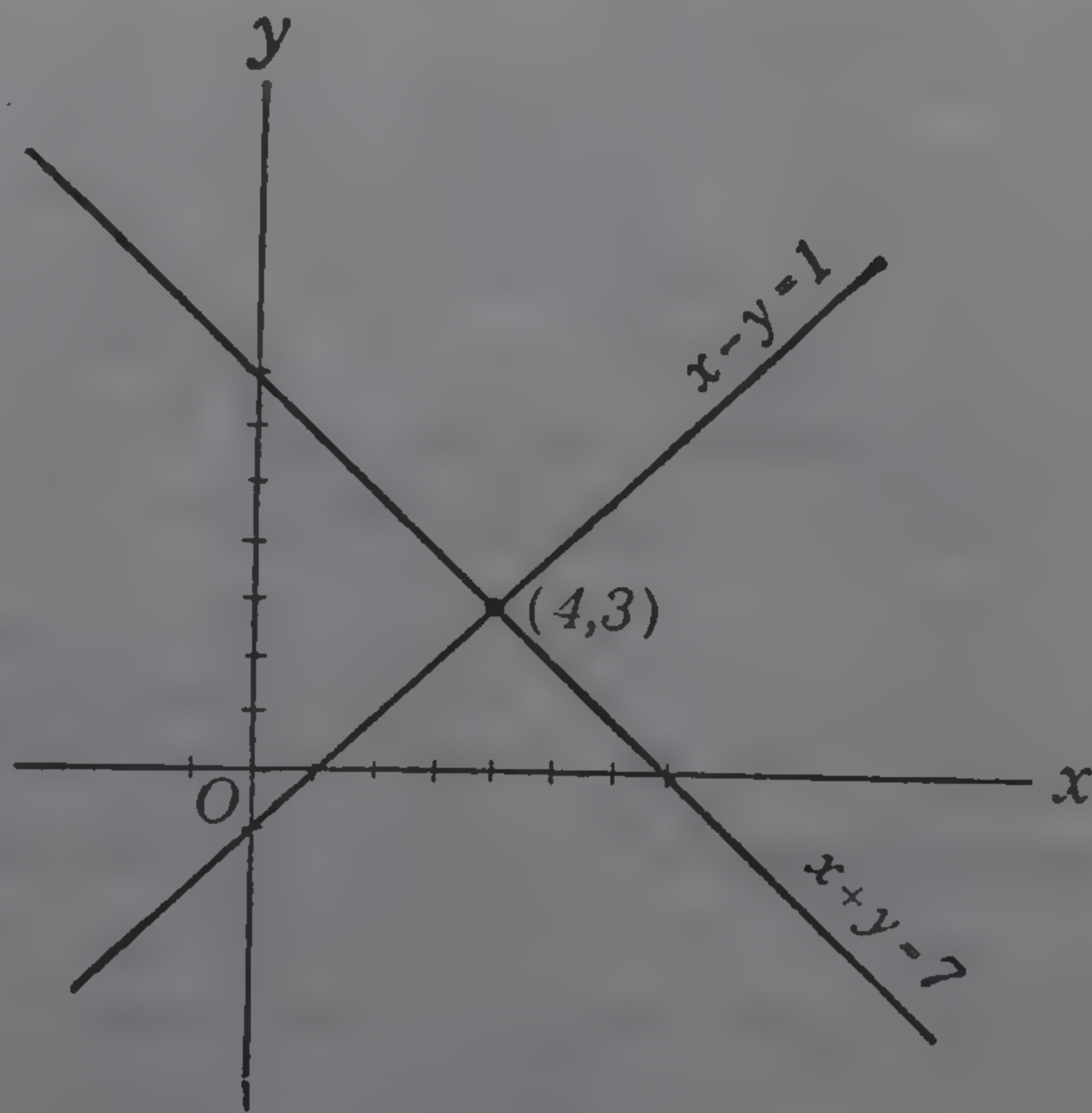


Fig. 26.

Not every pair of linear equations in  $x$  and  $y$  have a common solution. For example, it is clear that the equations

$$(4) \quad x + y = 7$$

and

$$(5) \quad x + y = 5$$



can have no common solution, since the sum of two numbers  $x$  and  $y$  cannot be both 7 and 5. Two equations which have no common solution are called **inconsistent** or **incompatible**. The graphs of inconsistent equations are parallel lines, since they have no point of intersection.

On the other hand, the system consisting of equation (5) and equation

$$(6) \qquad 2x + 2y = 10$$

have all their solutions in common. That is, they represent the same line, since any pair of numbers satisfying (5) also satisfies (6) because  $2x + 2y = 2(x + y)$ . Two linear equations in  $x$  and  $y$  which represent the same line are called **dependent** on each other.

### EXERCISES

*Decide which of the following systems are simultaneous, inconsistent, or dependent; solve if possible, and check by graphing each pair of equations:*

$$1. \begin{cases} 2x + 3y = 4, \\ 4x + 6y = 5. \end{cases}$$

$$2. \begin{cases} 3x - y = 4, \\ 9x - 3y = 8. \end{cases}$$

$$3. \begin{cases} x - 3y = 5, \\ 2x - 6y = 10. \end{cases}$$

$$4. \begin{cases} 2x + 5y = 7, \\ 6x + 15y = 21. \end{cases}$$

$$5. \begin{cases} 2x - 3y = 4, \\ 4x - 9y = 8. \end{cases}$$

$$6. \begin{cases} 3x + y = 3, \\ 6x + 3y = 6. \end{cases}$$

$$7. \begin{cases} 3x - y = 7, \\ 6x + 3y = 9. \end{cases}$$

$$8. \begin{cases} 2x + 5y = 1, \\ 5y - 2x = 9. \end{cases}$$

$$9. \begin{cases} 4x - 6y = 2, \\ 3y = 2x - 1. \end{cases}$$

$$10. \begin{cases} 2x + 5y = 1, \\ 15y = 3 - 6x. \end{cases}$$

$$11. \begin{cases} 6x + 4y = 8, \\ 2y = 2 - 3x. \end{cases}$$

$$12. \begin{cases} 4x - 8y = 6, \\ 4y = x - 1.5. \end{cases}$$

\*13. Show that the equations

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

are dependent if  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ . (Hint: Let the value of this common ratio be  $k$ .)

\*14. Show that the equations in exercise 13 are inconsistent if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

15. Solve each of the systems in exercises 1–10, section 59, graphically, and check by solving algebraically.

61. Systems of three linear equations in three unknowns. If the system of three linear equations

\* May be omitted without disturbing continuity.



$$\begin{cases} a_1x + b_1y + c_1z = k_1, \\ a_2x + b_2y + c_2z = k_2, \\ a_3x + b_3y + c_3z = k_3, \end{cases}$$

in  $x$ ,  $y$ , and  $z$  has a unique common solution, as is usually the case, it may be found by an extension of the methods used in section 59. To solve a system of three linear equations in three unknowns, we may proceed as follows:

(a) Between any two of the three equations, one unknown may be eliminated, resulting in an equation in the other two unknowns.

(b) Between any other pair of the three equations, the same unknown may be eliminated, resulting in a second equation in the other two unknowns.

(c) The two equations in the same two unknowns obtained in steps (a) and (b) may be solved.

(d) The values obtained in step (c) may be substituted in any one (the simplest) of the three given equations, which then may be solved for the third unknown.

(e) The solution should be checked by substituting in the three given equations.

*Example.* Solve the system

$$\begin{aligned} (1) & \quad x + 2y + 3z = 9, \\ (2) & \quad 2x - y + 2z = 11, \\ (3) & \quad 3x + 4y - 2z = -4. \end{aligned}$$

(a) Multiplying equation (1) by 2 and equation (2) by  $-3$ , we get

$$\begin{aligned} (4) & \quad 2x + 4y + 6z = 18, \\ (5) & \quad -6x + 3y - 6z = -33. \end{aligned}$$

Adding (4) and (5), we eliminate  $z$ , obtaining

$$(6) \quad -4x + 7y = -15.$$

(b) Adding (2) and (3), we eliminate  $z$ , obtaining

$$(7) \quad 5x + 3y = 7.$$

(c) Solving (6) and (7) by the methods of section 59, we get

$$(8) \quad x = 2 \text{ and } y = -1.$$

(d) Substituting in (1), we get  $2 - 2 + 3z = 9$ , or

$$(9) \quad z = 3.$$



(e) Substituting (8) and (9) in the three given equations, we find that they check.

The same procedure can be used to solve a system of 4 linear equations in 4 unknowns, and so on, by successively reducing the number of equations and the number of unknowns by one. Thus a system of 4 equations in 4 unknowns may be reduced to a system of 3 equations in 3 unknowns by eliminating the same unknown between 3 different pairs of the 4 given equations. The resulting system of 3 equations may be solved for the 3 unknowns as in the above example.

Another method for dealing with systems of linear equations will be taken up in Chapter XVI, where the theory of such systems will be discussed in greater detail.

## EXERCISES

*Solve and check:*

$$1. \begin{cases} x + y + z = 6 \\ 2x - y + 3z = 9 \\ 3x + 2y - 2z = 1. \end{cases}$$

$$3. \begin{cases} x + z + 4 = 0 \\ x + y = 1 \\ 3x + 2y - z = 4. \end{cases}$$

$$5. \begin{cases} u + v + w = 0 \\ 2u - v + 5w = 8 \\ -u - 2v + w = 0. \end{cases}$$

$$7. \begin{cases} 3A + B + C = 4 \\ 2C - 3B = 2A + 5 \\ 2A - 3C = B - 3. \end{cases}$$

$$9. \begin{cases} x + y - z = a \\ x - y + z = 3a \\ -x + y + z = 5a. \end{cases}$$

$$11. \begin{cases} x + y + z = 2 \\ 2x - z + t = 3 \\ 3x + 2y + t = 4 \\ 3y + 2z + 2t = 7. \end{cases}$$

$$13. \begin{cases} x - z = t \\ 2x + y + 2u - 3t = 2 \\ 3z + u + t = 2 \\ x + z = y \\ x + y + z + t + u = 5. \end{cases}$$

$$2. \begin{cases} x + y + z = 1 \\ x - 2z = 0 \\ y + 2z = 0. \end{cases}$$

$$4. \begin{cases} 2r - 3s + t = 8 \\ r + 3s + 6t = -3 \\ 2r - s + 2t = 2. \end{cases}$$

$$6. \begin{cases} 2u - 3v + 2w = 0 \\ 4u + v - w + 3 = 0 \\ 2u - v + 2w = 2. \end{cases}$$

$$8. \begin{cases} 3x + 3y + 2z = 7 \\ 2x - y + 2z = 4 \\ x - 5y + z = \frac{1}{2}. \end{cases}$$

$$10. \begin{cases} x + y + z = 0 \\ x - y + z = 2(c - b) \\ x + y - z = 2(a - c). \end{cases}$$

$$12. \begin{cases} 2t - 2x + y + 4 = 0 \\ 4t = y - z - 3 \\ x + 2y = 3z - 14 \\ 6t + 13 = 3x - 2y. \end{cases}$$

$$14. \begin{cases} x + y + z + t + u = 3 \\ 2x + t = 0 \\ 2x - u = 0 \\ x - y - z - t = 2 \\ -z + u - t = 2. \end{cases}$$

62. Systems of equations reducible to linear equations. Equations which are not linear themselves may lead to linear equations.



*Example 1.* Solve the system

$$\begin{cases} \frac{y-x}{2x-4} = \frac{1}{2} \\ \frac{x+3}{xy+4x} = \frac{1}{y} \end{cases}$$

*Solution.* Clearing fractions in each equation by multiplying both sides by the L.C.D., we get

$$\begin{cases} 2y - 2x = 2x - 4 \\ xy + 3y = xy + 4x, \end{cases}$$

or,

$$\begin{cases} 4x - 2y = 4 \\ 4x - 3y = 0. \end{cases}$$

Solving this system, we get  $x = 3$ ,  $y = 4$ , which checks in both given equations.

If the given equations are linear in the reciprocals of the unknowns, it is better not to clear fractions but rather to regard the reciprocals as the unknowns temporarily.

*Example 2.* Solve the system

$$\begin{aligned} (1) \quad & \left\{ \frac{1}{x} + \frac{2}{y} = \frac{5}{6} \right. \\ (2) \quad & \left. \frac{3}{x} - \frac{1}{y} = \frac{3}{4} \right. \end{aligned}$$

*Solution.* Multiplying (2) by 2 and adding to (1), we get  $\frac{7}{x} = \frac{28}{12}$ , or  $x = 3$ . Multiplying (1) by 3 and subtracting (2), we get  $\frac{7}{y} = \frac{7}{4}$ , or  $y = 4$ . The reader should check by substituting in both equations.

*Note.* If desired, the solution of example 2 may be obtained by letting  $u = \frac{1}{x}$  and  $v = \frac{1}{y}$ . Then (1) and (2) become

$$\begin{cases} u + 2v = \frac{5}{6} \\ 3u - v = \frac{3}{4} \end{cases}$$

respectively. Solving this system for  $u$  and  $v$ , we get  $u = 1/3$ ,  $v = 1/4$ . Hence  $x = 3$  and  $y = 4$ .



## EXERCISES

$$1. \begin{cases} \frac{6x-4}{12x+3} = \frac{2y+2}{4y-7} \\ \frac{4x+5}{2y+3} = \frac{8x+1}{4y+7} \end{cases}$$

$$3. \begin{cases} \frac{x-2}{x-1} = \frac{y-1}{y+1} \\ \frac{x+2}{x-1} = \frac{y-1}{y-3} \end{cases}$$

$$5. \begin{cases} \frac{2x-2}{3y+6} = \frac{2x+3}{3y-9} \\ \frac{2y}{3x+11} = \frac{4y-3}{6x+16} \end{cases}$$

$$7. \begin{cases} \frac{4}{x} - \frac{3}{y} = 1 \\ \frac{12}{x} + \frac{9}{y} = 9 \end{cases}$$

$$9. \begin{cases} \frac{6}{x} - \frac{7}{y} = 4 \\ \frac{2}{x} + \frac{3}{y} = -4 \end{cases}$$

$$11. \begin{cases} \frac{4}{x} - \frac{2}{y} = 14 \\ \frac{7}{x} + \frac{3}{y} = 5 \end{cases}$$

$$13. \begin{cases} \frac{1}{x} - \frac{1}{y} = 1 \\ \frac{3}{x} + \frac{7}{y} = 5 \end{cases}$$

$$15. \begin{cases} \frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 9 \\ \frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 11 \\ \frac{3}{x} + \frac{4}{y} - \frac{2}{z} = -4 \end{cases}$$

$$17. \begin{cases} \frac{1}{x} + \frac{1}{z} = -4 \\ \frac{2}{x} + \frac{1}{y} = 0 \\ \frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 4 \end{cases}$$

$$2. \begin{cases} \frac{4x+5}{6x-2y} = 2 \\ \frac{2x-3y}{4x+7y} = \frac{1}{15} \end{cases}$$

$$4. \begin{cases} \frac{x-4}{y+2} = \frac{x}{y-2} \\ \frac{x-4}{y-4} = \frac{x-3}{y-2} \end{cases}$$

$$6. \begin{cases} \frac{9x-y-8}{7x+2y-9} = \frac{3}{4} \\ 2x-5y = -25 \end{cases}$$

$$8. \begin{cases} \frac{6}{x} + \frac{1}{y} = 1 \\ \frac{9}{x} - \frac{2}{y} = 5 \end{cases}$$

$$10. \begin{cases} \frac{10}{x} + \frac{2}{y} = -1 \\ \frac{4}{x} - \frac{3}{y} = 3 \end{cases}$$

$$12. \begin{cases} \frac{2}{x} + \frac{3}{y} = 8 \\ \frac{1}{x} - \frac{2}{y} = -10 \end{cases}$$

$$14. \begin{cases} \frac{1}{x} - \frac{2}{y} = 4 \\ \frac{3}{x} + \frac{2}{y} = 4 \end{cases}$$

$$16. \begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 6 \\ \frac{2}{x} - \frac{1}{y} + \frac{3}{z} = 9 \\ \frac{3}{x} + \frac{2}{y} - \frac{2}{z} = 1 \end{cases}$$

$$18. \begin{cases} \frac{2}{x} - \frac{3}{y} + \frac{1}{z} = 8 \\ \frac{1}{x} + \frac{3}{y} + \frac{6}{z} = -3 \\ \frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 2 \end{cases}$$



$$19. \begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3 \\ \frac{2}{x} - \frac{1}{y} + \frac{3}{z} = -5 \\ \frac{3}{x} + \frac{2}{y} + \frac{4}{z} = 4. \end{cases}$$

$$20. \begin{cases} \frac{2}{x} - \frac{3}{y} + \frac{2}{z} = 0 \\ \frac{4}{x} + \frac{1}{y} - \frac{1}{z} = -3 \\ \frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 2. \end{cases}$$

$$21. \begin{cases} \frac{2}{x-1} + \frac{3}{y-1} = 4 \\ \frac{3}{x-1} + \frac{1}{y-1} = 13. \end{cases}$$

$$22. \begin{cases} \frac{2}{x-1} + \frac{5}{y+1} = 4 \\ \frac{1}{x-1} + \frac{2}{y+1} = 1. \end{cases}$$

$$23. \begin{cases} \frac{1}{3(x-1)} - \frac{1}{2(y+1)} = \frac{7}{6} \\ \frac{1}{3(x-1)} - \frac{1}{6(y+1)} = \frac{1}{6}. \end{cases}$$

$$24. \begin{cases} \frac{1}{x-3} + \frac{3}{4(y-2)} = \frac{1}{4} \\ \frac{1}{4(x-3)} + \frac{1}{2(y-2)} = 1. \end{cases}$$

**63. Verbal problems.** Many problems written in prose, practical and otherwise, lead to linear equations or systems of linear equations. The most important part of the work of solving such a problem is that of translating it from the clumsy language of everyday prose into the more convenient language of algebraic symbolism. Hence for each problem the student should write out a careful list or vocabulary of all symbols to be used and express all quantities to be considered in terms of those symbols.\* If you fail to write down what your symbols stand for, you are writing in a secret code, thus defeating the purpose of written language, which is to communicate ideas clearly. In particular, be careful to make note of the units in which the quantities mentioned are expressed and to convert them to similar units before working out the problem.

The following procedure should be used in solving all problems:

- (a) Read the problem, to get the general idea.
- (b) Read it again, paying attention to the relationships among the various quantities involved.
- (c) Choose symbols for what appear to be the basic quantities which have to be found (the unknowns).

\* Various schemes of arrangement of this vocabulary on the page, such as "boxes," or other bookkeeping devices, seem to be popular. While they may save a few seconds in the solution of certain types of problems, they are restricted to certain types and tend in general to hinder the reasoning even for the types to which they apply. It is better, in the author's opinion, for the student to learn to translate problems into algebra apart from classifying them into types, so that he will become unafraid to attack an unfamiliar problem.



(d) Read the problem again, phrase by phrase, and write down the list or vocabulary of all quantities involved together with their expressions in terms of the unknowns chosen in step (c).

(e) Read the problem again, paying careful attention to the parts which express relationships among the quantities which can be translated into equations.

(f) When you have as many equations as unknowns, solve the equations.

(g) Check the answers against the language of the original verbal problem, *not in the equations obtained in step (e)*.

Step (f) may take the most room on paper, but steps (a)–(e) merit at least as much attention.

*Example 1.* A man is 24 years older than his son. Eight years ago, he was twice as old as his son was. What are their present ages?

*First solution.* Let  $x$  be the age of the son now, expressed in years. Then  $x + 24$  is the father's age now, in years. Eight years ago the son was  $x - 8$  years old, and the father was  $x + 24 - 8 = x + 16$  years old. The second sentence of the problem may be translated into the equation

$$x + 16 = 2(x - 8).$$

Solving this equation, we get  $x = 32$ , the son's present age. Hence  $x + 24 = 56$ , the father's present age. Checking, we find that 8 years ago the son was 24 and the father 48, or twice 24.

*Second solution.* Let  $x$  be the father's age now, in years, and let  $y$  be the son's age now, in years. Eight years ago the father was  $x - 8$  years old and the son was  $y - 8$  years old. Then the first sentence of the problem may be translated into the equation

$$(1) \quad x = y + 24.$$

The second sentence of the problem may be translated into the equation

$$(2) \quad x - 8 = 2(y - 8).$$

Solving the system of equations (1) and (2), we get  $x = 56$  and  $y = 32$ . The check is as before.

*Example 2.* A can do a job in 6 days alone, and B can do it alone in 10 days. How long will it take them to do the job together?

*Solution.* Let  $x$  be the number of days required for both to do the job together. In one day, A can do  $1/6$  of the job, B can do



$1/10$  of the job, and both together can do  $1/x$  of the job. Therefore, we have the equation

$$\frac{1}{6} + \frac{1}{10} = \frac{1}{x}.$$

To solve this equation, we multiply both sides by the L.C.D.  $30x$ , obtaining  $5x + 3x = 30$  or  $x = 30/8 = 15/4 = 3\frac{3}{4}$  days. The check is left to the student.

*Example 3.* How much acid must be added to 48 grams of a solution 75% pure in order to obtain a new solution 76% pure?

*Solution.* Let  $x$  be the number of grams of acid to be added. The total weight of the new solution therefore will be  $48 + x$  grams. The number of grams of acid in the original solution is  $0.75 \times 48 = 36$ . Hence the number of grams of acid in the new solution is  $36 + x$ . Since the new solution is to be 76% pure, we have the equation

$$36 + x = 0.76(48 + x).$$

Solving the equation, we get  $36 + x = 36.48 + 0.76x$ , or  $0.24x = 0.48$ , or  $x = 2$  grams. The check is left to the student.

*Example 4.* A boat, operating uniformly at full power, goes 5 miles downstream in 60 minutes and returns in 90 minutes. What would be the speed of the boat in still water, and what is the rate of the current?

*Solution.* Let  $x$  be the speed of the boat in still water, and let  $y$  be the speed of the current, measured in miles per hour. Then the speed of the boat downstream is  $x + y$  and the speed upstream is  $x - y$ .\* Since rate times time is distance, for uniform motion, we have the equations

$$\begin{aligned} 1 \cdot (x + y) &= 5 \\ 1.5 \cdot (x - y) &= 5. \end{aligned}$$

Solving this system of equations, we find that  $x = 4\frac{1}{6}$  miles per hour and  $y = \frac{5}{6}$  miles per hour. The check is left to the student.

One reason for checking against the wording of the problem and not against the equation (or equations) is that the student may have solved the equation(s) correctly but may have formed the wrong equation(s). Another reason, apart from mistakes, is that

\* This sentence idealizes the actual situation and is in fact only a crude first approximation to the truth. For a more accurate discussion, one would have to take account of the shape of the boat and the advanced theory of the dynamics of fluids. Therefore we shall always make the above idealization here in all similar problems; it is good enough for our purpose.



a number may satisfy the equation but not the problem; for when we form the equation we assert that *if* the unknowns satisfy the conditions of the problem, *then* they must satisfy the equation. But this does *not* imply the converse proposition, that *if* a number satisfies the equation, *then* it must satisfy the problem. (The converse of a correct proposition may be incorrect.) Thus the solutions of the equation are the only *possible* solutions of the problem, but whether they are in fact *solutions* of the problem remains to be seen by checking. Thus checking the problem is an essential step.

*Note.* In practice, it can even happen that the problem asked is insoluble, although the resulting equation may have a solution. The information that the problem is insoluble may well be valuable.

*Example 5.* An absent-minded professor has a certain number of students in his class, but has forgotten how many. However, he does remember clearly that if he had three times as many, he would have seven more than he now has. How many students has he?

*Solution.* Let  $x$  be the number of students he has. Then  $3x = x + 7$ . Solving this equation, we get  $x = 7/2$ , or  $3\frac{1}{2}$  students. While  $7/2$  does check in the equation, it cannot possibly be an answer to the problem since the answer must be an integer, of necessity. But our reasoning has proved that  $7/2$  is the only possible answer. Hence the problem has no solution, and we can only conclude that the professor's memory is worse than we thought.

A more significant example of this phenomenon will be given in section 81.

To demonstrate the superiority of algebraic methods over those of arithmetic, one has only to try to solve the problems in the exercises below without using algebraic symbols. Until the 16th century no adequate algebraic symbolism was used.

## EXERCISES

1. The sum of two numbers is 30. Five times the smaller exceeds twice the larger by 24. Find the numbers.
2. A man's age is now four-sevenths of what it will be 15 years from now. How old is he now?
3. One-third of a man's age 6 years ago equals one-fifth of his age 10 years hence. How old is he now?
4. The sum of two numbers is 66 and their difference is 8. Find the numbers.



5. A collection of nickels and quarters totals \$2.80. The total number of coins is 20. How many of each kind are there?

6. A collection of nickels and quarters totals \$4.10. The total number of coins is 30. How many of each kind are there?

7. A rectangular field is four times as long as it is wide. If it were 5 feet shorter and 2 feet wider, its area would be increased by 20 square feet. Find its length and width.

8. A rectangle is 30 feet longer than it is wide. If its length were increased by 50 feet and its width were diminished by 8 feet, its area would be increased by 200 square feet. Find its dimensions.

9. A man invests part of \$5000 at 4% interest and the rest at 6%. His total annual income is \$280. How much has he invested at each rate?

10. A student has quiz grades of 62 and 78. What grade must he achieve on a third quiz in order to have an average of 80?

11. A student has an average grade of 78 on three quizzes. What grade must he achieve on a fourth quiz in order to have an average of 80?

12. A lady goes shopping and spends half her money in the first shop she visits. In the second shop she spends  $\frac{1}{3}$  of what remains. In the third shop she spends \$5. She then has \$11 left. How much had she at the start?

13. Find three consecutive integers whose sum is 84.

14. Find three consecutive even integers whose sum is 102.

15. Find three consecutive odd integers whose sum is 117.

16. Three years from now a man will be three times as old as his son will be. Two years ago he was four times as old as his son was. How old are they now?

17.  $A$  can do a job alone in 6 days and  $B$  can do it alone in 4 days. How long will it take them to do the job together?

18.  $A$  can do a job alone in 6 days,  $B$  can do it alone in 4 days, and  $C$  can do it alone in 8 days. How long will it take them to do the job together?

19.  $A$  can do a job in 6 days, and  $A$  and  $B$  together can do the job in 2 days. How long would it take  $B$  to do it alone?

20. One input pipe can fill a tank alone in 6 hours, another input pipe can fill it alone in 4 hours, and a drain pipe can empty the full tank alone in 8 hours. If the tank is empty and all three pipes are wide open, how long will it take to fill the tank?

21. How many gallons of cottonseed oil must be added to 45 gallons of a solution of olive oil which is 90% olive oil in order to make the resulting solution 80% olive oil?

22. A mixture contains 3 pounds of sugar and 5 pounds of flour. How much flour must be added to make a mixture that is 75% flour?

23. A freight train, running at the rate of 20 miles per hour, leaves a station 3 hours before an express, which travels in the same direction



at the rate of 50 miles per hour. How long after the express leaves will it catch the freight, and how far from the station will they be?

24. Achilles races with a tortoise, giving the tortoise a handicap of 990 yards. Achilles runs at the rate of 500 yards per minute while the tortoise runs at the rate of 5 yards per minute. How long will it take Achilles to catch the tortoise?

25. How much water must be evaporated from 100 pounds of a 12% solution of salt in order to obtain a 20% solution?

26. How many ounces of each of a 50% solution and a 10% solution must be mixed to obtain 24 ounces of a 25% solution?

27. Two boys balance one another on a teeterboard. The first boy weighs 60 pounds and is 4 feet from the fulcrum. The second boy is 3 feet from the fulcrum. How much does the second boy weigh? (*Hint:* If weights  $w_1$  and  $w_2$  are on opposite sides of the fulcrum of a teeterboard or lever, at distances  $d_1$  and  $d_2$  respectively from the fulcrum, then  $w_1d_1 = w_2d_2$ .)

28. How far from the fulcrum of a teeterboard must an 80-pound boy be to balance a 120-pound boy who is 2 feet from the fulcrum?

29. A boat travels 8 miles downstream in 20 minutes and makes the return trip in 30 minutes. Find the speed of the boat in still water and the rate of the current.

30. A man walks 31 miles, partly at the rate of 2 miles per hour and the rest at the rate of 5 miles per hour. If he had walked at the rate of 5 miles per hour for the same time as he actually walked at the rate of 2 miles per hour, and vice versa, he would have covered 15 miles more than he did. How long did it take him to walk the 31 miles?

31. At what time between 3 and 4 o'clock do the hands of a clock coincide?

32. The age of Diophantus, a brilliant Greek mathematician of about 250 A.D., may be calculated from an epitaph which reads as follows: Diophantus passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh more as a bachelor; five years after his marriage was born a son who died four years before his father at half his father's age.

33. If the length of a rectangle were increased by 2 feet and the width decreased by 2 feet, the area would be decreased by 12 square feet. If the length were decreased by 1 foot and the width were increased by 2 feet, the area would be increased by 13 square feet. Find the dimensions of the rectangle.

34. An airplane travels 990 miles in 3 hours with the wind, and makes the return trip against the wind in 3 hours and 40 minutes. Find the speed of the plane in still air and the speed of the wind.

35. A boat travels for 2 hours with a current of 2 miles per hour and returns the same distance against the current in 3 hours. How far did it travel each way and what is its speed in still water?



36. An airplane bucks a head wind of 40 miles per hour for 4 hours and returns the same distance assisted by a tail wind of 40 miles per hour in 3 hours. How far did it fly each way and what is its speed in still air?

37. A confectioner wishes to mix candy worth 70 cents a pound with candy worth 40 cents a pound to make 42 pounds of an assortment worth 50 cents a pound. How much of each kind should he use?

38. Three pounds of flour and 5 pounds of sugar cost 94 cents. Six pounds of flour and 2 pounds of sugar cost 76 cents. Find the cost per pound of flour and of sugar.

39. A certain two-digit number is equal to five times the sum of its digits. If 9 were added to the number, its digits would be reversed. Find the number. (*Hint*: If  $u$  is the units digit and  $t$  is the tens digit, the number is  $10t + u$ .)

40. Find the angles of a triangle if one angle is half the second, and the third is twice the sum of the first two. (*Hint*: the sum of all the angles of a triangle is 180 degrees.)



## CHAPTER IX

### Integral and Fractional Exponents

64. Negative integers and zero as exponents. In section 12 and section 23, we defined positive integral exponents as follows:

DEFINITION 1. If  $n$  is any positive integer,  $x^n = xx \cdots x$  where there are  $n$  factors on the right of the equals sign.

With this definition, we proved, in section 23, that positive integral exponents obey the following laws:

- I.  $x^a x^b = x^{a+b}$ . Example.  $x^2 x^3 = xx \cdot xxx = x^5$ .  
 II.  $(x^a)^b = x^{ab}$ . Example.  $(x^2)^3 = (x^2)(x^2)(x^2) = x^6$ .  
 III.  $(xy)^a = x^a y^a$ . Example.  $(xy)^3 = xyxyxy = xxxyyy = x^3 y^3$ .  
 IV. If  $y \neq 0$ ,  
 $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ . Example.  $\left(\frac{x}{y}\right)^3 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x^3}{y^3}$ .  
 V. If  $x \neq 0$ ,  
 (i)  $\frac{x^a}{x^b} = x^{a-b}$  if  $a > b$ . Example.  $\frac{x^5}{x^2} = \frac{xxxxx}{xx} = x^3$ .  
 (ii)  $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$  if  $a < b$ . Example.  $\frac{x^2}{x^5} = \frac{xx}{xxxxx} = \frac{1}{x^3}$ .  
 (iii)  $\frac{x^a}{x^b} = 1$  if  $a = b$ . Example.  $\frac{x^2}{x^2} = 1$ .

The occurrence of three different cases in V is a source of inconvenience, especially if we wish to use literal exponents rather than particular numbers, and it is natural to wish to unify them.

Let us boldly use  $V(i)$  mechanically even when  $a < b$ . For example,  $\frac{x^2}{x^5} = x^{2-5} = x^{-3}$ . But a negative exponent has no meaning under definition 1. The symbol  $x^{-3}$  cannot possibly be interpreted as the product of  $-3$   $x$ 's, except possibly at a séance; but  $x^{-3}$  will not stand for the product of the ghosts of 3 departed  $x$ 's. The symbol  $x^{-3}$  does not come under the jurisdiction of definition 1 at all and is at present a completely undefined symbol. This is fortunate because, since  $x^{-3}$  has no preassigned meaning, we are free to give it any meaning that suits our convenience without fear of contradicting previous results. It will be convenient to



assign to the symbol  $x^{-3}$  as its meaning the right answer to our problem, that is  $\frac{1}{x^3}$ , for then we can dispense with  $V(ii)$  altogether.

Hence we are led to make the following definition.

**DEFINITION 2.** *If  $-n$  is a negative integer,*

$$x^{-n} = \frac{1}{x^n} \quad (x \neq 0).$$

*Example 1.*  $\frac{x^2}{x^5} = x^{-3} = \frac{1}{x^3}.$

*Example 2.*  $2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$

Similarly let us apply  $V(i)$  when  $a = b$ . For example, we may write  $\frac{x^2}{x^2} = x^{2-2} = x^0$ . But the symbol  $x^0$  has no sense under either definition 1 or definition 2, since 0 is neither positive nor negative. This is again fortunate, since we are free to assign to the symbol  $x^0$  any meaning that suits our convenience. Again we choose the right answer, 1, since this allows us to use  $V(i)$  in all cases. Hence we make the following definition.

**DEFINITION 3.** *If  $x \neq 0$ ,  $x^0 = 1$ .*

*Example 3.*  $5^0 = 1.$

We can now replace  $V(i)(ii)(iii)$  above by the single law:

$$V. \frac{x^a}{x^b} = x^{a-b}, \quad (x \neq 0).$$

It can be proved that the laws *I, II, III, IV*, and the new *V* hold for all integral exponents, positive, negative, or zero.\*

## EXERCISES

*Simplify each of the following by removing non-positive exponents:*

1.  $2x^{-3}.$

2.  $(2x)^{-3}.$

3.  $3x^0.$

4.  $(3x)^0.$

5.  $2x^{-2}y^3.$

6.  $2a^{-3}b^0.$

7.  $a^{-1}, b^{-1}.$

8.  $a^{-1}b^{-1}.$

9.  $\frac{a^{-1} + b^{-1}}{a^{-1} - b^{-1}}.$

10.  $\frac{x^{-1}}{x^{-1} + y^{-1}}.$

11.  $\frac{2a^{-1}}{2a^{-1} - b^{-1}}.$

12.  $\frac{x}{y^{-1}} + \frac{y}{x^{-1}}.$

\* For indications of the proof, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, pp. 178-9



*Evaluate each of the following:*

13.  $2^{-3}$ .      14.  $6^0$ .      15.  $\left(\frac{1}{2}\right)^{-3}$ .      16.  $(-2)^{-3}$ .      17.  $\left(\frac{1}{2}\right)^0$ .  
 18.  $2 \cdot 3^{-1}$ .      19.  $(2 \cdot 3)^{-1}$ .      20.  $3 \cdot 10^0$ .      21.  $(3 \cdot 10)^0$ .

*Simplify, using the laws I–V for exponents:*

22.  $(3x^2y^3)(2x^{-3}y^{-2})$ .      23.  $(x^{-2})^{-3}$ .      24.  $\frac{2x^3y^2}{4x^{-2}y^{-2}}$ .  
 25.  $\frac{(xy^2)^{-3}}{(x^2y)^{-2}}$ .      26.  $x^{-2}(x^6 + 2x^3 + 3x^2)$ .  
 27.  $\frac{(ab^2)^{-3}(a^{-2}b^{-1})^2}{(a^3b^4)^{-1}}$ .      28.  $10^3 \cdot 10^0 \cdot 10^{-2}$ .      29.  $\frac{10^4 \cdot 10^{-6}}{10^{-3}}$ .  
 30.  $\frac{10^7 \cdot 10^{-5}}{10^2}$ .      31.  $(2x^0y^2)^3(y^{-2})^2(x^2y^2)^0$ .  
 32.  $\frac{(3 \cdot 10^4)(4 \cdot 10^{-1})}{6 \cdot 10^2}$ .      33.  $\frac{(2 \cdot 10^3)^2(6 \cdot 10^{-1})^2}{2 \cdot 10^3}$ .

*Simplify:*

34.  $\frac{a^{-1} - b^{-1}}{a^2 - b^2}$ .      35.  $(x^{-1} + y^{-1})(x + y)^{-1}$ .  
 36.  $\frac{1}{x^{-1} + y^{-1}}$ .      37.  $\frac{1}{x^{-2} - y^{-2}}$ .

**65. A use of powers of ten in scientific writing.** When very large or very small numbers are used in scientific writing, it is customary to express them in terms of powers of ten, as follows. We write  $1,000,000 = 10^6$ ,  $256,000,000 = 256 \times 10^6$  or  $25.6 \times 10^7$  or  $2.56 \times 10^8$ ; we use the  $\times$  sign for multiplication to avoid confusion with the decimal point. Similarly, we write  $0.0001 = 10^{-4}$ ,  $0.000256 = 256 \times 10^{-6}$  or  $25.6 \times 10^{-5}$  or  $2.56 \times 10^{-4}$ . Multiplying a number by  $10^4$  moves the decimal point four places to the right, while multiplying by  $10^{-4}$  moves the decimal point four places to the left.

It can be proved that every positive real number can be expressed as a number between 1 and 10 multiplied by a suitable integral power of 10; when this is done, the number is said to be expressed in standard form.

For example, 256,000,000 has the standard form  $2.56 \times 10^8$ ; 0.000256 has the standard form  $2.56 \times 10^{-4}$ ; and 2.56 has the standard form  $2.56 \times 10^0$ .

This standard form is more compact and comprehensible than



the everyday way of writing such numbers with large numbers of zeros before or after the decimal point. It also lends itself to calculations with cumbersome numbers, large or small, which are made easier by the use of the laws of exponents.

*Example.* Calculate  $\frac{256,000,000 \times 0.0004}{2000}$ . We write

$$\frac{2.56 \times 10^8 \times 4 \times 10^{-4}}{2 \times 10^3} = 5.12 \times 10^{8-4-3} = 5.12 \times 10^1 = 51.2.$$

While the powers of ten are combined according to the laws of exponents, their coefficients are multiplied and divided in the ordinary way. How this may be avoided by expressing them too as (fractional) powers of ten will be discussed in the chapter on logarithms.

## EXERCISES

*Express each of the following in standard form:*

1. 300,000.      2. 240,000.      3. 65,700,000.      4. 234,000,000.
5. 0.341.      6. 0.0357.      7. 3.52.      8. 41.3.
9. 0.0569.      10. 0.00034.      11. 0.000000561.
12. 0.00000000341.

*Express each of the following in ordinary decimal (positional) notation:*

13.  $10^6$ .      14.  $10^8$ .      15.  $10^{-4}$ .      16.  $10^{-7}$ .      17.  $6.54 \times 10^7$ .
18.  $5.63 \times 10^9$ .      19.  $3.42 \times 10^{-5}$ .      20.  $8.72 \times 10^{-3}$ .
21.  $7.53 \times 10^{-6}$ .      22.  $8.54 \times 10^0$ .

*Calculate, using the laws for exponents, and express the result in ordinary positional (decimal) notation:*

23.  $\frac{(8 \times 10^7) \times (3 \times 10^{-2})}{2 \times 10^4}$ .      24.  $\frac{(5.2 \times 10^6) \times (4 \times 10^{-8})}{13 \times 10^{-4}}$ .

25. If  $c$  represents the velocity of radiant energy in a vacuum,  $L$  its wave length, and  $f$  its frequency, then  $f = c/L$ . Suppose  $c = 3 \times 10^{10}$  centimeters per second and  $L = 6 \times 10^{-5}$  centimeters. Find  $f$ . (*Hint: write  $3 \times 10^{10} = 30 \times 10^9$ .*)

26. An angstrom is a unit of length equal to  $10^{-8}$  centimeters. If one centimeter equals 0.3937 inch, express in inches the wave length of red light with a wave length of 8000 angstroms.

27. A light-year is a unit of length equal to  $5.88 \times 10^{12}$  miles. The distance of the cluster of stars called the Pleiades is  $1.2936 \times 10^{15}$  miles. How many light-years is this?



28. The Great Nebula in Andromeda is approximately  $5.292 \times 10^{22}$  miles away. How many light-years is this? Use the data of exercise 27.

29. If a gram equals 0.002205 pound and the mass of the earth is  $5.97 \times 10^{27}$  grams, find the mass of the earth in pounds.

30. The mass of the sun is  $1.98 \times 10^{33}$  grams. How many pounds is this? Use the data of exercise 29.

31. One coulomb equals  $3.00 \times 10^9$  statecoulombs. The charge on an electron is  $4.80 \times 10^{-10}$  statecoulombs. How many coulombs is this?

**66. Fractional exponents.** In section 64 we extended the concept of exponent to include all integers as exponents. It is natural to try to extend it still further, to include fractions. In making such an extension (since, if we do it at all, we do it for our own convenience) we wish to preserve the validity of the basic laws *I–V* of section 64; just as in extending the concept of number itself, in chapter I, from positive integers to other kinds of numbers, we wished to preserve the validity of the basic laws such as the associative, distributive, and commutative laws.

The symbol  $x^{\frac{1}{2}}$  has no meaning under any of the previous definitions of exponent, since  $\frac{1}{2}$  is not an integer. Therefore we are free to give it any meaning that suits our convenience. If law *II* (section 64) is to operate, then  $(x^{\frac{1}{2}})^2$  will have to yield  $x^{\frac{1}{2} \cdot 2} = x^1 = x$ .

Hence  $x^{\frac{1}{2}}$  will have to stand for some quantity which when squared yields  $x$ ; but this can only be  $+\sqrt{x}$  or  $-\sqrt{x}$ . Hence we choose

the definition:  $x^{\frac{1}{2}} = \sqrt{x}$ . This fits also with law *I* (section 64), for  $x^{\frac{1}{2}} x^{\frac{1}{2}} = x^{\frac{1}{2} + \frac{1}{2}} = x^1 = x$  represents the fact that  $\sqrt{x} \sqrt{x} = x$ . More

generally we define  $x^{\frac{1}{q}} = \sqrt[q]{x}$ , so that law *II* will operate correctly; thus,  $(x^{\frac{1}{q}})^q = x^{\frac{1}{q} \cdot q} = x^1 = x$ . Finally, if law *II* is to operate, we must have

$$x^{p/q} = x^{\frac{1}{q} \cdot p} = (x^{\frac{1}{q}})^p = (\sqrt[q]{x})^p$$

and

$$x^{p/q} = x^{p \cdot \frac{1}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}.$$

Thus we are led to make the following definition.

**DEFINITION 4.** If  $p/q$  is any rational number, expressed with positive denominator  $q$ , we define  $x^{p/q} = (\sqrt[q]{x})^p = \sqrt[q]{x^p}$ .



Example 1.  $x^{\frac{1}{2}} = \sqrt{x} = (\sqrt{x})^2$ . In particular,  $8^{-\frac{2}{3}} = \sqrt[3]{8^{-2}} = \sqrt[3]{\frac{1}{64}} = \frac{1}{4}$  or  $8^{-\frac{2}{3}} = (\sqrt[3]{8})^{-2} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ .

Note that any rational number can be expressed with positive denominator, as  $\frac{3}{-2} = -\frac{3}{2}$ , so that this is no restriction.

It can now be proved\* that fractional exponents obey laws I-V, and also that any fractional exponent may be replaced by an equal fraction. Thus  $x^{\frac{2}{3}} = x^{\frac{4}{6}}$ .

It is desirable for the present to confine ourselves to non-negative values of  $x$  in fractional powers of  $x$  in order to avoid difficulties involving imaginary numbers and other difficulties whose explanation is best left for more advanced courses. For example, from  $(-1)^{\frac{1}{2}} = (-1)^{\frac{2}{4}} = (-1)^{\frac{1}{2}}$  we might be tempted to conclude that  $\sqrt{-1} = \sqrt[4]{-1}$  or  $\sqrt{-1} = \sqrt{-1}^2 = 1$ , or  $-1 = +1$ .

Fractional exponents may be used to simplify radical expressions, as in the following examples, but their real importance lies in the fact that they form a basis for the theory of logarithms, discussed in chapter XXIII, which may be taken up next if desired.

Example 2.  $\sqrt[3]{x^4} \cdot \sqrt[3]{x^5} = x^{\frac{4}{3}} \cdot x^{\frac{5}{3}} = x^{\frac{4}{3} + \frac{5}{3}} = x^{\frac{9}{3}} = \sqrt[3]{x^9}$  or  $(\sqrt[3]{x})^9$ .

While positive integral exponents were used by René Descartes (French, 1596-1650), negative and fractional exponents were not used extensively until the time of Isaac Newton (English, 1642-1727). The concept of exponent can be extended further to include all real and even all complex numbers as exponents, but this is beyond the scope of this book.

## EXERCISES

Perform each of the following:

1.  $10^3$
2.  $10^4$
3.  $10^{-3}$
4.  $10^{-4}$
5.  $8^{10}$
6.  $2^{10}$
7.  $9^{10}$
8.  $2^{-10}$
9.  $100^{10}$
10.  $100^{-10}$
11.  $10^{10}$
12.  $100^{-10}$
13.  $17^{10}$
14.  $37^{10}$
15.  $10^3 \times 10^4$
16.  $10^3 \div 10^4$
17.  $\frac{10^3 \times 10^4}{10^5}$
18.  $\frac{10^3 \times 10^4}{10^5}$
19.  $10^{10}$
20.  $10^{10}$
21.  $10^{10}$
22.  $10^{10}$

\*The proof of this is given in § 14.10.101, *The Theory of Logarithms*, which may be taken up next if desired.



*Simplify each of the following:*

23.  $(16a^4)^{1/2}$ .

24.  $(8a^6)^{1/3}$ .

25.  $(x^4x^{1/2})^{-2}$ .

26.  $(x^{1/2}/x^{1/3})^6$ .

27.  $\left(\frac{a^{1/2}b^{4/3}}{a^{-2}b}\right)^3$ .

28.  $(4x^{-2}y^4)^{3/2}$ .

29.  $\left(\frac{9x^3y}{4x^{-1}y^{-3}}\right)^{1/2}$ .

30.  $\left(\frac{27a^{-4}y^6}{a^2y^{-3}}\right)^{2/3}$ .

31.  $\sqrt[4]{x^3}(\sqrt[3]{x})^2$ .

32.  $\sqrt[5]{x^2} \cdot (\sqrt[5]{x})^3$ .

33.  $\frac{(\sqrt[3]{x})^5}{\sqrt[3]{x^2}}$ .

34.  $\sqrt[3]{x^4}/\sqrt[4]{x^3}$ .



## CHAPTER X

### Radicals

**67. Basic laws for radicals.** If  $x^n = a$ , where  $n$  is an integer greater than 1, then  $x$  is called an  $n$ th root of  $a$ . It will be proved in chapter XIV that every complex number has  $n$  distinct  $n$ th roots. To avoid confusion, we single one out, to be called the **principal  $n$ th root**, as follows.

If  $a$  is a positive (real) number, the principal  $n$ th root will be the positive  $n$ th root. If  $n$  is odd and  $a$  is a negative (real) number, the principal  $n$ th root will be the negative  $n$ th root. For the present no other cases will be considered. It can be proved that the principal  $n$ th root is uniquely determined.

The radical  $\sqrt[n]{a}$  shall stand for the principal  $n$ th root. The quantity  $a$  is called the **radicand**,  $n$  is called the **index** or **order** of the radical. The index 2 is usually omitted from the symbol.

*Examples.*  $\sqrt{9} = 3$ , not  $-3$ . To designate  $-3$  we would write  $-\sqrt{9}$ . Thus  $\pm\sqrt{9} = \pm 3$ . Also,  $\sqrt[3]{-8} = -2$ .

Radicals obey the following laws:

- |                                                                                   |                                                                                                                                                                  |
|-----------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>I. <math>(\sqrt[n]{a})^n = a</math>.</p>                                       | <p><i>Example:</i> <math>(\sqrt[3]{6})^3 = 6</math>.</p>                                                                                                         |
| <p>II. <math>\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}</math>.</p>                   | <p><i>Example:</i> <math>\sqrt[3]{24} = \sqrt[3]{8 \cdot 3}</math><br/> <math>= \sqrt[3]{8} \sqrt[3]{3} = 2 \sqrt[3]{3}</math>.</p>                              |
| <p>III. <math>\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}</math>.</p> | <p><i>Example:</i> <math>\sqrt[3]{\frac{9}{8}} = \frac{\sqrt[3]{9}}{\sqrt[3]{8}} = \frac{\sqrt[3]{9}}{2}</math><br/> <math>= \frac{1}{2} \sqrt[3]{9}</math>.</p> |
| <p>IV. <math>\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}</math>.</p>                     | <p><i>Example:</i> <math>\sqrt[4]{\sqrt[3]{a}} = \sqrt[12]{a}</math>.</p>                                                                                        |
| <p>V. <math>\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}</math>.</p>                 | <p><i>Example:</i> <math>\sqrt[4]{a^3} = (\sqrt[4]{a})^3 = a^{3/4}</math>.</p>                                                                                   |

To avoid difficulties with imaginary numbers, we shall hereafter in this chapter suppose that all letters represent positive (real) numbers.

*Proof of I.* By definition.

*Proof of II.* If  $x = \sqrt[n]{a}$  and  $y = \sqrt[n]{b}$ , then  $x^n = a$  and  $y^n = b$ . Thus  $(xy)^n = x^n y^n = ab$ . But  $x$  and  $y$  are positive; hence  $xy$  is positive. Therefore  $xy$  is the principal  $n$ th root of  $ab$ , or  $xy = \sqrt[n]{ab}$ . This completes the proof. If  $a$  or  $b$  or both are negative and  $n$  is odd, a similar proof can be given.



*Proof of III.* Similar to proof of II. We leave this to the student as an exercise.

*Proof of IV.* Let  $x = \sqrt[n]{a}$ . Then  $x^n = a$ . Let  $y = \sqrt[m]{x}$ . Then  $y^m = x$ . Raising both sides to the  $n$ th power, we have  $(y^m)^n = x^n$ , or  $y^{mn} = a$ . But  $y$  is positive and hence is the principal  $m$ nth root of  $a$ , or  $y = \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ .

*Proof of V.* This is left to the student as an exercise.

**68. The simplest form of a radical.** A rational number is called a **perfect  $n$ th power** if it is the  $n$ th power of some rational number. A rational expression is called a **perfect  $n$ th power** if it is the  $n$ th power of some rational expression.

*Examples.*  $\frac{4}{9}$  is a perfect square because  $\frac{4}{9} = \left(\frac{2}{3}\right)^2$ . Similarly,  $8a^6b^9$  is a perfect cube because  $8a^6b^9 = (2a^2b^3)^3$ .

An irrational  $n$ th root of a rational number is sometimes called a **surd**. If the index of the root is 2, the surd is called a **quadratic surd**.

Thus,  $\sqrt{3}$  and  $\sqrt{\frac{1}{3}}$  are quadratic surds.

The following are the usual ways in which radicals are simplified.

(A) *Removal of factors from the radicand.* To remove factors from a radicand, express the radicand as a product of factors which are, as far as possible, perfect  $n$ th powers. Then use law II and extract the  $n$ th roots of all perfect  $n$ th powers.

*Example 1.*  $\sqrt{18a^3b^5} = \sqrt{9a^2b^4 \cdot 2ab} = \sqrt{9a^2b^4} \sqrt{2ab} = 3ab^2 \sqrt{2ab}$ .

*Example 2.*  $\sqrt[3]{-5x^7} = \sqrt[3]{(-1)x^6 \cdot 5x} = \sqrt[3]{(-1)x^6} \sqrt[3]{5x} = -x^2 \sqrt[3]{5x}$ .

It is sometimes possible to compute a root approximately by removing factors as above, and then using table I.

*Example 3.*  $\sqrt{125} = \sqrt{25 \cdot 5} = 5\sqrt{5}$ . From table I we find  $\sqrt{5} = 2.236$  approximately. Hence  $\sqrt{125} = 5(2.236) = 11.180$  approximately.

(B) *Rationalizing the denominator.* It is usually more convenient to remove denominators of radicands, as follows:

*Example 4.*  $\sqrt[3]{\frac{5}{2x^2}} = \sqrt[3]{\frac{5}{2x^2} \cdot \frac{4x}{4x}} = \sqrt[3]{\frac{20x}{8x^3}} = \frac{\sqrt[3]{20x}}{\sqrt[3]{8x^3}} = \frac{\sqrt[3]{20x}}{2x} = \frac{1}{2x} \sqrt[3]{20x}$ .

This process is called **rationalizing the denominator**. To rationalize the denominator of a radicand of index  $n$ , multiply



numerator and denominator of the radicand by the simplest expression that will make the denominator a perfect  $n$ th power. Then use law *III*, as in the above example.

*Example 5.* Compute  $\sqrt{\frac{2}{3}}$  by use of table I.

*Solution.*  $\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3} \cdot \frac{3}{3}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{1}{3} \sqrt{6} = \frac{1}{3} (2.449) = 0.816$

approximately.

(C) *Lowering the index of a radical.* This can sometimes be done by expressing the radicand as a power of some quantity and using law *V*.

*Example 6.*  $\sqrt[6]{27} = \sqrt[6]{3^3} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}.$

A radical will be said to be in **simplest form** if we have

- (a) reduced the radicand to a single fraction in lowest terms;
- (b) rationalized the denominator;
- (c) removed all factors which are perfect  $n$ th powers, where  $n$  is the index of the radical;
- (d) lowered the index of the resulting radical as far as possible.

*Example 7.*  $\sqrt{\frac{1}{a} + (2 + a)} = \sqrt{\frac{1 + 2a + a^2}{a}} = \sqrt{\frac{(1 + a)^2}{a} \cdot \frac{a}{a}}$   
 $= \frac{1 + a}{a} \sqrt{a}.$

## EXERCISES

*Reduce to simplest form:*

- |                                                     |                                         |                                         |                                         |
|-----------------------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|
| 1. $(\sqrt[3]{2})^3.$                               | 2. $(\sqrt[5]{a})^5.$                   | 3. $\sqrt{2} \cdot \sqrt{8}.$           | 4. $\sqrt{5} \cdot \sqrt{5}.$           |
| 5. $\sqrt{3} \sqrt{12}.$                            | 6. $\sqrt{4} \cdot \sqrt{9}.$           | 7. $\sqrt[3]{2} \sqrt[3]{4}.$           | 8. $\sqrt[5]{4} \cdot \sqrt[5]{8}.$     |
| 9. $\frac{\sqrt[3]{24}}{\sqrt[3]{3}}.$              | 10. $\frac{\sqrt[3]{54}}{\sqrt[3]{2}}.$ | 11. $\frac{\sqrt{72}}{\sqrt{2}}.$       | 12. $\frac{\sqrt{24}}{\sqrt{6}}.$       |
| 13. $\sqrt{12}.$                                    | 14. $\sqrt{48}.$                        | 15. $\sqrt{50}.$                        | 16. $\sqrt{98}.$                        |
| 17. $\sqrt{108}.$                                   | 18. $\sqrt{96}.$                        | 19. $\sqrt[3]{32}.$                     | 20. $\sqrt[3]{54}.$                     |
| 21. $\sqrt{\frac{1}{2}}.$                           | 22. $\sqrt{\frac{1}{3}}.$               | 23. $\sqrt{\frac{3}{8}}.$               | 24. $\sqrt{\frac{5}{12}}.$              |
| 25. $\sqrt[3]{\sqrt{2}}.$                           | 26. $\sqrt[3]{\sqrt{3}}.$               | 27. $\sqrt[5]{5^3}.$                    | 28. $\sqrt[3]{7^4}.$                    |
| 29. $\sqrt[5]{8}.$                                  | 30. $\sqrt[12]{27}.$                    | 31. $\sqrt{\frac{1 - \frac{1}{2}}{2}}.$ | 32. $\sqrt{\frac{1 + \frac{1}{2}}{2}}.$ |
| 33. $\sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2}.$ |                                         |                                         |                                         |



34.  $\sqrt{1 - \left(\frac{3}{5}\right)^2}$ .

35.  $\sqrt[3]{-8}$ .

36.  $\sqrt[3]{-64}$ .

37.  $\sqrt[5]{-32}$ .

38.  $\sqrt[5]{-243}$ .

39.  $\sqrt{9x^6y^8}$ .

40.  $\sqrt{16a^{16}x^2}$ .

41.  $\sqrt{4x^{36}y^9}$ .

42.  $\sqrt[3]{-8a^8b^9}$ .

43.  $\sqrt[3]{-27x^8y^6}$ .

44.  $\sqrt[3]{16a^6b^8}$ .

45.  $\sqrt[3]{54x^{10}y^8}$ .

46.  $\sqrt{64x^9y^{16}}$ .

47.  $\sqrt{\frac{a^6}{9b^8}}$ .

48.  $\sqrt{\frac{8x^7}{y^{10}}}$ .

49.  $\sqrt{\frac{25}{4x^6y^7}}$ .

50.  $\sqrt{\frac{16x^5}{y^3}}$ .

51.  $\sqrt{\frac{32x^7}{9y^5}}$ .

52.  $\sqrt{\frac{72x^3y^8}{75z^5}}$ .

53.  $\sqrt[3]{\frac{8x^8}{y^6}}$ .

54.  $\sqrt[3]{\frac{16a^5}{y^8}}$ .

55.  $\sqrt{a^2b^2 + a^2c^2}$ .

56.  $\sqrt{x^2y^2 - x^2z^2}$ .

57.  $\sqrt{9x^2 + 9}$ .

58.  $\sqrt{16x^2 - 64}$ .

59.  $\sqrt{\frac{1}{a} + (6 + 9a)}$ .

60.  $\sqrt{\frac{1}{a} + (4 + a)}$ .

61.  $\sqrt{\frac{9}{2a} + 6 + 2a}$ .

62.  $\sqrt{\frac{4}{a} - 12 + 9a}$ .

Compute by use of table I:

63.  $\sqrt{\frac{2}{5}}$ .

64.  $\sqrt{\frac{9}{2}}$ .

65.  $\sqrt{\frac{7}{3}}$ .

66.  $\sqrt{\frac{9}{7}}$ .

67.  $\sqrt{\frac{13}{6}}$ .

68.  $\sqrt{\frac{11}{8}}$ .

69.  $\sqrt[3]{\frac{5}{3}}$ .

70.  $\sqrt[3]{\frac{2}{5}}$ .

69. Addition and subtraction of radicals. Two or more radicals are called **similar** if they have the same index and radicand. An algebraic sum of terms involving radicals may be simplified by reducing each radical to simplest form and collecting terms with similar radicals, as follows.

*Example.*  $\sqrt{8} + \sqrt{18} - \sqrt{32} = 2\sqrt{2} + 3\sqrt{2} - 4\sqrt{2} = (2 + 3 - 4)\sqrt{2} = \sqrt{2}$ .

## EXERCISES

*Simplify and collect terms:*

1.  $2\sqrt{3} + 4\sqrt{3}$ .

2.  $5\sqrt{2} - 2\sqrt{2}$ .

3.  $3\sqrt[3]{5} + 4\sqrt[3]{5} - 2\sqrt[3]{5}$ .

4.  $7\sqrt{2} + 4\sqrt{2} - 3\sqrt{2}$ .

5.  $\sqrt{8} + \sqrt{18}$ .

6.  $\sqrt{32} - \sqrt{8}$ .

7.  $\sqrt{12} - \sqrt{48} + \sqrt{75}$ .

8.  $\sqrt{50} - \sqrt{18} + \sqrt{98}$ .



9.  $\sqrt{72} - \sqrt{\frac{1}{2}}$       10.  $\sqrt{48} - \sqrt{27} + \sqrt{\frac{4}{3}}$
11.  $3\sqrt{12} - 2\sqrt{27} + \sqrt{\frac{1}{3}}$       12.  $\sqrt{\frac{9}{2}} + \sqrt{\frac{1}{2}} - 2\sqrt{18}$
13.  $\sqrt{16a} - \sqrt{36a}$       14.  $\sqrt{72x^3} + \sqrt{8x^3}$
15.  $\sqrt{32a^2x} - \sqrt{8b^2x}$       16.  $\sqrt{98x} - \sqrt{18a^2x}$
17.  $\sqrt[3]{16a^3x} + \sqrt[3]{54b^3x}$       18.  $\sqrt{\frac{25}{2x}} - 2\sqrt{\frac{1}{8x}}$
19.  $a\sqrt{\frac{b}{a}} - \sqrt{\frac{a^3b}{4}}$
20.  $\sqrt{\frac{a+1}{a-1}} + 2\sqrt{1 - \frac{1}{a^2}} - \sqrt{\left(a - \frac{1}{a}\right)\frac{1}{a}}$
21.  $\sqrt{\frac{a+b}{a-b}} - \sqrt{\frac{a-b}{a+b}}$       22.  $\sqrt{1 - \frac{x}{y}} + 2\sqrt{\frac{y^2 - xy}{y^2}}$
23.  $\sqrt{12} + \sqrt{18} + 2\sqrt{\frac{1}{2}} + 6\sqrt{\frac{1}{3}}$
24.  $\sqrt{36x^3y} + \sqrt{16xy^3} + \sqrt{18xy} - \sqrt{8x^3y}$
25. If  $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$  and  $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$  find  $r_1 + r_2$ .

70. Multiplication with radicals of the same index. This is done by means of law II, section 67, together with the usual rules of algebra.

*Example 1.*  $\sqrt[3]{4x} \sqrt[3]{2x^2y^4} = \sqrt[3]{8x^3y^4} = 2xy \sqrt[3]{y}$ .

*Example 2.* Multiply  $(2\sqrt{3} + \sqrt{2})(\sqrt{3} - 3\sqrt{2})$ .

*Solution:*

$$\begin{array}{r}
 2\sqrt{3} + \sqrt{2} \\
 \sqrt{3} - 3\sqrt{2} \\
 \hline
 -6\sqrt{6} - 3\sqrt{4} \\
 2\sqrt{9} + \sqrt{6} \\
 \hline
 6 - 5\sqrt{6} - 6 = -5\sqrt{6}.
 \end{array}$$

## EXERCISES

Multiply and simplify:

1.  $\sqrt{5} \cdot \sqrt{2}$     2.  $\sqrt{3a} \sqrt{5b}$     3.  $\sqrt{6} \sqrt{8}$     4.  $\sqrt{10} \sqrt{6}$
5.  $(2\sqrt{3})(5\sqrt{2})$     6.  $(3\sqrt{2})(7\sqrt{3})$     7.  $(2\sqrt{6})(3\sqrt{8})$
8.  $(3\sqrt{15})(\sqrt{5})$     9.  $(\sqrt[3]{4})(2\sqrt[3]{6})$     10.  $(3\sqrt[3]{12})(2\sqrt[3]{9})$
11.  $\sqrt{12x} \cdot \sqrt{3xy}$     12.  $(4\sqrt{x})(\sqrt{12x^5})$     13.  $(3\sqrt{6x})^2$



14.  $(2\sqrt{3x^3})^2$ .      15.  $(4\sqrt{3x^2y})^2$ .      16.  $(5\sqrt{2x^3y^2})^2$ .  
 17.  $(3\sqrt{a^2 + b^2})^2$ .      18.  $(4\sqrt{x^2 - y^2})^2$ .  
 19.  $[(x + y)\sqrt{x + y}]^2$ .      20.  $\sqrt{x + y} \cdot \sqrt{x - y}$ .  
 21.  $(2\sqrt[3]{3x})^3$ .      22.  $(3\sqrt[3]{2x})^3$ .      23.  $\sqrt{2}(\sqrt{3} + \sqrt{5})$ .  
 24.  $\sqrt{3}(\sqrt{6} - \sqrt{2})$ .      25.  $\sqrt{6}(\sqrt{2} - \sqrt{3})$ .  
 26.  $2\sqrt{2}(\sqrt{6} - \sqrt{3})$ .      27.  $(2 + \sqrt{3})(2 - \sqrt{3})$ .  
 28.  $(3 - \sqrt{2})(2 - \sqrt{3})$ .      29.  $(2\sqrt{3} - 3\sqrt{2})^2$ .  
 30.  $(3\sqrt{5} - \sqrt{2})^2$ .      31.  $(3\sqrt{5} - 2)(\sqrt{5} + 3)$ .  
 32.  $(5\sqrt{8} - 2)^2$ .      33.  $(\sqrt{2} - 2\sqrt{3})(3\sqrt{2} + \sqrt{3})$ .  
 34.  $(2\sqrt{3} - 3)(3\sqrt{2} + 2)$ .      35.  $(3\sqrt{2} - \sqrt{6})(\sqrt{3} + 3)$ .  
 36.  $(2\sqrt{3} - \sqrt{2})(3\sqrt{2} - \sqrt{3})$ .

Find the value of:

37.  $x^2 + 2x + 4$ , when  $x = 3 + \sqrt{2}$ .  
 38.  $x^2 - 2x - 1$ , when  $x = 1 - \sqrt{2}$ .  
 39.  $x^2 - 4x - 1$ , when  $x = 2 - \sqrt{5}$ .  
 40.  $2x^2 - 5x - 3$ , and  $x = \sqrt{3} + 2$ .  
 41.  $2x^2 - 3xy$ , when  $x = \sqrt{2} + 3$ ,  $y = \sqrt{2} - 3$ .  
 42.  $x^2 + y^2 - 3xy$ , when  $x = 1 - \sqrt{2}$ ,  $y = 1 + \sqrt{2}$ .  
 43. If  $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$  and  $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ , find  $r_1 r_2$ .  
 44. Simplify  $(3 - \sqrt{x - 2})^2$ .

**71. Division with radicals. Rationalizing denominators.** Division of two radicals of the same index can be done by law III, section 67, and simplifying.

$$\text{Example 1. } \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}} = \sqrt{\frac{3}{2} \cdot \frac{2}{2}} = \frac{1}{2} \sqrt{6} = \frac{\sqrt{6}}{2}.$$

An alternative procedure is to rationalize the denominator directly by removing radicals from the denominator by means of multiplying by whatever is needed to make the radicand in the denominator a perfect power. Thus in example 1, we could write

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

$$\text{Example 2. } \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

The rationalized form, having no radicals in the denominator, is considered simpler, because it is more convenient for computing



decimal values. Thus to compute  $\frac{1}{\sqrt{8}}$  directly requires long division of 1 by  $\sqrt{8} = 2.818$  approximately, while  $\frac{\sqrt{2}}{4}$  requires only mental division of  $\sqrt{2} = 1.414$  approximately, by 4. Thus the answer is  $\frac{1}{4}(1.414) = 0.354$  approximately.

A denominator consisting of a binomial quadratic surd  $\sqrt{a} + \sqrt{b}$  can be rationalized by multiplying by the conjugate surd  $\sqrt{a} - \sqrt{b}$ , and vice versa.

$$\text{Example 3. } \frac{1}{\sqrt{5} - \sqrt{2}} = \frac{1}{\sqrt{5} - \sqrt{2}} \cdot \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}} = \frac{\sqrt{5} + \sqrt{2}}{3}.$$

$$\text{Example 4. } \frac{1}{7 + \sqrt{3}} = \frac{1}{7 + \sqrt{3}} \cdot \frac{7 - \sqrt{3}}{7 - \sqrt{3}} = \frac{7 - \sqrt{3}}{46}.$$

### EXERCISES

*Rationalize denominators and simplify, leaving answers in radical form unless directed otherwise:*

1.  $\frac{\sqrt{5}}{\sqrt{2}}$
2.  $\frac{2\sqrt{7}}{\sqrt{3}}$
3.  $\frac{\sqrt{15}}{\sqrt{3}}$
4.  $\frac{\sqrt{3}}{\sqrt{6}}$
5.  $\frac{\sqrt{12x^3}}{\sqrt{2x}}$
6.  $\frac{\sqrt{3x}}{\sqrt{8x^3}}$
7.  $\frac{\sqrt{5xy}}{\sqrt{12x^5}}$
8.  $\frac{\sqrt{6x^3y}}{\sqrt{18xy^2}}$
9.  $\frac{\sqrt[3]{7}}{\sqrt[3]{4}}$
10.  $\frac{2\sqrt[3]{11x^2}}{\sqrt[3]{9x}}$
11.  $\frac{1}{3 - \sqrt{2}}$
12.  $\frac{1}{4 + \sqrt{3}}$
13.  $\frac{3}{\sqrt{3} + \sqrt{2}}$
14.  $\frac{2}{\sqrt{5} - \sqrt{2}}$
15.  $\frac{\sqrt{2}}{\sqrt{5} - \sqrt{3}}$
16.  $\frac{\sqrt{6} - \sqrt{2}}{\sqrt{3}}$
17.  $\frac{\sqrt{7} + \sqrt{8}}{\sqrt{2}}$
18.  $\frac{\sqrt{5} - 2}{\sqrt{5} + 2}$
19.  $\frac{\sqrt{3} - 1}{\sqrt{2} + 1}$
20.  $\frac{\sqrt{5} - 2}{3 - \sqrt{5}}$
21.  $\frac{2\sqrt{5} - \sqrt{2}}{3\sqrt{5} + 2\sqrt{2}}$
22.  $\frac{\sqrt{3} - \sqrt{6}}{2\sqrt{6} - \sqrt{3}}$
23.  $\frac{\sqrt{5} + \sqrt{3}}{2\sqrt{3} - \sqrt{5}}$
24.  $\frac{2\sqrt{7} - 3\sqrt{2}}{3\sqrt{7} + 2\sqrt{2}}$
25.  $\frac{\sqrt{x-1}}{1 - \sqrt{x-1}}$
26.  $\frac{\sqrt{x-a}}{b + \sqrt{x-a}}$

27. Compute, by use of table I, the answers to the first four exercises.

72. A use of fractional exponents. Certain operations with radicals are best done by means of fractional exponents. For example, to multiply or divide two radicals of different orders, we may convert these radicals to fractional powers of the radicand,



change the exponents of those fractional powers to equivalent fractions with their least common denominator, convert back to radical form, and simplify.

*Example 1.* Simplify  $\sqrt[3]{x^2} \sqrt[5]{y^3}$ . We write  $x^{2/3} y^{3/5} = x^{10/15} y^{9/15} = \sqrt[15]{x^{10} y^9} = \sqrt[15]{x^{10} y^9}$ .

*Example 2.* Simplify  $\frac{\sqrt{6}}{\sqrt[3]{2}}$ . We write  $\frac{6^{1/2}}{2^{1/3}} = \frac{6^{2/6}}{2^{2/6}} = \frac{\sqrt[6]{6^2}}{\sqrt[6]{2^2}} = \sqrt[6]{\frac{6^2}{2^2}} = \sqrt[6]{\frac{36}{4}} = \sqrt[6]{9} = \sqrt[6]{3^2} = \sqrt[3]{3}$ .

## EXERCISES

*Simplify:*

1.  $\sqrt{x} \cdot \sqrt[3]{x}$ .
2.  $\sqrt[3]{x} \sqrt[4]{x}$ .
3.  $\sqrt[3]{x^2} \sqrt[4]{x^3}$ .
4.  $\sqrt{x} \sqrt[4]{x^3}$ .
5.  $\frac{\sqrt{10}}{\sqrt[3]{2}}$ .
6.  $\sqrt{2} \sqrt[3]{4}$ .
7.  $\sqrt{8}/\sqrt[3]{4}$ .
8.  $\sqrt[4]{\sqrt[3]{7}}$ .
9.  $\sqrt{\sqrt[3]{5}}$ .
10.  $\frac{\sqrt[4]{6}}{\sqrt[3]{4}}$ .
11.  $\sqrt[3]{32}/\sqrt[4]{8}$ .
12.  $\frac{\sqrt[3]{x^2}}{\sqrt{5xy}}$ .
13.  $\sqrt[5]{x^2y} \sqrt{y^3}$ .
14.  $\frac{\sqrt[3]{9a}}{\sqrt{3a}}$ .



## CHAPTER XI\*

### Quadratic Equations and Quadratic Functions

73. Solution of quadratic equations by factoring. Every quadratic equation in  $x$  can be written in the form

$$(1) \quad ax^2 + bx + c = 0 \quad (a \neq 0)$$

or, dividing both members by  $a$ , as

$$(2) \quad x^2 + px + q = 0$$

where  $p = b/a$  and  $q = c/a$ .

If the first-degree term is missing, the equation is called a **pure quadratic**. To solve such an equation is easy, for the roots of  $x^2 = N$  are obviously  $x = \pm \sqrt{N}$ . Thus the roots of  $x^2 - 9 = 0$  are  $\pm 3$ . The roots of  $x^2 - 10 = 0$  are  $\pm \sqrt{10}$ . The roots of  $x^2 + 4 = 0$  are  $\pm \sqrt{-4}$  or†  $\pm 2i$ .

If the left member of (1) or (2) can be expressed as a product of two factors of the first degree, the equation may be solved by setting each factor equal to zero separately, as in the following example.

*Example 1.* Solve  $x^2 - 5x + 4 = -2$ .

*Solution.* We write the equation as  $x^2 - 5x + 6 = 0$ . Factoring, we get  $(x - 2)(x - 3) = 0$ . Setting each factor equal to zero, we get  $x - 2 = 0$  and  $x - 3 = 0$ . Hence  $x = 2$  and  $x = 3$ . Checking, by substitution, we get: for  $x = 2$ ,  $2^2 - 5 \cdot 2 + 4 = -2$ , or  $-2 = -2$ ; for  $x = 3$ ,  $3^2 - 5 \cdot 3 + 4 = -2$ , or  $-2 = -2$ .

*Justification of the procedure.* By theorem 1, section 11, the product of several factors can be zero when and only when at least one of the factors is zero. To solve the equation  $x^2 - 5x + 6 = 0$  (to which the original equation is equivalent‡) means to find all values of  $x$  for which the left member has the value zero. But the left member can be expressed as the product of  $(x - 2)$  and  $(x - 3)$ . This product therefore is zero when and only when  $x - 2 = 0$  or  $x - 3 = 0$ , that is, when  $x = 2$  or  $3$ . Therefore if

\* The student should review sections 12-16 at this point.

† The student should review section 16.

‡ See section 29 for the definition of *equivalent* equations.



$x$  satisfies the original equation, then  $x$  can only be either 2 or 3. The converse "if  $x$  is either 2 or 3, then  $x$  satisfies the original equation" is certainly correct because all steps taken are reversible, but it is advisable to check by substitution in any case.

A quadratic equation has two roots, one arising from each factor; if the roots arising from both factors are the same, we still say there are two roots, and their common value is called a **double root** of the equation.

*Example 2.* The equation  $x^2 - 6x + 9 = 0$  has the double root 3, since the left member is  $(x - 3)(x - 3)$ .

## EXERCISES

*Solve and check:*

1.  $x^2 - x - 6 = 0$ .

2.  $x^2 + 7x + 6 = 0$ .

3.  $x^2 - x - 10 = 2$ .

4.  $x^2 - 3x - 12 = x$ .

5.  $x^2 - 16 = 0$ .

6.  $x^2 - 49 = 0$ .

7.  $4x^2 - 9 = 0$ .

8.  $25x^2 - 4 = 0$ .

9.  $x^2 + 9x - 36 = 0$ .

10.  $x^2 + 7x - 18 = 0$ .

11.  $x^2 = 16x + 36$ .

12.  $x^2 - 8x - 20 = 2x + 4$ .

13.  $x^2 + 17x + 29 = 3x + 5$ .

14.  $x^2 - 20x - 50 = 2x - 2$ .

15.  $6x^2 + 7x - 5 = 0$ .

16.  $4x^2 + 8x - 5 = 0$ .

17.  $6x^2 + 5x - 4 = 0$ .

18.  $6x^2 - 19x + 3 = 0$ .

19.  $2x^2 + 7x + 3 = 0$ .

20.  $8x^2 - 22x + 15 = 0$ .

21.  $x^2 - 3 = 0$ .

22.  $4x^2 - 5 = 0$ .

23.  $9x^2 - 10 = 0$ .

24.  $2x^2 - 1 = 0$ .

25.  $x^2 - 2kx + k^2 = 0$ .

26.  $x^2 - 5kx + 6k^2 = 0$ .

27.  $21x^2 + 5kx - 4k^2 = 0$ .

28.  $3x^2 - 19kx + 20k^2 = 0$ .

29.  $x^4 - 13x^2 + 36 = 0$ . (*Hint: Let  $x^2 = u$ ; solve for  $u$ , then for  $x$ .)*

30.  $x^4 - 29x^2 + 100 = 0$ .

31.  $x^4 - 4x^2 + 3 = 0$ .

32.  $x^4 - 11x^2 + 18 = 0$ .

33.  $x^2 + 4 = 0$ .

34.  $x^2 + 9 = 0$ .

35.  $9x^2 + 4 = 0$ .

36.  $25x^2 + 9 = 0$ .

37.  $(x - 8)^2 = 100$ .

38.  $(x + 7)^2 = 25$ .

39.  $(x - 2)^2 - 5(x - 2) + 6 = 0$ .

40.  $(x + 3)^2 + 9(x + 3) - 36 = 0$ .

41.  $x^2 - 5x = 0$ .

42.  $2x^2 - 8x = 0$ .

43.  $2x^2 + 7x = 0$ .

44.  $3x^2 + 8x = 0$ .

45.  $x^2 - 16x + 64 = 0$ .

46.  $x^2 - 10x + 25 = 0$ .

47.  $4x^2 + 20x + 25 = 0$ .

48.  $9x^2 + 12x + 4 = 0$ .

49. Solve  $s = \frac{1}{2}gt^2$  for  $t$ .

50. Solve  $T = \frac{1}{2}mv^2$  for  $v$ .

51. Solve  $A = \pi r^2$  for  $r$ .

52. Solve  $V = \pi r^2 h$  for  $r$ .

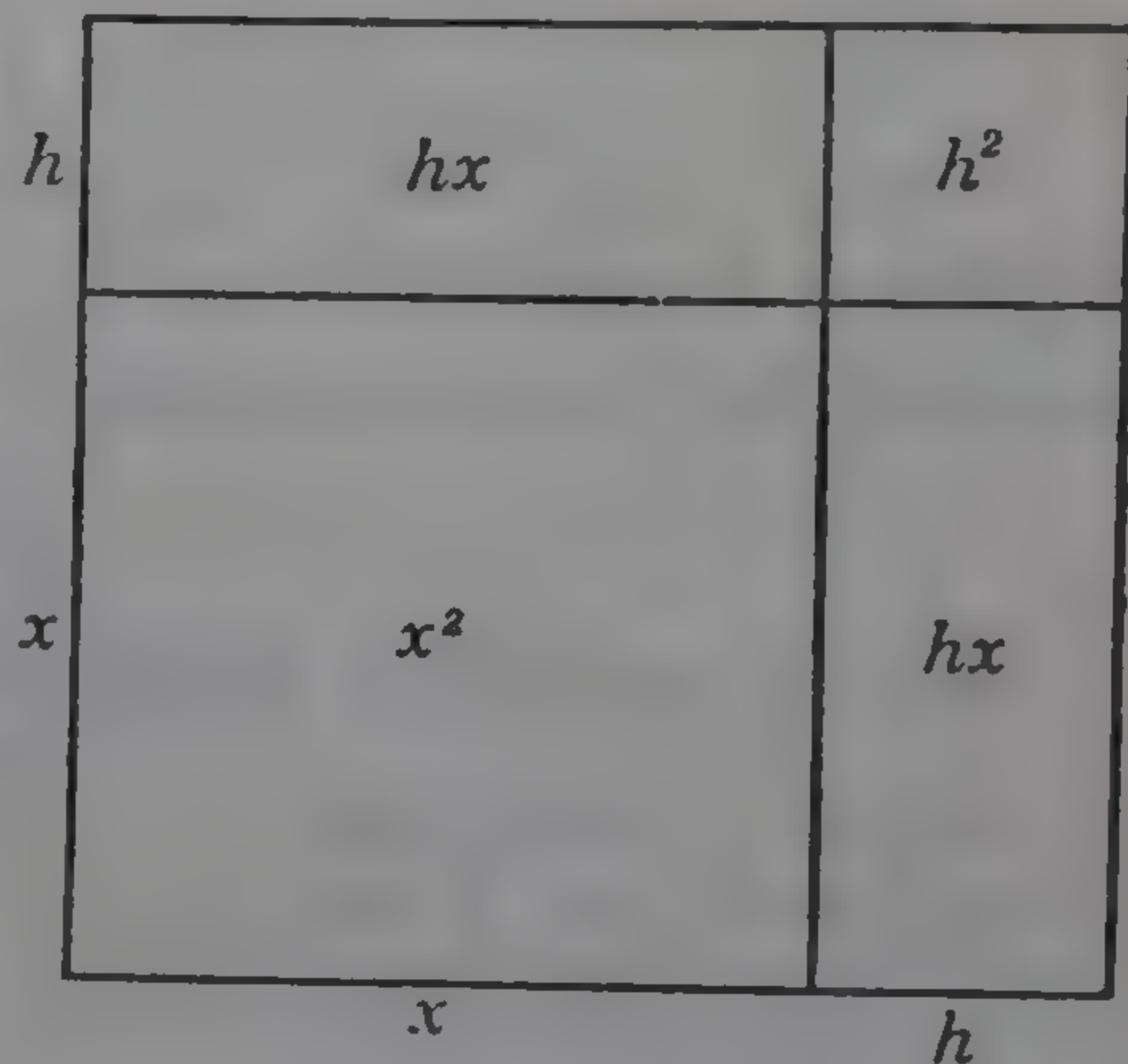
74. **Completing the square.** Since  $x^2 + 2hx + h^2 = (x + h)^2$ , it follows that if the square of half the coefficient of  $x$  be added to



an expression of the form\*  $x^2 + px$ , it will become a perfect square. That is,  $x^2 + px + \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2$ . The geometric meaning of the phrase "completing the square" will be obvious in terms of the areas in the jigsaw puzzle of Fig. 27.

When a quadratic equation cannot be readily expressed as a product of factors, it may be solved by completing the square, as follows.

*Example 1.* Solve the equation  $x^2 - 6x - 1 = 0$ .



$$(x+h)^2 = x^2 + 2hx + h^2$$

Fig. 27.

*Solution.* Transpose the constant term, obtaining  $x^2 - 6x = 1$ . We may complete the square on the left side of the equation by adding the square of half the coefficient of  $x$ , that is,  $3^2$  or 9. But if we add 9 to the left member, we must of course also add 9 to the right member. Thus we get  $x^2 - 6x + 9 = 1 + 9$  or  $(x - 3)^2 = 10$ . By III, section 29, we may write  $x - 3 = \pm \sqrt{10}$ , or  $x = 3 \pm \sqrt{10}$ . Checking, we get: for  $x = 3 + \sqrt{10}$ ,  $(3 + \sqrt{10})^2 - 6(3 + \sqrt{10}) - 1 = 0$ , or  $9 + 6\sqrt{10} + 10 - 18 - 6\sqrt{10} - 1 = 0$ , or  $0 = 0$ ; for  $x = 3 - \sqrt{10}$  we get  $(3 - \sqrt{10})^2 - 6(3 - \sqrt{10}) - 1 = 9 - 6\sqrt{10} + 10 - 18 + 6\sqrt{10} - 1$ , or  $0 = 0$ .

The roots  $3 \pm \sqrt{10}$  of the equation of example 1 are irrational.† If a decimal approximation is desired, it may be obtained by working out  $\sqrt{10}$  (or from table I), obtaining  $3 \pm 3.162$ , or 6.162 and -0.162 approximately.

It can also happen that the roots may be imaginary‡ as in the following example.

*Example 2.* Solve the equation  $x^2 + 2x + 4 = 0$ .

*Solution.* We write  $x^2 + 2x = -4$ . Completing the square we get  $x^2 + 2x + 1 = -4 + 1$ , or  $(x + 1)^2 = -3$ . Hence  $x + 1 = \pm \sqrt{-3}$ , or  $x + 1 = \pm i\sqrt{3}$ . Hence  $x = -1 \pm i\sqrt{3}$ . Checking we get: for  $x = -1 + i\sqrt{3}$ ,  $(-1 + i\sqrt{3})^2 + 2(-1 + i\sqrt{3}) + 4 = 0$ , or  $1 - 2i\sqrt{3} - 3 - 2 + 2i\sqrt{3} + 4 = 0$ , or  $0 = 0$ ; for  $x = -1 - i\sqrt{3}$  we get  $(-1 - i\sqrt{3})^2 + 2(-1 - i\sqrt{3}) + 4 = 0$ , or  $1 + 2i\sqrt{3} - 3 - 2 - 2i\sqrt{3} + 4 = 0$ , or  $0 = 0$ .

\* Writing  $p$  for  $2h$ , or  $p/2$  for  $h$ .

† See sections 13, 14, and 15.

‡ The student should review section 16.



If the coefficient of  $x^2$  is different from 1, divide through by this coefficient before completing the square, as follows.

*Example 3.* Solve  $2x^2 - 7x + 3 = 0$ .

*Solution.* Transposing the 3, and dividing by 2, we get  $x^2 - \frac{7}{2}x = -\frac{3}{2}$ . Adding  $\left(\frac{7}{4}\right)^2 = \frac{49}{16}$  to both sides, we get  $\left(x - \frac{7}{4}\right)^2 = \frac{25}{16}$  or  $x - \frac{7}{4} = \pm \frac{5}{4}$ . Hence  $x = 3$  and  $x = \frac{1}{2}$ . The check is left to the student.

## EXERCISES

*Solve by completing the square, and check:*

1.  $x^2 - 16x = 36$ .

3.  $x^2 + 14x + 24 = 0$ .

5.  $6x^2 + 7x - 5 = 0$ .

7.  $6x^2 + 5x = 4$ .

9.  $2x^2 + 7x + 3 = 0$ .

2.  $x^2 - 10x - 24 = 0$ .

4.  $x^2 - 22x - 48 = 0$ .

6.  $4x^2 = 5 - 8x$ .

8.  $6x^2 - 19x + 3 = 0$ .

10.  $8x^2 = 22x - 15$ .

*Solve by completing the square, and check, (a) leaving answers in simplest radical form if irrational, or in terms of  $i$  if imaginary; (b) if the roots are irrational, approximate them to the nearest thousandth:*

11.  $x^2 - 4x + 1 = 0$ .

13.  $2x^2 - 10x + 11 = 0$ .

15.  $x^2 + 2x - 4 = 0$ .

17.  $x^2 - 2x + 5 = 0$ .

19.  $x^2 + 4x + 13 = 0$ .

21.  $x^2 - 5x + 1 = 0$ .

23.  $3x^2 - 11x - 20 = 0$ .

25.  $(x + 1)(x + 5) = 13$ .

27.  $x^2 + px + q = 0$ .

12.  $x^2 - 7x + 2 = 0$ .

14.  $3x^2 - 6x + 1 = 0$ .

16.  $2x^2 + 3x - 7 = 0$ .

18.  $x^2 - 6x + 34 = 0$ .

20.  $x^2 + x + 2 = 0$ .

22.  $2x^2 - 5x - 3 = 0$ .

24.  $6x^2 + 7x - 3 = 0$ .

26.  $3x^2 + x + 5 = 0$ .

28.  $ax^2 + bx + c = 0$ ,

where  $a \neq 0$ .

**75. The quadratic formula.** By completing the square, we can prove that the roots of the general quadratic equation

$$(1) \quad ax^2 + bx + c = 0 \quad (a \neq 0)$$

are given by the formula

$$(2) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

*Proof.* Dividing both members of (1) by  $a$ , and transposing the constant term, we get



$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

To complete the square, we add  $\left(\frac{b}{2a}\right)^2$  to both sides, obtaining

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

or 
$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

or 
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Therefore, 
$$x + \frac{b}{2a} = \frac{\pm \sqrt{b^2 - 4ac}}{2a}$$

or 
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The student should check by substituting both answers (one with the + sign and the other with the - sign before the radical) in (1); both answers check.

To solve any quadratic equation whatever, we have only to substitute the values of its coefficients  $a$ ,  $b$ , and  $c$  into the formula (2) and simplify. As before, we may check by substitution.

*Example 1.* Solve  $x^2 - 5x + 6 = 0$ . Here  $a = 1$ ,  $b = -5$ , and  $c = 6$ . Hence the roots are

$$\frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = \frac{5 \pm 1}{2} = \begin{cases} 3 \\ 2. \end{cases}$$

Check as before.

*Example 2.* Solve  $x^2 - 6x - 1 = 0$ . Here  $a = 1$ ,  $b = -6$ , and  $c = -1$ . The roots are

$$\frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{6 \pm \sqrt{40}}{2} = \frac{6 \pm 2\sqrt{10}}{2} = 3 \pm \sqrt{10}.$$

Check by substituting as before.

Note that the roots  $3 \pm \sqrt{10}$  are irrational.\*

If a decimal approximation is desired, we may work out  $\sqrt{10}$  (or use table I), obtaining  $3 \pm 3.162$ , or 6.162 and -0.162, approximately.

*Example 3.* Solve  $x^2 + 2x + 4 = 0$ . Here  $a = 1$ ,  $b = 2$ ,  $c = 4$ . The roots† are

\* The student should review sections 13, 14, and 15.

† The student should review section 16.



$$\begin{aligned} \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} &= \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{-3}}{2} \\ &= \frac{2(-1 \pm \sqrt{-3})}{2} = -1 \pm \sqrt{-3} = -1 \pm i\sqrt{3}. \end{aligned}$$

Check by substituting as before.

*Note.* A pure quadratic equation should be solved by taking square roots, with the  $\pm$  sign. If a trinomial quadratic equation can be factored easily, it should be solved by factoring. In all other cases it is probably best to solve by formula.

## EXERCISES

*Solve by use of the quadratic formula, and check:*

1–26. The equations of exercises 1–26, section 74, inclusive.

27.  $12 - x^2 = -x$ .

28.  $5x = 3 - 2x^2$ .

29.  $9x^2 = 12x - 4$ .

30.  $9 - 24x + 16x^2 = 0$ .

31.  $2x^2 - \sqrt{3}x + 1 = 0$ .

32.  $x^2 - \sqrt{2}x + 3 = 0$ .

33.  $\sqrt{2}x^2 - x + \sqrt{8} = 0$ .

34.  $\sqrt{2}x^2 - 3x + \sqrt{18} = 0$ .

35.  $\sqrt{3}x^2 = 2x + \sqrt{12}$ .

36.  $1 + 3x^2 - 2\sqrt{3}x = 0$ .

37.  $x^2 - 2hx + k = 0$ .

38.  $x^2 + 2mx - n = 0$ .

39.  $ax^2 + 2dx + c = 0$ .

40.  $ax^2 - 2dx - k = 0$ .

41.  $x^2 - 2ax + (a^2 - b) = 0$ .

42.  $x^2 - 2ax + (a^2 + b^2) = 0$ .

43.  $x^2 + 2ix - 3 = 0$ .

44.  $2ix^2 + 5x + i = 0$ .

76. **Character of the roots.** Up to this point in the present chapter, no restriction has been imposed on the coefficients  $a$ ,  $b$ , and  $c$  of the quadratic equation  $ax^2 + bx + c = 0$  ( $a \neq 0$ ). That is, everything said so far is true no matter what complex\* numbers  $a$ ,  $b$ , and  $c$  represent. Let us now consider only equations with real coefficients.

The two roots, which we shall denote by  $r_1$  and  $r_2$  respectively, are given by the formulas

$$(1) \quad r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

From (1) and the fact that the sum, product, difference, or quotient of two real numbers (division by zero excluded) is a real number, it follows that the roots are both imaginary if  $b^2 - 4ac$  is negative, both real if  $b^2 - 4ac$  is not negative, and both equal if  $b^2 - 4ac$  is zero. The quantity  $b^2 - 4ac$  is called the **discriminant**

\* The student should review sections 11–16, inclusive.



of the equation  $ax^2 + bx + c = 0$ . Hence we have the following summary:

### Quadratic Equations with Real Coefficients

| Discriminant           | positive         | zero           | negative              |
|------------------------|------------------|----------------|-----------------------|
| Character of the roots | real and unequal | real and equal | imaginary and unequal |

If we restrict ourselves further to equations with rational coefficients, we may draw further conclusions. Since the sum, product, difference, or quotient of two rational numbers is a rational number (division by zero excluded), it follows that the discriminant is a rational number. The roots will be rational if and only if the discriminant is a perfect square. If the discriminant is not a perfect square, its square roots are irrational. Hence we have the following summary:

### Quadratic Equations with Rational Coefficients

| Discriminant           | positive and a perfect square | positive but not a perfect square | zero                      | negative              |
|------------------------|-------------------------------|-----------------------------------|---------------------------|-----------------------|
| Character of the roots | real, rational, and unequal   | real, irrational, and unequal     | real, rational, and equal | imaginary and unequal |

*Example 1.* Without solving the equation, describe the character of the roots of  $x^2 - 6x - 1 = 0$ .

*Solution.* The coefficients are rational. The discriminant  $b^2 - 4ac = (-6)^2 - 4 \cdot 1 \cdot (-1) = 40$ . This is positive but not a perfect square. Hence the roots are real, irrational, and unequal.

*Example 2.* Find the values of  $k$  for which the equation  $x^2 + kx + 9 = 0$  has equal roots.

*Solution.* The roots can be equal if and only if the discriminant is zero, that is, when  $k^2 - 36 = 0$  or  $k = \pm 6$ . *Check:* when  $k = 6$ , the given equation is  $x^2 + 6x + 9 = 0$ , which has the roots  $-3, -3$ ; when  $k = -6$ , the given equation is  $x^2 - 6x + 9 = 0$  which has the roots  $3, 3$ .



*Note.* From (1), it follows that the roots  $r_1$  and  $r_2$  are equal if and only if  $b^2 - 4ac = 0$ , regardless of what kind of numbers the coefficients are.

## EXERCISES

*Calculate the discriminant and describe the character of the roots without solving, if possible; check by solving:*

- |                                   |                                                 |
|-----------------------------------|-------------------------------------------------|
| 1. $x^2 - 4x - 12 = 0$ .          | 2. $x^2 - x - 30 = 0$ .                         |
| 3. $6x^2 + 7x - 5 = 0$ .          | 4. $4x^2 - 5 + 8x = 0$ .                        |
| 5. $2 + x^2 - 7x = 0$ .           | 6. $x^2 + 2x + 4 = 0$ .                         |
| 7. $x^2 + 3x + 5 = 0$ .           | 8. $x^2 - 8x + 16 = 0$ .                        |
| 9. $9x^2 + 12x + 4 = 0$ .         | 10. $x^2 + 2x - 4 = 0$ .                        |
| 11. $x^2 + 6x + 18 = 0$ .         | 12. $x^2 + 4x + 13 = 0$ .                       |
| 13. $x^2 - 5x = 0$ .              | 14. $2x^2 + 7x = 0$ .                           |
| *15. $2x^2 - \sqrt{3}x + 1 = 0$ . | *16. $\sqrt{2}x^2 + \sqrt{3}x - \sqrt{8} = 0$ . |
| *17. $x^2 + \sqrt{5}x - 5 = 0$ .  | *18. $\sqrt{2}x^2 - x + \sqrt{2} = 0$ .         |
| *19. $x^2 + 2ix - 5 = 0$ .        | *20. $2ix^2 + 5x + i = 0$ .                     |
| *21. $x^2 + 4ix - 4 = 0$ .        | *22. $9x^2 - 6ix - 1 = 0$ .                     |

*Find the value or values of  $k$  for which the equation will have equal roots:*

- |                             |                              |
|-----------------------------|------------------------------|
| 23. $kx^2 - 12x + 4 = 0$ .  | 24. $9x^2 - 30x + k = 0$ .   |
| 25. $16x^2 + kx + 9 = 0$ .  | 26. $2kx^2 - 20x + 25 = 0$ . |
| 27. $2kx^2 + 6kx + 9 = 0$ . | 28. $kx^2 - 8x + 16k = 0$ .  |
| 29. $x^2 - 4kx + 16k = 0$ . | 30. $4x^2 - 4kx + 5k = 0$ .  |

*Find the value or values of  $k$  for which the function is a perfect square:*

- |                         |                         |
|-------------------------|-------------------------|
| 31. $9x^2 + 6kx + 25$ . | 32. $9x^2 + 6kx + 4k$ . |
| 33. $kx^2 - 3kx + 9$ .  | 34. $4kx^2 - 6kx + 9$ . |

*Find the values of  $k$  for which the roots of the equation are (a) real and unequal, (b) imaginary and unequal:*

- |                          |                           |
|--------------------------|---------------------------|
| 35. $x^2 + 4x + k = 0$ . | 36. $x^2 + 2x + k = 0$ .  |
| 37. $x^2 - kx + 9 = 0$ . | 38. $x^2 + kx + 25 = 0$ . |

77. The graph of a quadratic function. Graphical solution of quadratic equations.\* It can be proved that the graph of every quadratic function  $y = ax^2 + bx + c$  ( $a \neq 0$ ) with real coefficients is a parabola. (See Fig. 28.) To plot the graph we substitute various values for  $x$ , calculate the corresponding values of  $y$ , plot the points  $(x, y)$  thus tabulated, and join them by a smooth curve.

\* May be omitted without disturbing continuity.

\* The student should review sections 34-37.



*Example 1.* Plot the graph of  $y = x^2 + x - 6$ .

*Solution.* We get the table

|     |    |    |    |   |    |   |    |    |
|-----|----|----|----|---|----|---|----|----|
| $x$ | 0  | 1  | -1 | 2 | -2 | 3 | -3 | -4 |
| $y$ | -6 | -4 | -6 | 0 | -4 | 6 | 0  | 6  |

Plotting these points, we get the graph of Fig. 28.

The real roots of the quadratic equation  $ax^2 + bx + c = 0$  with real coefficients are the real values of  $x$  for which  $y = ax^2 + bx + c = 0$ . Thus, they are the  $x$ -coordinates of the points where the graph crosses the  $x$ -axis.

For example, the real roots of  $x^2 + x - 6 = 0$  are 2 and -3. (See Fig. 28.)

*Note.* When both roots are real and equal, the parabola is tangent to the  $x$ -axis. When the roots are imaginary, the graph has no point in common with the  $x$ -axis. If the roots are real and unequal, the graph crosses the  $x$ -axis in two distinct points. The student should explain how one can tell which of these three cases will arise by means of calculating the discriminant  $b^2 - 4ac$ . See section 76.

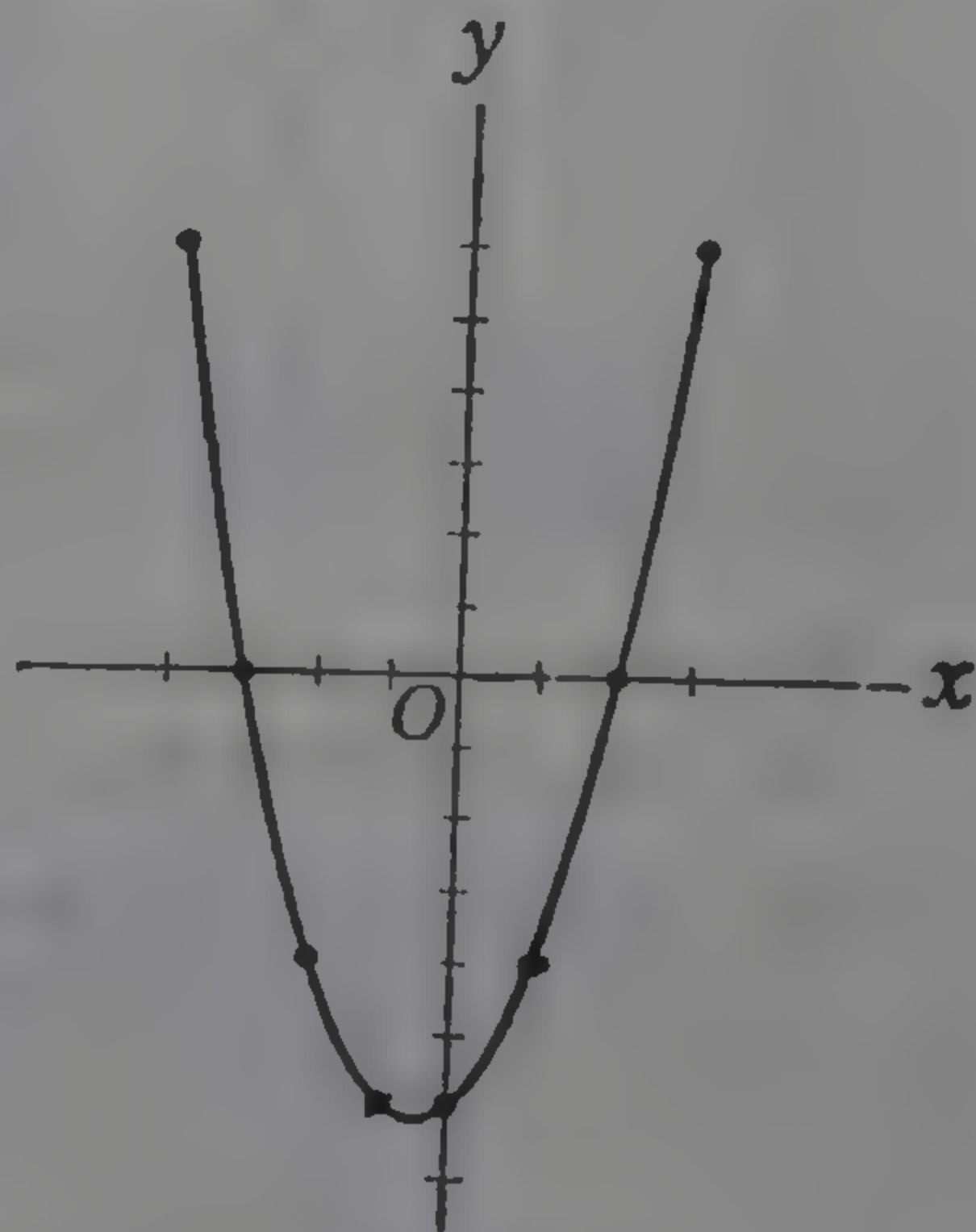


Fig. 28.

## EXERCISES

Draw the graph of each of the following functions, plotting at least eight points for each having  $x$ -coordinates which are successive integers:

- |                     |                           |                       |                           |
|---------------------|---------------------------|-----------------------|---------------------------|
| 1. $x^2 - 2$ .      | 2. $\frac{1}{2}x^2 + 3$ . | 3. $2 - x^2$ .        | 4. $3 - \frac{1}{2}x^2$ . |
| 5. $x^2 - x - 2$ .  | 6. $2 - x - x^2$ .        | 7. $x^2 + x + 1$ .    |                           |
| 8. $x^2 - 6x + 9$ . | 9. $1 + 7x - 2x^2$ .      | 10. $2x^2 - 5x + 2$ . |                           |

Estimate the real roots, if any, of each equation graphically:

- |                           |                                |
|---------------------------|--------------------------------|
| 11. $x^2 - x - 2 = 0$ .   | 12. $x^2 - 2x - 3 = 0$ .       |
| 13. $3 - 2x - x^2 = 0$ .  | 14. $6 - x - x^2 = 0$ .        |
| 15. $4x^2 - 4x + 1 = 0$ . | 16. $4x^2 - 9 = 0$ .           |
| 17. $16 - 9x^2 = 0$ .     | 18. $9x^2 - 12x + 4 = 0$ .     |
| 19. $x^2 + 2x + 4 = 0$ .  | 20. $\frac{1}{3}x^2 + 2 = 0$ . |
| 21. $x^2 - 3x = 0$ .      | 22. $x^2 + 3x = 0$ .           |

Plot the graphs of both functions on the same set of axes:

23.  $x^2 - x - 2$  and  $-x^2 + x + 2$ .  
 24.  $x^2 - 2x - 3$  and  $-x^2 + 2x + 3$ .



25.  $x^2 - 2$  and  $-x^2 + 2$ .  
 26.  $x^2 + 1$  and  $-x^2 - 1$ .  
 27.  $x^2 - 4x + 4$  and  $-x^2 + 4x - 4$ .  
 28.  $x^2 - 3x$  and  $-x^2 + 3x$ .

Without plotting the graph, decide by means of the discriminant whether the graph will cross the  $x$ -axis in two distinct points, will be tangent to the  $x$ -axis, or will not meet the  $x$ -axis at all:

29.  $4x^2 - 12x + 9$ .      30.  $x^2 - 7x + 12$ .      31.  $x^2 - 8x + 15$ .  
 32.  $x^2 - 6x + 9$ .      33.  $x^2 + 3x + 6$ .      34.  $4x^2 + 9$ .  
 35.  $x^2 - 2x - 5$ .      36.  $2x^2 + 3x - 4$ .

Find the values of  $k$  for which the graph of each function will (a) be tangent to the  $x$ -axis; (b) cross the  $x$ -axis in two distinct points; (c) not meet the  $x$ -axis at all:

37.  $kx^2 - 2x + 1$ .      38.  $2x^2 - 4x + k$ .  
 39.  $4x^2 - 3kx + 9$ .      40.  $4x^2 + 4kx + 1$ .

78. The sum and product of the roots. The roots  $r_1$  and  $r_2$  of  $ax^2 + bx + c = 0$  ( $a \neq 0$ ) are given by the formulas

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

By direct calculation, which should be checked by the student we find

$$(1) \quad r_1 + r_2 = -\frac{b}{a}$$

and

$$(2) \quad r_1 r_2 = \frac{c}{a}.$$

*Example 1.* The sum of the roots of  $x^2 + x - 6 = 0$  is  $-1/1 = -1$ . The product of the roots is  $-6/1 = -6$ .

*Example 2.* If one root of  $x^2 + x + c = 0$  is 2, find the other root.

*First solution.* The sum of the roots is  $-1$ . Let  $r_1 = 2$ . Then  $2 + r_2 = -1$ . Hence  $r_2 = -3$ .

*Second solution.* Since  $x = 2$  must satisfy the equation, we have  $2^2 + 2 + c = 0$ . Hence  $c = -6$ . Solving  $x^2 + x - 6 = 0$ , we find that the second root is  $-3$ .

*Example 3.* If one root of  $x^2 + bx - 6 = 0$  is 2, find the other root.



*First solution.* The product of the roots is  $-6$ . Let  $r_1 = 2$ . Then  $2 \cdot r_2 = -6$ . Hence  $r_2 = -3$ .

*Second solution.* Since  $x = 2$  must satisfy the equation, we have  $2^2 + 2b - 6 = 0$ . Hence  $b = 1$ . Solving  $x^2 + x - 6 = 0$ , we find that the second root is  $-3$ .

By means of (1) and (2) we can prove the following theorem:

**THEOREM 1.** *If  $r_1$  and  $r_2$  are the roots of  $ax^2 + bx + c = 0$ , then  $ax^2 + bx + c = a(x - r_1)(x - r_2)$ .*

*Proof.* We may write  $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a(x^2 + [-r_1 - r_2]x + r_1r_2)$ . By (5), section 45, the latter expression is equal to  $a(x - r_1)(x - r_2)$ .

*Example 4.* Form a quadratic equation with integral coefficients having the roots  $\frac{1}{2}$  and  $-\frac{1}{3}$ .

*Solution.* By theorem 1, we may write  $a(x - \frac{1}{2})(x + \frac{1}{3}) = 0$  as an equation having the desired roots. To get integral coefficients, we may let  $a = 6 = 2 \cdot 3$ , and write  $2 \cdot 3(x - \frac{1}{2})(x + \frac{1}{3}) = 0$ , or  $2(x - \frac{1}{2}) \cdot 3(x + \frac{1}{3}) = 0$ , or  $(2x - 1)(3x + 1) = 0$ , or, finally,  $6x^2 - x - 1 = 0$ .

## EXERCISES

*Without solving, find the sum and product of the roots:*

1.  $x^2 - 5x + 6 = 0$ .

2.  $x^2 - 3x + 7 = 0$ .

3.  $2x^2 + 5x - 8 = 0$ .

4.  $3x^2 - 2x + 12 = 0$ .

5.  $x = 3 - 2x^2$ .

6.  $5 - 4x = 2x^2$ .

7.  $2x^2 - 9 = 0$ .

8.  $5x^2 + 2 = 0$ .

9.  $3x^2 + 5x = 0$ .

10.  $2x^2 - 3x = 0$ .

11. If one root of  $x^2 - 2x + h = 0$  is 3, find the other root, and the value of  $h$ .

12. If one root of  $x^2 + 3x - k = 0$  is 2, find the other root, and the value of  $k$ .

13. If one root of  $2x^2 + kx - 6 = 0$  is  $-2$ , find the other root, and the value of  $k$ .

14. If one root of  $6x^2 + kx - 3 = 0$  is  $1/3$ , find the other root, and the value of  $k$ .

15. If the sum of the roots of  $2hx^2 - 24x + 15h + 5 = 0$  is 6, find the roots, and the value of  $h$ .

16. If the product of the roots of  $(h - 4)x^2 - 11x + 2h = 0$  is 6, find the roots, and the value of  $h$ .



17. If one root of  $4x^2 - 12x + h = 0$  exceeds the other root by 2, find the roots, and the value of  $h$ .

18. If one root of  $3x^2 - 10x + 2h + 1 = 0$  is nine times the other, find the roots, and the value of  $h$ .

19. If one root of  $2hx^2 + (3h - 6)x - 9 = 0$  is the negative of the other, find the roots, and the value of  $h$ .

20. If the roots of  $9x^2 - 24x + h = 0$  differ by 2, find the roots, and the value of  $h$ .

Write a quadratic equation with integral coefficients having the given numbers as roots:

21. 5, -2.

22. 3, -7.

23.  $1/2$ ,  $-1/3$ .

24. 3,  $-1/2$ .

25.  $3/2$ ,  $-2/3$ .

26.  $-4/3$ , 2.

27.  $3\sqrt{2}$ ,  $-3\sqrt{2}$ .

28.  $2\sqrt{3}$ ,  $-2\sqrt{3}$ .

29.  $5i$ ,  $-5i$ .

30.  $\pm i\sqrt{3}$ .

31.  $\frac{2 \pm \sqrt{3}}{4}$ .

32.  $\frac{3 \pm \sqrt{5}}{2}$ .

33.  $3 \pm 2i$ .

34.  $\frac{2 \pm 3i}{4}$ .

35.  $a \pm \sqrt{b}$ , where  $a$  and  $b$  are integers.

36.  $a \pm bi$ , where  $a$  and  $b$  are integers.

79. Equations leading to quadratics. (A) *Fractional equations often lead to quadratics.*

*Example 1.* Solve the equation

$$\frac{x^2}{(x-2)(x-3)} = \frac{2}{x-2} + \frac{6}{(x-2)(x-3)}.$$

*Solution.* Multiplying both members of the equation by the L.C.D.  $(x-2)(x-3)$ , we get

$$x^2 = 2(x-3) + 6,$$

or

$$x^2 - 2x = 0,$$

whose roots are 2 and 0. This proves that if  $x$  satisfies the original equation, it can only be either 2 or 0. That is, 2 and 0 are the only eligible candidates for the position of root. To decide whether they are really roots, we must investigate the correctness of the converse\* propositions: "If  $x = 2$ , then  $x$  satisfies the original equation," and "If  $x = 0$ , then  $x$  satisfies the original equation." This is most easily done by substituting the candidates for the rootship in the original equation. We find that 0 is a root, since

\* The student should review section 29.



$$\frac{0^2}{(0-2)(0-3)} = \frac{2}{0-2} + \frac{6}{(0-2)(0-3)},$$

$$\frac{0}{6} = -1 + \frac{6}{6}, \text{ or } 0 = -1 + 1.$$

But 2 is not a root, since division by 0 is excluded.

The unsuccessful candidates for the rootship, if any, are often called extraneous roots, but this is merely a euphemistic way of saying that they are not roots at all. The occurrence of extraneous roots shows clearly that checking is not merely a superfluous step but an essential part of the work.

(B) Sometimes an equation may become quadratic upon making a change of variable, as in the following example.

**Example 2.** Solve  $\frac{1}{x^2} - \frac{5}{x} + 6 = 0$ .

**Solution.** Let  $u = \frac{1}{x}$ . Then the equation becomes  $u^2 - 5u + 6 = 0$ , which has the solutions  $u = 2$  and  $u = 3$ . Hence  $\frac{1}{x} = 2$  and  $\frac{1}{x} = 3$ . From  $\frac{1}{x} = 2$  we obtain  $x = \pm \frac{1}{2} = \pm \frac{\sqrt{2}}{2}$ . From  $\frac{1}{x} = 3$  we obtain  $x = \pm \frac{1}{3} = \pm \frac{\sqrt{3}}{3}$ . The student should check all four roots in the original equation.

## EXERCISES

*Do as many as you can.*

1.  $x = \frac{1}{x-2}$

2.  $6x = \frac{x-6}{x-2}$

3.  $\frac{x}{x-2} = \frac{2}{x-4}$

4.  $\frac{x}{2-x} = \frac{3}{x-2}$

5.  $\frac{x^2}{(x-3)(x+1)} = \frac{2}{x-3} + \frac{3}{(x-3)(x+1)}$

6.  $\frac{2x^2}{(x+2)(x-1)} = \frac{1}{x+2} + \frac{1}{(x+2)(x-1)} = 0$

7.  $\frac{x+2}{x+1} = \frac{1}{2x}$

8.  $\frac{1}{x+5} + \frac{2}{x+1} = 1$

9.  $\frac{1}{x+1} + \frac{2}{x+3} = \frac{1}{2x-1}$

10.  $\frac{x-2}{x-1} = \frac{6}{x-1} - \frac{8}{x-1} + 5$

11.  $x^4 - 13x^2 + 36 = 0$

12.  $x^4 - 20x^2 + 100 = 0$



13.  $\frac{144}{x^4} - \frac{97}{x^2} + 9 = 0.$

14.  $\frac{36}{x^4} - \frac{25}{x^2} + 4 = 0.$

15.  $(x - 4)^4 - 11(x - 4)^2 + 18 = 0.$  (*Hint: Let  $u = (x - 4)^2$ .*)

16.  $(x - 2)^4 - 7(x - 2)^2 + 12 = 0.$

17.  $(x^2 - 2x)^2 - 7(x^2 - 2x) - 8 = 0.$  (*Hint: Let  $u = x^2 - 2x$ .*)

18.  $(x^2 + 3x)^2 - 14(x^2 + 3x) + 40 = 0.$

19.  $(x^3 + 8) + (2x^2 + x - 6) = 0.$  (*Hint: Factor each expression in parentheses and extract common factor.*)

20.  $(x^3 - 1) + (x^2 - 5x + 4) = 0.$

21.  $x^6 + 7x^3 - 8 = 0.$

22.  $8x^6 + 7x^3 - 1 = 0.$

**80. Irrational equations.** An irrational equation is one in which an irrational expression\* is involved. In solving equations of this type, we shall again find it necessary to check the converse proposition by substitution in the original equation.†

*Example 1.* Solve the equation  $x - 7 = \sqrt{x - 5}$ .

*Solution.* Squaring both sides we get  $x^2 - 14x + 49 = x - 5$ , or  $x^2 - 15x + 54 = 0$ . The roots of the latter equation are 6 and 9. We have so far proved that if  $x$  satisfies the original equation, then  $x$  can only be 6 or 9. That is, 6 and 9 are the only candidates eligible for the position of root. To see whether or not they are roots, we must check the correctness of the converse propositions: "If  $x = 6$ , then  $x$  satisfies the original equation," and "If  $x = 9$ , then  $x$  satisfies the original equation." This is done by substituting in the original equation. For  $x = 6$ , we get  $6 - 7 = \sqrt{6 - 5}$ , or ‡  $-1 = +1$ . Thus 6 is not a root. But for  $x = 9$ , we get  $9 - 7 = \sqrt{9 - 5}$ , or  $2 = 2$ . Thus, 9 is a root.

*Example 2.* Solve  $\sqrt{x - 2} = \sqrt{x} + 2$ .

*Solution.* Squaring both sides, we get  $x - 2 = x + 4\sqrt{x} + 4$ . Since we have not yet gotten rid of the radicals completely, we transpose, obtaining  $-6 = 4\sqrt{x}$ , or  $-3 = 2\sqrt{x}$ , and square again. Thus  $9 = 4x$ , or  $x = 9/4$ . Therefore  $9/4$  is the only possible root. But substitution reveals that it is not a root, since

$$\sqrt{\frac{9}{4} - 2} = \sqrt{\frac{9}{4}} + 2 \text{ yields } \sqrt{\frac{1}{4}} = \sqrt{\frac{9}{4}} + 2, \text{ or } \frac{1}{2} = \frac{3}{2} + 2, \text{ which is}$$

false. Hence the given equation has no root at all.

Since extraneous roots occur, some step in the argument must fail to be reversible.§ Which step is it?

\* See section 24 for definition.

† Compare the preceding section, and section 29.

‡ Recall that the radical sign represents the principal root only.

§ See section 29.



## EXERCISES

*Solve and check:*

1.  $\sqrt{x+2} = 3$ .
2.  $\sqrt{x+2} = -3$ .
3.  $\sqrt{x} = x - 2$ .
4.  $\sqrt{x} = 2 - x$ .
5.  $\sqrt{3x+4} = 2 + \sqrt{2x-4}$ .
6.  $\sqrt{x-4} = 9 - \sqrt{x+5}$ .
7.  $\sqrt{x-4} = \sqrt{x+5} - 9$ .
8.  $\sqrt{x+4} + \sqrt{x+11} = 7$ .
9.  $\sqrt{x+2} + \sqrt{2x+5} = 1$ .
10.  $\sqrt{3x+1} = 1 + \sqrt{2x-1}$ .
11.  $\sqrt{2x+4} = 1 + \sqrt{2x}$ .
12.  $\sqrt{3x-1} + \sqrt{3x+6} = 7$ .
13.  $\sqrt{3-2x} = 3 + \sqrt{2+2x}$ .
14.  $\sqrt{x+5} + \sqrt{2x+3} = 3$ .
15.  $\sqrt{2x+1} + \sqrt{2x-1} = 4$ .
16.  $\sqrt{2x-1} - \sqrt{x} = 2$ .
17.  $\sqrt{3x-2} - \sqrt{x} + \sqrt{x-1} = 0$ .
18.  $\frac{1}{\sqrt{x-2}} - \sqrt{\frac{2x-3}{x-2}} = 2$ .
19.  $\sqrt{3x-3} - \sqrt{x+1} = \sqrt{2x}$ .
20.  $\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$ .

81. Verbal problems. Many verbal problems lead to the solution of quadratics. The student should review carefully the seven steps in solving a verbal problem given in section 63.

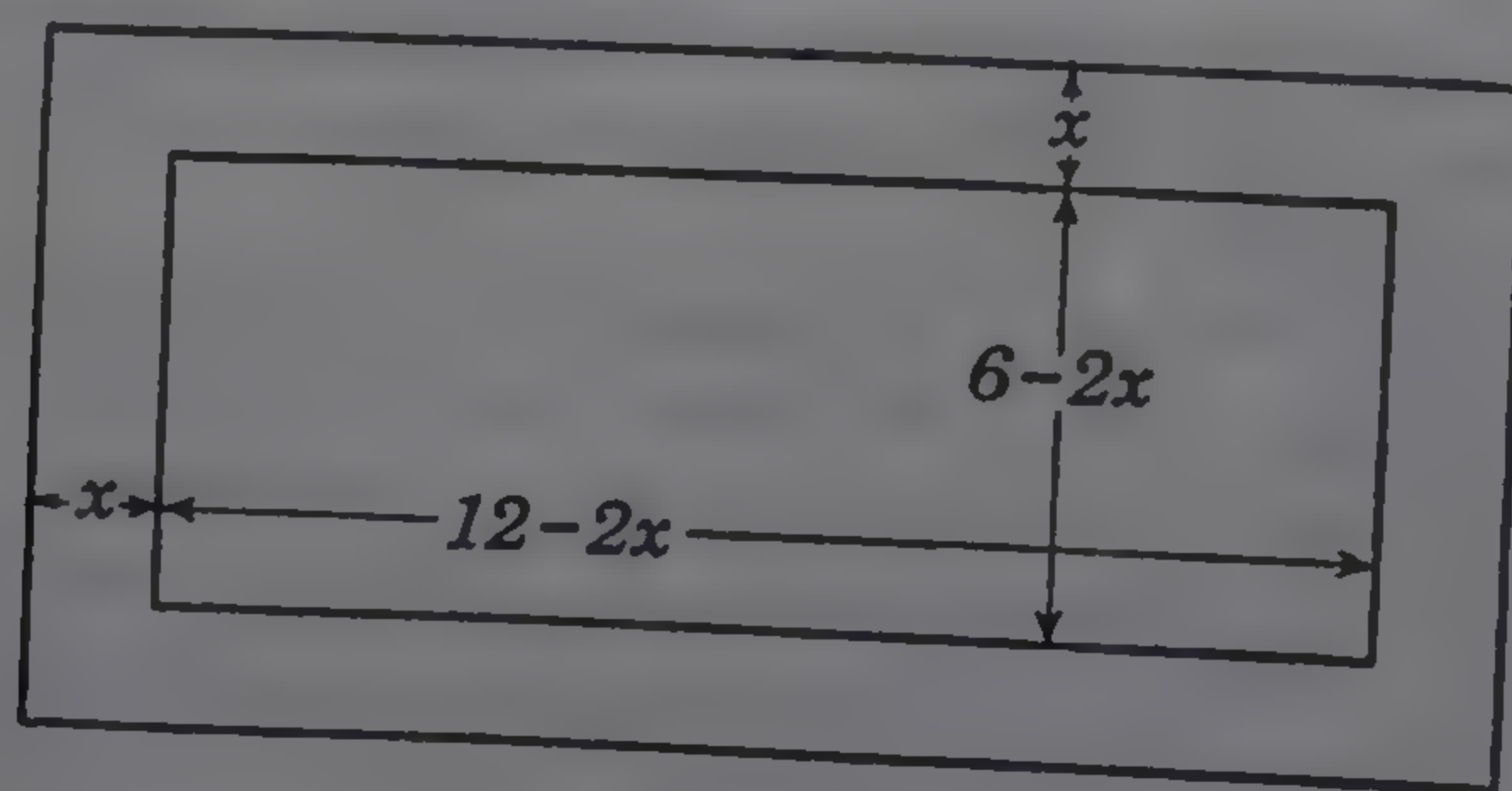


Fig. 29.

*Example.* Within a flower garden 6 yards wide and 12 yards long, we are to pave a path of uniform width around the boundary so as to leave an area of 40 square yards for flowers. How wide should we make the path?

*Solution.* Let  $x$  = width of path, in yards. Then the width of the actual flower plot is  $6 - 2x$ , and the length is  $12 - 2x$ . The area therefore is  $(6 - 2x)(12 - 2x)$ . Hence the equation is

$$(1) \quad (6 - 2x)(12 - 2x) = 40.$$

The solutions of (1) are 1 and 8.



Clearly 1 satisfies the conditions of the verbal problem, since the flower plot will then have the dimensions 4 yards by 10 yards. But while 8 is a root of equation (1), it cannot satisfy the conditions of the problem, since you cannot pave a path 8 yards wide in a garden only 6 yards wide. How can this happen? See section 63.

## EXERCISES

1. The area of a rectangle is 56 square yards. Its perimeter is 30 yards. Find its length and width.

2. The area of a rectangle is 400 square feet, and its perimeter is 100 feet. Find its length and width.

3. The two legs of a right triangle differ by 7 feet. The hypotenuse is 13 feet long. Find the legs. (*Hint: Use the Pythagorean theorem.*)

4. One leg of a right triangle is 1 foot less than twice the other leg. The hypotenuse is 17 feet long. Find the legs.

5. Find two consecutive numbers whose product is 552.

6. Find two consecutive even numbers whose product is 1088.

7. Find two consecutive odd numbers whose product is 783.

8. The length of a rectangle is 24 feet greater than the width. The area is 1620 square feet. Find the length and the width.

9. The number of inches in the perimeter of a certain square is equal to the number of square inches in its area. Find the length of the side of the square.

10. If the price of eggs rises 10 cents per dozen, one will be able to get 2 dozen fewer eggs with \$6.00 than was possible at the lower price. What was the lower price?

11. A car travels 120 miles. A second car travels 10 miles per hour faster than the first and makes the same trip in 2 hours less. Find the speed of each.

12. A stone is dropped from the top of a vertical cliff 100 feet high. Using the formula  $s = 16t^2$  for the distance  $s$  (measured in feet) through which a body will fall in  $t$  seconds, find the number of seconds it takes for the stone to reach bottom.

13. A stone is dropped from the top of a vertical wall 36 feet high. Using the formula of exercise 12, find the number of seconds it takes for the stone to reach bottom.

14. A stone is dropped from a cliff overlooking a lake, and the sound of the splash is heard 9 seconds after the stone was dropped. If the velocity of sound were 1024 feet per second, how high is the cliff above the lake? (*Hint: use the formula of exercise 12 together with other considerations.*)

15. Find a positive number which when diminished by 20 is equal to 69 times the reciprocal of the number.

16. Find a positive number which when increased by 24 is equal to 52 times the reciprocal of the number.



17. *A* alone can do a job in 6 days less than it takes *B* alone to do the same job. Working together they complete the job in 4 days. How long would it take each to do the job alone?

18. One pipe alone can fill a tub in 4 minutes less than it takes a second pipe alone to fill the tub. Both pipes together can fill the tub in 2 minutes. How long does it take each pipe alone to fill the tub?

19. Within a garden 10 yards wide and 20 yards long we wish to pave a walk of uniform width so as to leave an area of 96 square yards for flowers. How wide should the walk be?

20. A train travels 200 miles at a uniform speed. If its speed had been 10 miles per hour less, the trip would have taken 1 hour more. Find the actual speed of the train.

21. The sum of the digits of a two-digit number is 10. The tens digit is 2 less than the square of the units digit. Find the number.

22. A pet shop bought a litter of puppies for \$80. All but 3 of them were sold, at a profit of \$6 each, for a total of \$80. How many puppies were in the litter?

23. The length of a rectangle is 6 feet greater than its width. Its area is 616 square feet. Find its dimensions.

24. If a body is thrown vertically upward with an initial velocity of  $v$  feet per second, its height  $h$  above the starting level, measured in feet, after  $t$  seconds, is given approximately by the formula  $h = vt - 16t^2$ , neglecting air resistance. (a) Solve this formula for  $t$ ; (b) find  $t$  if  $v = 128$  ft. per sec. and  $h = 192$  ft.

25. In calculating the current in a simple series electrical circuit with inductance  $L$  henrys, resistance  $R$  ohms, and capacitance  $C$  farads, it becomes necessary to solve the equation  $Lx^2 + Rx + \frac{1}{C} = 0$ . Find  $x$  in terms of  $L$ ,  $R$ , and  $C$ .

26. The average speed of a stream is 3 miles per hour. A boat takes 3-4 hour longer to travel 35 miles upstream than it does for the return trip downstream. What is the speed of the boat in still water and how long did the trip take each way?

27. A boat travels 30 miles down-stream and 30 miles back upstream in 4 hours. If the speed of the current is 4 miles per hour, what is the boat's speed in still water, and how long did the trip take each way?

28. Using the formula of exercise 24, how long will it take for a ball thrown vertically upward with an initial velocity of 60 feet per second to reach the point 54 feet above starting level? Explain the physical significance of the fact that there are two answers.

29. The sum  $s$  of the first  $n$  consecutive positive integers 1, 2, 3, . . . ,  $n$  is given by the formula  $s = \frac{1}{2}n(n + 1)$ . How many of these consecutive integers must be added to obtain a sum of 703?

30. The sum  $s$  of the first  $n$  consecutive even numbers 2, 4, 6, . . . ,  $2n$  is given by the formula  $s = n(n + 1)$ . How many of these consecutive even numbers must be added to obtain a sum of 306?



31. If \$2000 invested at interest compounded annually amounts to \$2121.80 in two years, what is the rate of interest?

32. Two straight roads cross at right angles. Two men start simultaneously from the point of intersection,  $A$  walking at the rate of 2.5 miles per hour on one road,  $B$  walking at the rate of 6 miles per hour on the other road. How long will it take for them to be 26 miles apart?

**\*82. Maxima and minima of quadratic functions.** The graph of a quadratic function  $y = ax^2 + bx + c$  with real coefficients ( $a \neq 0$ ) is a parabola and obviously has a highest point if it opens downward, or a lowest point if it opens upward. The coordinates of this **maximum** or **minimum** point may be found by completing the square as follows.

*Example 1.* Find the coordinates of the maximum or minimum point of  $y = x^2 - 6x - 1$ .

*Solution.* We add and subtract the square of half the coefficient of  $x$ , obtaining  $y = x^2 - 6x + 9 - 1 - 9$  or

$$(1) \quad y = (x - 3)^2 - 10.$$

Since  $(x - 3)^2$  is never negative for real values of  $x$ , the minimum  $y$  is clearly  $-10$  and is attained when  $(x - 3)^2 = 0$ , or  $x = 3$ . Hence the minimum point has the coordinates  $(3, -10)$ .

*Example 2.* Find the coordinates of the maximum or minimum point of  $y = 5 - 12x - 3x^2$ .

*Solution.* We write  $y = -3(x^2 + 4x) + 5$ . To complete the square within the parentheses we must insert 4. Hence we must add 12 outside the parentheses, obtaining  $y = -3(x^2 + 4x + 4) + 5 + 12$ , or

$$(2) \quad y = -3(x + 2)^2 + 17.$$

Since  $(x + 2)^2$  is never negative for real values of  $x$ , the maximum value of  $y$  is clearly 17, and this is attained when  $(x + 2)^2 = 0$ , or  $x = -2$ . Hence the maximum point has the coordinates  $(-2, 17)$ .

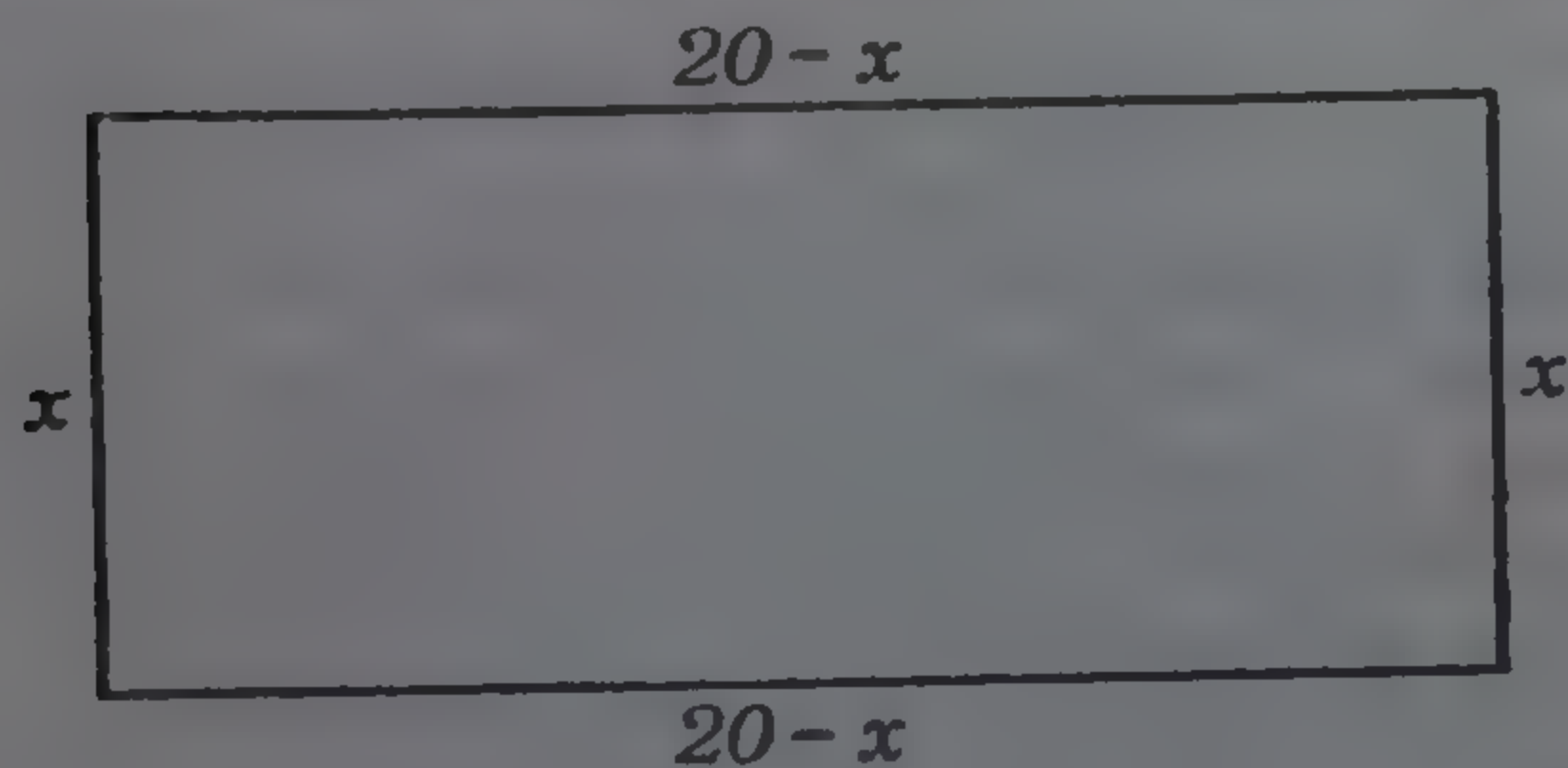


Fig. 30.

Finding the maximum or minimum values of a function  $y = f(x)$  and the value of  $x$  for which it is attained is clearly a

problem of paramount practical importance. In engineering, business, etc., we want to get the best result, expend the least

\* This section may be omitted without disturbing the continuity of the book.



energy, and so on. Whenever the function involved is quadratic, we may solve such a problem as above. A general method (for quadratic or non-quadratic functions) is provided in the subject known as *differential calculus*.\*

*Example 3.* Find the dimensions of the rectangle with maximum area whose perimeter is 40 feet, and find the maximum area.

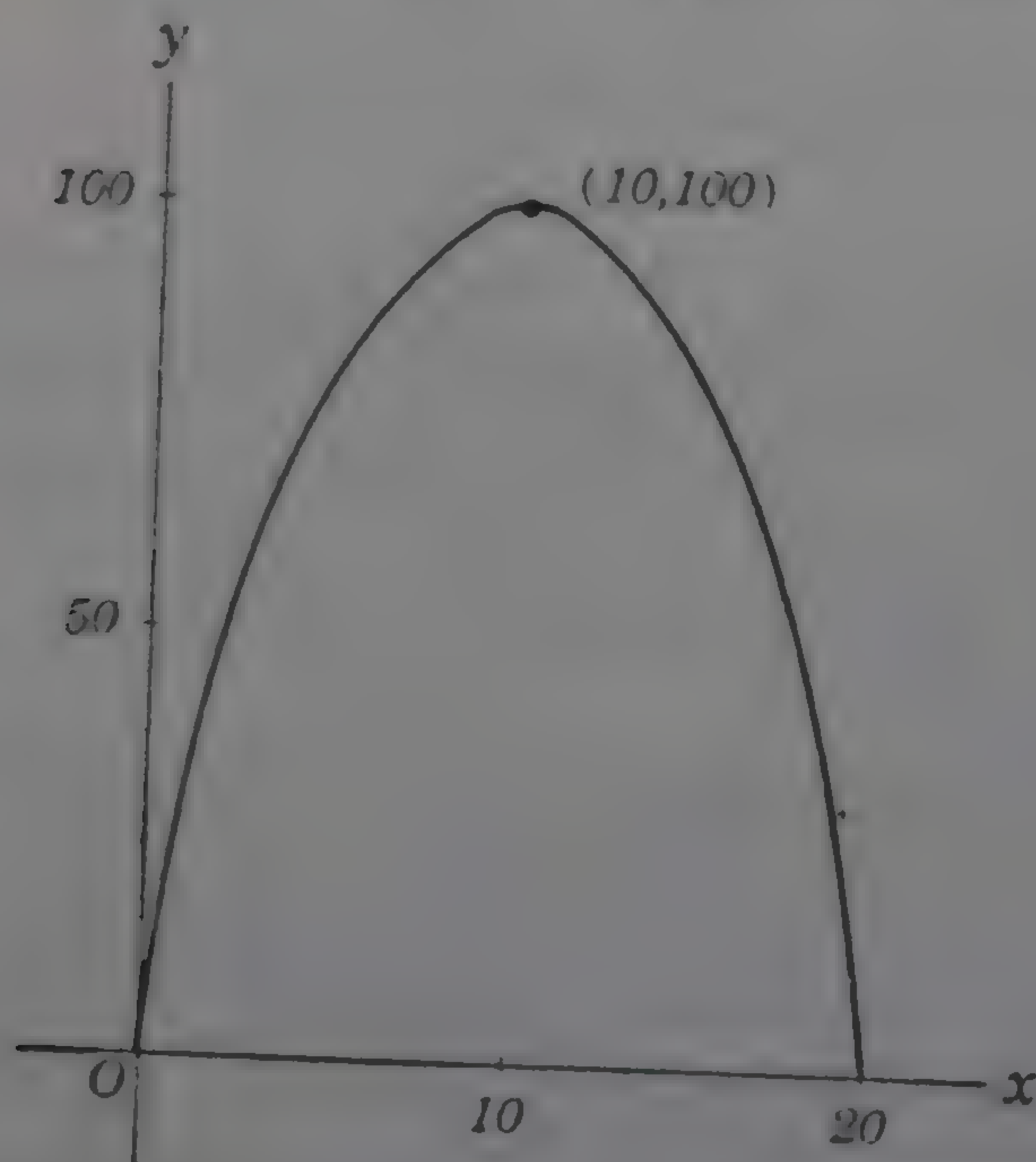


Fig. 31.

*Solution.* Let  $x$  be the width of the rectangle, measured in feet. Then  $20 - x$  is the length and  $y = x(20 - x)$  is the area (Fig. 30). Thus the function whose maximum we wish is the function

$$y = 20x - x^2,$$

or

$$y = -(x^2 - 20x).$$

Completing the square, we get  $y = -(x^2 - 20x + 100) + 100$  or

$$y = -(x - 10)^2 + 100.$$

Hence the maximum area is 100 square feet and is attained when  $x = 10$ . The required rectangle therefore is a square. The graph

\* For a brief introduction to this subject, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, Chapter XI.



of the function (3) given in Fig. 31 indicates the relationship between this example and the preceding ones.

## EXERCISES

*Find the coordinates of the maximum or minimum points, as the case may be, state whether the point determined is a maximum or minimum point, and sketch the graph of each of the following:*

1.  $y = x^2 - 6x + 2.$

2.  $y = x^2 + 4x + 3.$

3.  $y = 4x - x^2 + 3.$

4.  $y = 6x - x^2 + 2.$

5.  $y = 2x^2 - 12x + 7.$

6.  $y = 5 + 6x - 3x^2.$

7.  $y = 3 + 4x + 4x^2.$

8.  $y = 2 + 2x - 4x^2.$

9.  $y = 15 - 6x - 6x^2.$

10.  $y = 3 - 3x + 3x^2.$

11. Find two numbers whose sum is 20 and whose product is a maximum.

12. Find the dimensions of the rectangle with maximum area whose perimeter is 60 feet. What is the maximum area?

13. Find two numbers whose sum is 20 and such that the sum of their squares is minimum.

14. Find the dimensions of the rectangle whose perimeter is 40 feet and such that the square of its diagonal is a minimum.

15. A body is thrown straight up from the ground with an initial velocity of 96 feet per second. Neglecting air resistance, its height  $h$ , measured in feet, after  $t$  seconds, is given by the formula  $h = 96t - 16t^2$ . After how many seconds does the body reach its maximum height, and what is the maximum height?

16. A projectile is thrown straight up from a height of 6 feet, with an initial velocity of 192 feet per second. Neglecting air resistance, its height  $h$ , measured in feet, after  $t$  seconds is given by the formula  $h = 6 + 192t - 16t^2$ . After how many seconds does the projectile reach its maximum height, and what is the maximum height?

17. If 400 people will attend a moving picture theater when the admission price is 30 cents, and if the attendance decreases by 40 for each 10 cents added to the price, then what price of admission will yield the greatest gross receipts?

18. A telephone company can get 1000 subscribers at a monthly rate of \$5.00 each. It will get 100 more subscribers for each 10-cent decrease in the rate. What rate will yield the maximum gross monthly income and what will this income be?

19. It is desired to enclose a rectangular field adjacent to the straight bank of a river. No fence is needed along the river bank. If 200 yards of fence is to be used, what are the dimensions of the rectangle which will enclose the maximum area, and what is the maximum area?

20. The base of a triangle is 20 feet and its altitude is 12 feet. Find the dimensions of the rectangle with maximum area that can be inscribed in the triangle with one side along the base (Fig. 32).



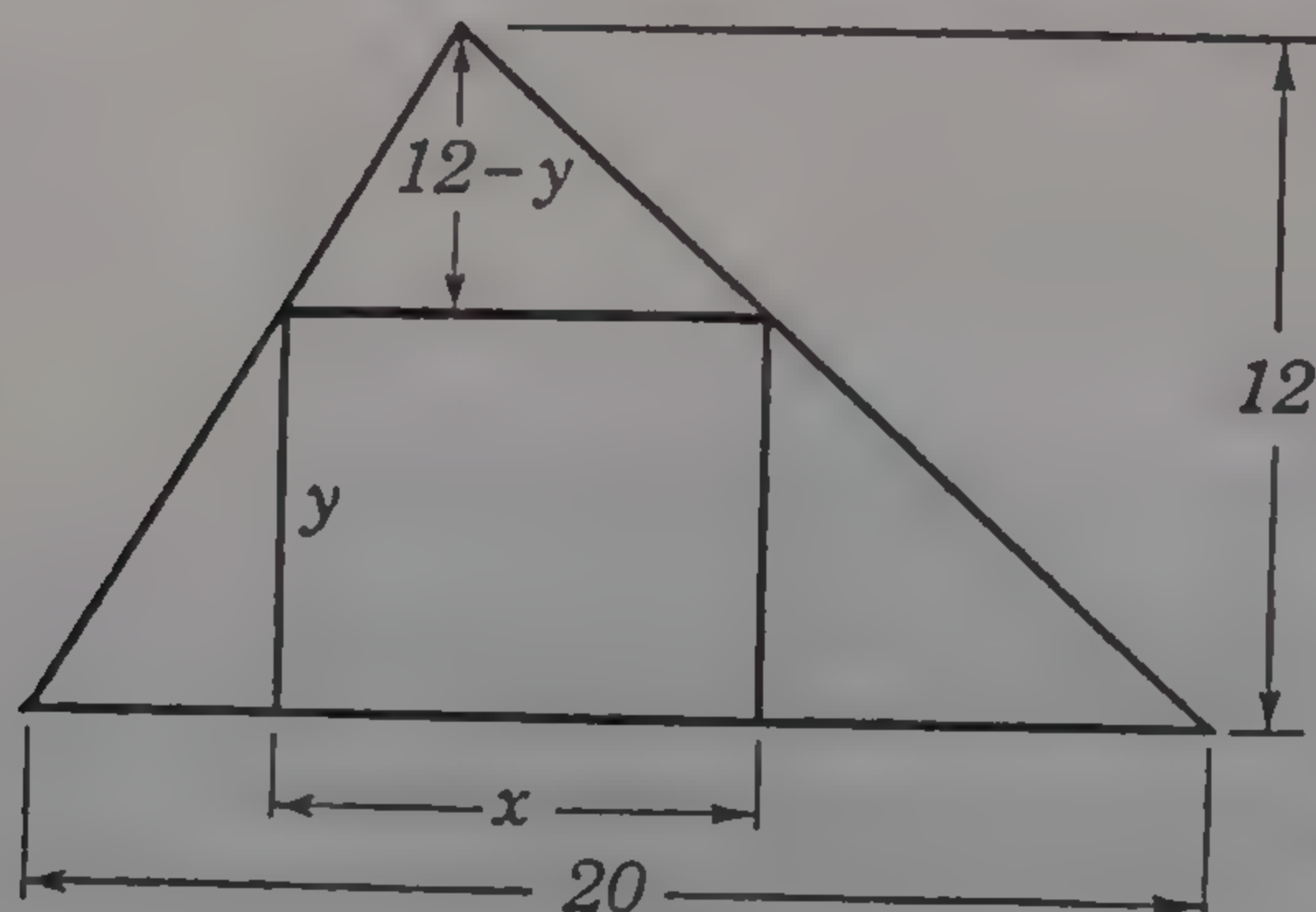


Fig. 32.

21. Show that if  $y = ax^2 + bx + c$  ( $a \neq 0$ ) has real coefficients, then the maximum or minimum point of the graph has the coordinates  $x = -b/2a$ ,  $y = (4ac - b^2)/4a$ , and that the point is a maximum or a minimum according as  $a$  is negative or positive.

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## CHAPTER XII

### Systems of Equations in Two Unknowns Involving Quadratics

83. The graph of a quadratic equation in  $x$  and  $y$ . To plot the graph of an equation of the second degree in  $x$  and  $y$  with real coefficients, it is usually desirable to solve the equation for  $y$  in terms of  $x$ , or for  $x$  in terms of  $y$ , and calculate a table, as in the following example. Square roots may be worked out, or read from table I. The table should always include the values  $x = 0$  and  $y = 0$  if possible; these will yield the points where the curve crosses the  $y$ -axis and the  $x$ -axis, respectively.

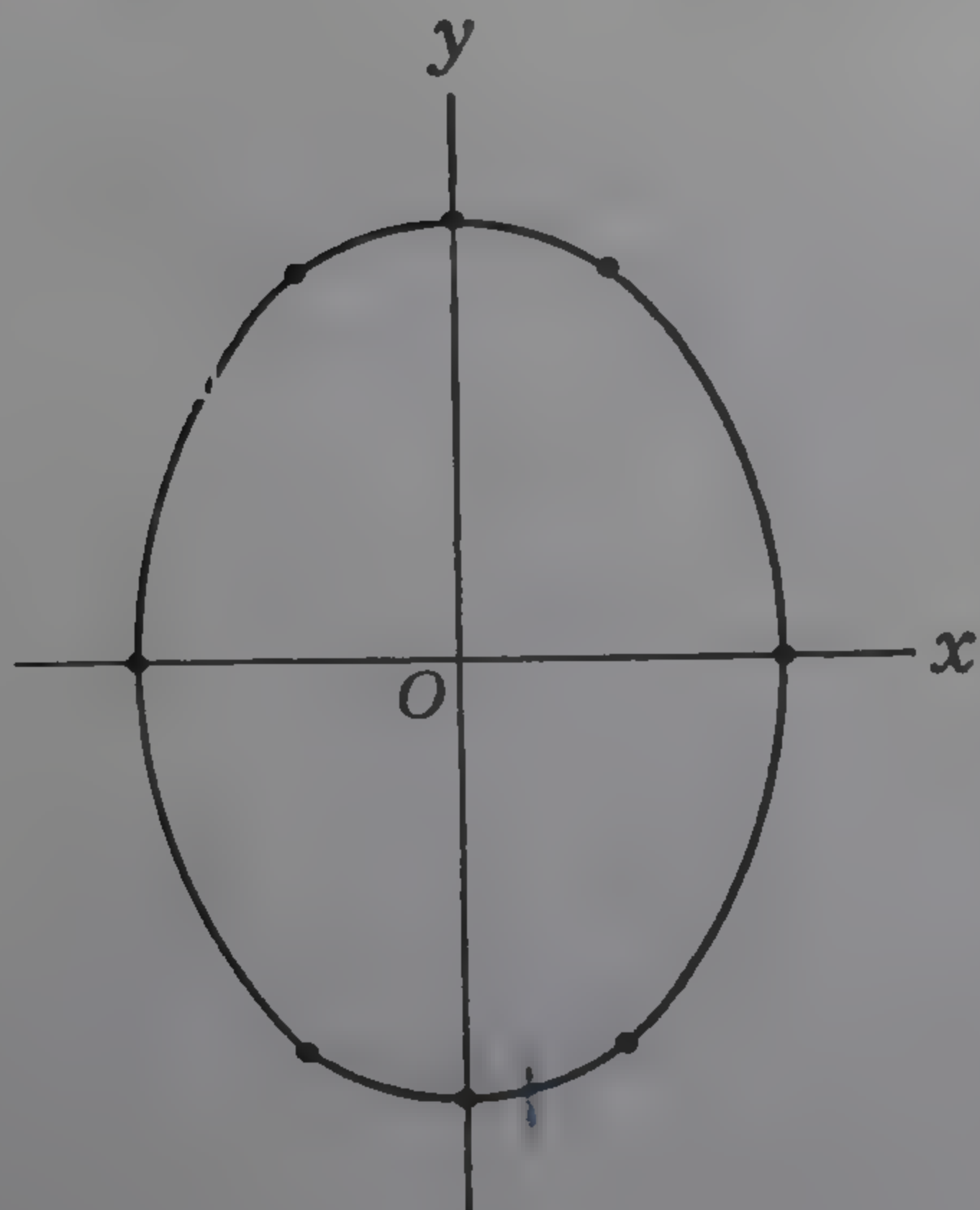


Fig. 33.

*Example 1.* Plot the graph of the equation  $9x^2 + 4y^2 = 36$ .

*Solution.* Solving for  $y$ , we get

$$y = \pm \frac{1}{2} \sqrt{36 - 9x^2}. \quad \text{Substituting}$$

values for  $x$ , we get the table:

| $x$ | 0       | $\pm 2$ | +1                                   | -1                                   |
|-----|---------|---------|--------------------------------------|--------------------------------------|
| $y$ | $\pm 3$ | 0       | $\pm \frac{3}{2} \sqrt{3} = \pm 2.6$ | $\pm \frac{3}{2} \sqrt{3} = \pm 2.6$ |

This yields eight points, which are joined to form the curve in Fig. 33. It is clear that no points of the curve exist for  $x > 2$ , or  $x < -2$ , since in either case  $y$  is imaginary.

*Example 2.* Plot the graph of  $xy = 1$ .

*Solution.* Solving for  $y$  in terms of  $x$ , we get  $y = \frac{1}{x}$ . Clearly no points of the curve exist with  $x = 0$  or  $y = 0$ . We get the table

| $x$ | 4             | 3             | 2             | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | -4             | -3             | ... |
|-----|---------------|---------------|---------------|---|---------------|---------------|---------------|----------------|----------------|-----|
| $y$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | 1 | 2             | 3             | 4             | $-\frac{1}{4}$ | $-\frac{1}{3}$ | ... |



This yields the graph of Fig. 34.

The general equation of second degree in  $x$  and  $y$  may be written as

$$(1) \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The following facts, proved in textbooks on analytic geometry, will be helpful in plotting graphs.

If the coefficients are all real, the graph is either an ellipse (Fig. 33), a hyperbola (Fig. 34), a circle, a parabola (Fig. 28), or, in special cases, a pair of straight lines (distinct or coincident), a single point, or non-existent. In particular:

If  $B^2 - 4AC$  is positive, the graph is a hyperbola, or, in special cases, a pair of intersecting straight lines;

If  $B^2 - 4AC$  is zero, the graph is a parabola, or, in special cases, two parallel or coincident straight lines, or non-existent;

If  $B^2 - 4AC$  is negative, the graph is an ellipse, or, in special cases, a circle, a single point, or non-existent;

If  $B = 0$  and  $A = C$ , the graph is a circle, or, in special cases, a single point or non-existent.

All the graphs of second-degree equations in  $x$  and  $y$  are called **conic sections** because they are the intersections of a right circular cone and a plane. These curves were studied by the ancient Greeks without thought of practical applications. A great book about them was written by Apollonius of Perga (3rd century B.C.). Their equations were first used to study them in the 17th century by Descartes, Fermat, Wallis, and others. They have many practical applications in physics. For example, the path of a projectile is, neglecting air resistance, a parabola; the orbits of the planets about the sun are, neglecting slight perturbations, ellipses.\*

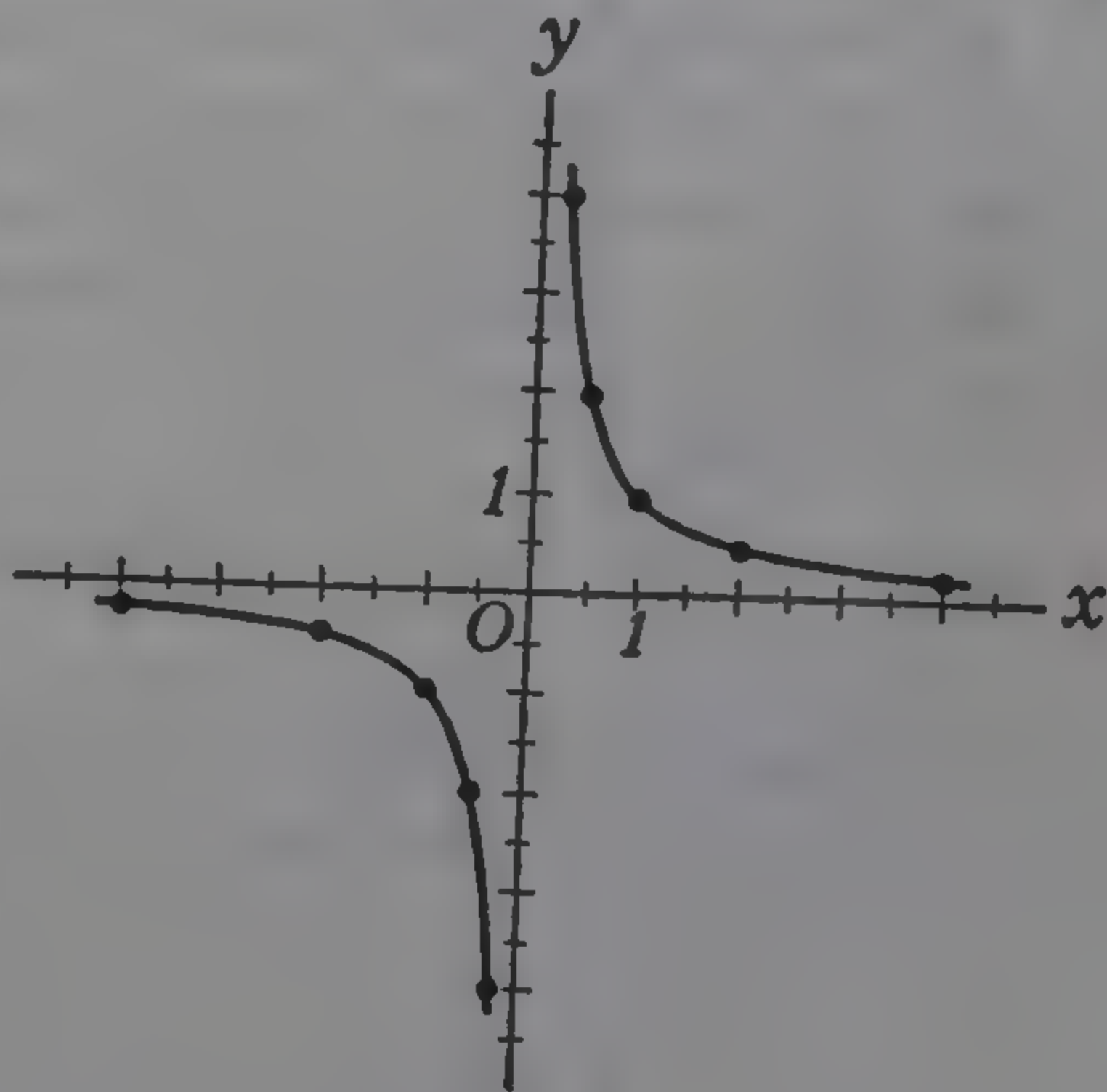


Fig. 34.

\* For other applications, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, section 82, and textbooks on analytic geometry.



## EXERCISES

Plot the graph of each of the following and identify the curve:

1.  $x^2 + y^2 = 25$ .
2.  $x^2 + y^2 - 4x + 2y - 4 = 0$ .
3.  $y = 3x^2 + 4x - 1$ .
4.  $9x^2 - 24xy + 16y^2 - 200x - 150y = 0$ .
5.  $x^2 - y^2 = 1$ .
6.  $xy - 3y = 1$ .
7.  $x^2 - y^2 - 4x - 6y - 6 = 0$ .
8.  $4x^2 + 9y^2 = 36$ .
9.  $9x^2 + 4y^2 - 18x + 16y - 11 = 0$ .
10.  $36x^2 + 24xy + 29y^2 = 5$ .
11.  $x^2 + y^2 = 0$ .
12.  $x^2 + y^2 = 1$ .
13.  $x^2 + y^2 = -1$ .
14.  $4x^2 + y^2 = 16$ .
15.  $x^2 - y^2 = 0$ . (*Hint*: The equation is equivalent to the system  $x + y = 0, x - y = 0$ .)
16.  $(x + 3)(y - 1) = 1$ .
17.  $x^2 - 2xy + y^2 = 0$ . (*Hint*: The equation is equivalent to  $(x - y)^2 = 0$ .)

84. Graphical solution of simultaneous equations. A common (real) solution of two equations in  $x$  and  $y$  are the coordinates of a

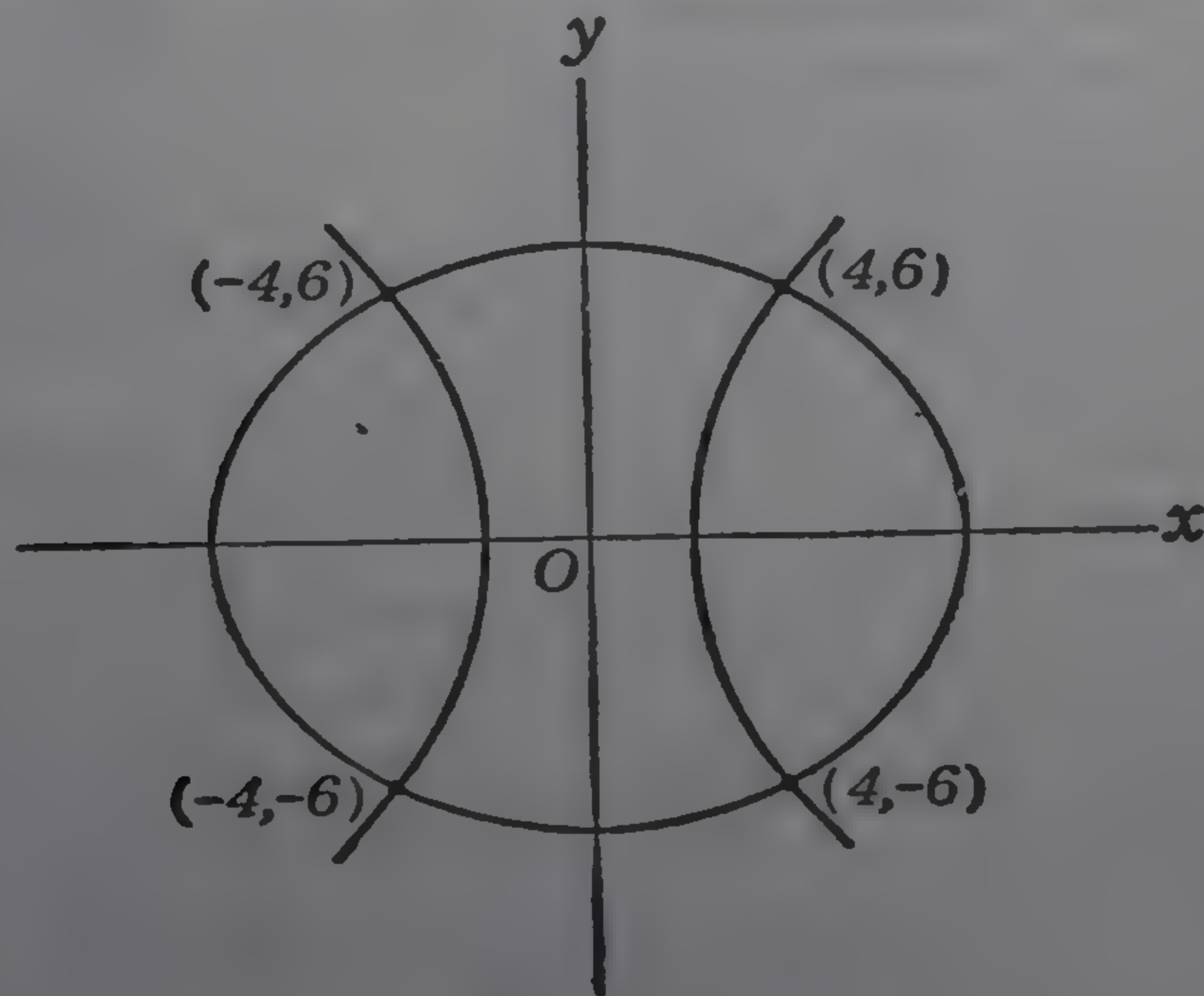


Fig. 35.

point belonging to both graphs, that is, a point of intersection of the two curves. Hence the common real solutions can be estimated from the graph.

*Example 1.* Find graphically the common solutions of

$$3x^2 + 4y^2 = 192$$

$$3x^2 - y^2 = 12.$$

and



Plotting the graphs of the two equations, we get Fig. 35, from which we read (approximately) the common solutions:

$$\frac{x}{y} \left| \frac{4}{6} \right| \frac{4}{-6} \left| \frac{-4}{6} \right| \frac{-4}{-6}.$$

The graphical solution yields only the *real* common solutions, since the  $x$ - and  $y$ -coordinates on the graph are always real numbers, and even these solutions are usually found only approximately, since they are based on drawing and eyesight. To find accurate solutions in all cases we must use algebraic methods, as in the following sections. However, it is desirable to plot the graphs in any case, since they serve as a rough check on the algebraic solution. For many practical problems, a graphical solution is sufficiently accurate. Two quadratic equations in  $x$  and  $y$  can have no more than four points of intersection, and they may have fewer.

### EXERCISES

*Estimate graphically the real solutions of each of the following systems of equations:*

1.  $\begin{cases} x^2 + y^2 = 25 \\ 4y = 3x. \end{cases}$
2.  $\begin{cases} x^2 + y^2 = 34 \\ x - y = 2. \end{cases}$
3.  $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10. \end{cases}$
4.  $\begin{cases} x^2 + y^2 = 13 \\ 2x - y = 4. \end{cases}$
5.  $\begin{cases} x^2 + 3y^2 = 12 \\ x + 3y = 6. \end{cases}$
6.  $\begin{cases} 25x^2 + 4y^2 = 100 \\ 5x - 2y = 10. \end{cases}$
7.  $\begin{cases} xy = 12 \\ x + y = 7. \end{cases}$
8.  $\begin{cases} xy = 1 \\ 3y - 5x = 2. \end{cases}$
9.  $\begin{cases} y^2 = 4x \\ y + 2x = 4. \end{cases}$
10.  $\begin{cases} y^2 = 4x \\ x + y = 3. \end{cases}$
11.  $\begin{cases} x^2 - y^2 = 5 \\ 3x + y = 11. \end{cases}$
12.  $\begin{cases} 7y^2 - 6xy = 8 \\ 2y - 3x = 5. \end{cases}$
13.  $\begin{cases} y^2 = 2x \\ x^2 = 2y. \end{cases}$
14.  $\begin{cases} y^2 = x \\ y = 4x - 3. \end{cases}$
15.  $\begin{cases} x^2 + y^2 = 25 \\ 4y^2 = 9x. \end{cases}$
16.  $\begin{cases} (x - 2)^2 + y^2 = 5 \\ x^2 + (y - 1)^2 = 10. \end{cases}$
17.  $\begin{cases} x^2 + y^2 = 10 \\ x^2 + 9y^2 = 18. \end{cases}$
18.  $\begin{cases} x^2 + 2y^2 = 18 \\ 2x^2 + y^2 = 33. \end{cases}$
19.  $\begin{cases} x^2 - y^2 = 5 \\ 2x^2 + y^2 = 22. \end{cases}$
20.  $\begin{cases} x^2 + 2y^2 = 33 \\ y^2 - x^2 = 15. \end{cases}$
21.  $\begin{cases} y^2 = 9x \\ y^2 - x^2 = 8. \end{cases}$
22.  $\begin{cases} 3y^2 = 4x \\ 4x^2 + 9y^2 = 72. \end{cases}$
23.  $\begin{cases} x^2 - y^2 = 7 \\ x^2 + y^2 = 25. \end{cases}$
24.  $\begin{cases} y^2 = 2x + 10 \\ x^2 + y^2 = 25. \end{cases}$

85. Algebraic solution when one equation is linear and one quadratic. In this case, the linear equation can be solved for one



variable in terms of the other and this expression substituted in the quadratic equation, thus yielding a quadratic in the other variable only. The solution is completed as in the following example.

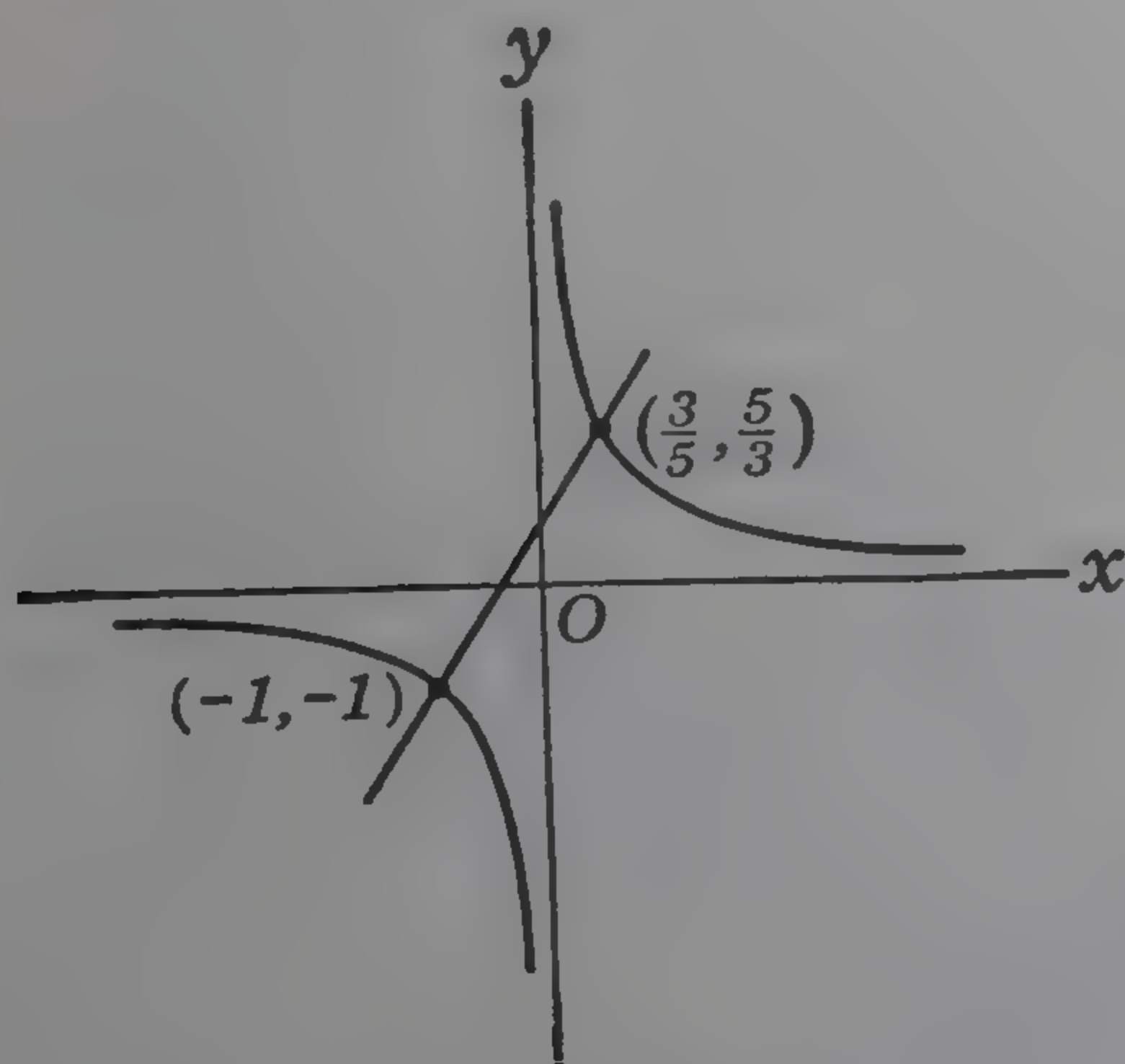


Fig. 36.

*Example 1.* Solve algebraically the system:

$$(1) \quad xy = 1$$

$$(2) \quad 3y - 5x = 2.$$

*Solution.* From the linear equation (2) we get

$$(3) \quad y = \frac{5x + 2}{3}.$$

Substituting this in the quadratic equation (1), we get

$$(4) \quad \frac{5x^2 + 2x}{3} = 1.$$

Solving (4), we get  $x = 3/5$  and  $x = -1$ . Substituting these results in (3), we get

$$\text{for } x = \frac{3}{5}, y = \frac{5}{3}$$

and

$$\text{for } x = -1, y = -1.$$

Both solutions should be checked by substituting in both equations.

## EXERCISES

*Solve algebraically and check:*

1-12. Solve exercises 1-12 of section 84.

$$13. \begin{cases} (x-2)^2 + y^2 = 5 \\ 2x - y = 4. \end{cases}$$

$$15. \begin{cases} (x-2)^2 + y = 5 \\ y = x + 1. \end{cases}$$

$$17. \begin{cases} x^2 + y^2 - 2y = 9 \\ 2x + y + 4 = 0. \end{cases}$$

$$19. \begin{cases} x^2 + xy - y^2 = 1 \\ 3y - 2x = 5. \end{cases}$$

$$14. \begin{cases} (y+1)^2 = x - 2 \\ y = 4x - 12. \end{cases}$$

$$16. \begin{cases} (x+1)^2 + (y+1)^2 = 5 \\ y = 2x + 1. \end{cases}$$

$$18. \begin{cases} x^2 + y^2 + 2y = 9 \\ y = 2x + 4. \end{cases}$$

$$20. \begin{cases} x^2 + y^2 - 2xy = 9 \\ 3x - 2y = 7. \end{cases}$$

21. Prove algebraically that the graphs of  $x^2 + y^2 = 1$  and  $x + y = 7$  do not intersect.

22. Prove algebraically that the graphs of  $2x^2 + y^2 = 2$  and  $x - y = 10$  do not intersect.



23. Find the positive value of  $k$  for which the graph of  $x + y = k$  is tangent to the graph of  $x^2 + y^2 = 1$ , and find the coordinates of the point of tangency.

24. Find the positive value of  $k$  for which the graph of  $3x + 5y = k$  is tangent to the graph of  $16x^2 + 25y^2 = 400$ , and find the coordinates of the point of tangency.

86. Algebraic solution when both equations are quadratic. The task of eliminating one variable was easily accomplished in the preceding section because one equation was linear, and from it one could get a simple expression for  $y$  in terms of  $x$  or for  $x$  in terms of  $y$ . If both equations are quadratic, the same procedure could be followed but one would in general get from one equation by means of the quadratic formula a cumbersome expression for  $y$  in terms of  $x$  which, when substituted in the other equation, would yield a complicated equation involving radicals. A better method of eliminating one variable from two polynomial equations of any degree in  $x$  and  $y$  will be discussed in section 131. For the present, we shall present a few special devices for the solution of special cases only. It can be proved that a system consisting of two quadratics in  $x$  and  $y$  can have at most four different solutions.

Case A. If the equation  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  is such that  $B = 0$  and  $A = C \neq 0$ , then it can be shown that its graph is a circle (except for trivial exceptional cases). If both equations of the system are of this type, the common solutions may be found as follows. By multiplication and addition we may eliminate the second-degree terms; then we may solve the resulting linear equation with either one of the given quadratic equations.

Example 1. Solve the system

$$\begin{array}{l} (1) \\ (2) \end{array} \quad \begin{cases} x^2 - 4x + y^2 = 1 \\ x^2 + y^2 - 2y = 9. \end{cases}$$

Solution. Multiplying the first equation by  $-1$  and adding the result to the second equation, we eliminate the second-degree terms obtaining

$$4x - 2y = 8$$

or

$$(3) \quad 2x - y = 4.$$

Solving (3) and (1) simultaneously as in the preceding section, we obtain the solutions  $x = 3, y = 2$  and  $x = 1, y = -2$ . These solutions should be checked in both original equations.



## EXERCISES

*Solve and check:*

1.  $\begin{cases} x^2 + y^2 + 4x = 1 \\ x^2 + y^2 - 2y = 9. \end{cases}$

2.  $\begin{cases} x^2 + y^2 + 2x + 2y = 3 \\ (x + 3)^2 + y^2 = 10. \end{cases}$

3.  $\begin{cases} (x + 2)^2 + y^2 = 5 \\ x^2 + (y + 1)^2 = 10. \end{cases}$

4.  $\begin{cases} (x + 2)^2 + (y - 1)^2 = 5 \\ x^2 + y^2 = 10. \end{cases}$

5.  $\begin{cases} x^2 + y^2 + 6x - 4y = 0 \\ (x - 5)^2 + (y - 2)^2 = 29. \end{cases}$

\*6. Show that if two circles intersect in two distinct points, then the linear equation obtained by eliminating the second-degree terms must be the equation of the straight line joining the points of intersection.

*Case B.* When both equations have the form  $Ax^2 + Cy^2 + F = 0$ , the system is linear in the variables  $x^2$  and  $y^2$  and can be solved by multiplication and addition, as in the following example.

*Example 2.* Solve the system:

$$\begin{cases} 3x^2 + 4y^2 = 192 \\ 3x^2 - y^2 = 12. \end{cases}$$

*Solution.* Multiplying the second equation by  $-1$  and adding to the first, we get  $5y^2 = 180$ , or  $y^2 = 36$ , or  $y = \pm 6$ . Multiplying the second by 4 and adding to the first, we get  $15x^2 = 240$ , or  $x^2 = 16$ , or  $x = \pm 4$ . Hence there are four solutions:

$$\frac{x}{y} \left| \begin{array}{c|c|c|c|c} 4 & 4 & -4 & -4 \\ \hline 6 & -6 & 6 & -6 \end{array} \right|. \quad \text{See Fig. 35, section 84.}$$

## EXERCISES

*Solve and check:*

7.  $\begin{cases} 2x^2 - 3y^2 = 6 \\ x^2 + 2y^2 = 17. \end{cases}$

8.  $\begin{cases} y^2 - 8x^2 = 8 \\ x^2 + 2y^2 = 33. \end{cases}$

9.  $\begin{cases} x^2 + 2y^2 = 10 \\ 2x^2 + y^2 = 11. \end{cases}$

10.  $\begin{cases} 8x^2 - y^2 + 9 = 0 \\ 3x^2 + y^2 = 31. \end{cases}$

11.  $\begin{cases} 2x^2 + y^2 = 5 \\ 2x^2 - 3y^2 = 3. \end{cases}$

12.  $\begin{cases} 2x^2 - 3y^2 = 1 \\ 3x^2 + 2y^2 = 21. \end{cases}$

13.  $\begin{cases} 3x^2 + 2y^2 = 1 \\ 12x^2 - y^2 = 1. \end{cases}$

14.  $\begin{cases} 3x^2 + 8y^2 = 21 \\ x^2 + 4y^2 = 10. \end{cases}$

*Case C.* When one equation can be factored, the solution may be accomplished as in the following examples.

*Example 3.* Solve

$$\begin{aligned} (4) \quad & \begin{cases} x^2 + y^2 = 25 \\ 12x^2 + 7xy - 12y^2 = 0. \end{cases} \\ (5) \quad & \end{aligned}$$

\* May be omitted without disturbing continuity.



*Solution.* The second equation can be factored, obtaining  $(4x - 3y)(3x + 4y) = 0$ . This is equivalent to the two linear equations

$$(6) \quad 4x - 3y = 0$$

and

$$(7) \quad 3x + 4y = 0.$$

Hence the given system is equivalent to two systems of the type studied in section 85, namely (4) and (6) together, and (4) and (7) together. Solving (4) and (6), we get  $x = 3, y = 4$ , and  $x = -3, y = -4$ . Solving (4) and (7), we get  $x = 4, y = -3$ , and  $x = -4, y = 3$ . Hence the given system has all four solutions.

If no first-degree terms are present, the system can sometimes be reduced to the preceding case by eliminating the constants.

*Example 4.* Solve

$$\begin{aligned} (8) \quad & 3x^2 + xy = 3 \\ (9) \quad & 4y^2 - xy = 4. \end{aligned}$$

*Solution.* To eliminate the constants, we multiply (8) by 4 and (9) by  $-3$  and add, obtaining

$$(10) \quad 12x^2 + 1xy - 12y^2 = 0.$$

This may be factored into  $(4x - 3y)(3x + 4y) = 0$ . We may then solve either of the given equations together with both the equations obtained by setting these factors equal to zero. Thus we may solve the systems

$$(11) \quad \begin{cases} 3x^2 + xy = 3 \\ 4x - 3y = 0 \end{cases}$$

and

$$(12) \quad \begin{cases} 3x^2 + xy = 3 \\ 3x + 4y = 0. \end{cases}$$

Solving (11), we get

$$3x^2 + \frac{4x^2}{3} = 3$$

$$9x^2 + 4x^2 = 9$$

$$13x^2 = 9$$

$$x = \pm \frac{3}{\sqrt{13}} = \pm \frac{3\sqrt{13}}{13}$$

$$y = \pm \frac{12\sqrt{13}}{39} = \pm \frac{4\sqrt{13}}{13}.$$



Solving (12), we get

$$\begin{aligned} 3x^2 - \frac{3x^2}{4} &= 3 \\ 12x^2 - 3x^2 &= 12 \\ 9x^2 &= 12 \\ x &= \pm \frac{\sqrt{12}}{3} = \pm \frac{2\sqrt{3}}{3} \\ y &= \mp \frac{\sqrt{3}}{2}. \end{aligned}$$

Hence the given system has the four solutions

$$\begin{array}{c|c|c|c|c} x & \frac{3\sqrt{13}}{13} & -\frac{3\sqrt{13}}{13} & +\frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} \\ \hline y & \frac{4\sqrt{13}}{13} & -\frac{4\sqrt{13}}{13} & -\frac{\sqrt{3}}{2} & +\frac{\sqrt{3}}{2} \end{array}$$

Other cases will be left to the ingenuity of the student, or they may be solved by means of the general method taken up in section 131.

## EXERCISES

*Solve the following systems, leaving irrational answers in radical form and check:*

$$15. \begin{cases} x^2 + y^2 = 25 \\ 12x^2 - 25xy + 12y^2 = 0. \end{cases}$$

$$16. \begin{cases} x^2 + y^2 = 16 \\ x^2 - 3xy + 2y^2 = 0. \end{cases}$$

$$17. \begin{cases} x^2 + y^2 = 25 \\ (x - y + 5)(3x - 4y) = 0. \end{cases}$$

$$18. \begin{cases} 16x^2 + 25y^2 = 400 \\ (4x + 5y - 20)(8x + 5y - 40) = 0. \end{cases}$$

$$19. \begin{cases} x^2 - y^2 = 9 \\ 4x^2 - (5 + 4\sqrt{2})xy + 5\sqrt{2}y^2 = 0. \end{cases}$$

$$20. \begin{cases} 5x^2 + 9y^2 = 161 \\ 6x^2 - 23xy + 20y^2 = 0. \end{cases}$$

$$21. \begin{cases} 3x^2 - 5y^2 = 7 \\ 2x^2 - 7xy + 6y^2 = 0. \end{cases}$$

$$22. \begin{cases} 5x^2 + 12y^2 = 128 \\ 3x^2 + 4xy - 4y^2 = 0. \end{cases}$$

$$23. \begin{cases} x^2 + 2xy + y^2 = 1 \\ xy + 2y^2 = 2. \end{cases}$$

$$24. \begin{cases} 3x^2 - 5xy = 2 \\ 2y^2 + xy = 4. \end{cases}$$

$$25. \begin{cases} 2x^2 + xy + y^2 = 8 \\ 3xy + y^2 = 10. \end{cases}$$

$$26. \begin{cases} 3x^2 + 4xy + y^2 = 5 \\ 3xy + 10y^2 = 4. \end{cases}$$

$$27. \begin{cases} x^2 + y^2 = 5 \\ xy = 2. \end{cases}$$

$$28. \begin{cases} x^2 + y^2 = 10 \\ xy = 3. \end{cases}$$



87. Verbal problems. The following verbal problems may lead to systems of equations involving quadratics. The student should follow generally the same steps outlined in section 63.

Note. The problem of finding the intersection of two curves is one of considerable practical importance beyond the problems below. Thus in navigation, a point is located usually as the intersection of two arcs of circles, or of two straight lines, or even, of two hyperbolas. In practical engineering problems, it is often necessary to find the coordinates of points of intersection of two graphs.

## EXERCISES

1. Find two positive numbers whose difference is 2 and whose product is 144.

2. Find two positive numbers whose sum is 18 and the sum of whose squares is 174.

3. The area of a right triangle is 60 square feet and its hypotenuse is 17 feet long. Find the legs.

4. The area of a rectangle is 240 square inches and a diagonal is 25 inches long. Find the length and width of the rectangle.

5. The perimeter of a rectangle is 50 feet and its area is 144 square feet. Find its length and width.

6. The perimeter of a rectangle is 124 inches and a diagonal is 31 inches long. Find its length and width.

7. If a bus travelling between two towns were to increase its usual speed by 40 miles per hour, it would make the trip in 30 minutes less than its usual time. But if its usual speed were decreased by 10 miles per hour, the trip would take 1 hour more. Find the usual speed of the bus and the length of the trip.

8. A man walks 40 miles in 40 minutes less time than his son. The son's speed is 4/3 mile per hour less than the father's speed. Find the sum of their

9. The hypotenuse of a right triangle is 20.5 feet long. If each leg were increased by 1.5 feet, the hypotenuse would be increased by 6 feet. Find the legs of the original triangle.

10. The diagonal of a rectangle is 25 inches long. If the length of the rectangle were decreased by 2 inches and its width increased by 4 inches, the diagonal would be decreased by 3 inches. Find the length and width of the original rectangle.

11. A rectangular flower plot has area 100 square feet, and is surrounded by a path 2 feet wide. The area of the path is 212 square feet. Find the length and width of the flower plot.

12. A and B can do a job in 6 days working together. A alone would take twice as long to do the job as B alone would take. How long would each alone require to do the job?



13. A man spent \$4.80 for a certain number of items. If the price per item were decreased by 5 cents, he could have bought 16 more items for the same amount of money. How many items were bought and at what price per item?

14. A rectangular piece of cardboard has an area of 120 square inches. By cutting a square 2 inches on each side from each of the four corners and folding up the sides, an open box is formed having a volume of 96 cubic inches. Find the length and width of the original cardboard.

*Question*  
*Rahman V. S. S. S.*  
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## CHAPTER XIII

### Ratio, Proportion, and Variation

**88. Ratio and proportion.** The ratio of one number  $a$  to another number  $b$  ( $\neq 0$ ) is the quotient  $\frac{a}{b}$ . The ratio  $a/b$  is sometimes written  $a:b$ . Four numbers  $a, b, c, d$  are said to be in proportion if  $a/b = c/d$  (or  $a:b = c:d$ ). A proportion is a statement that two ratios are equal. The proportion  $a:b = c:d$  is often read “ $a$  is to  $b$  as  $c$  is to  $d$ .” In the proportion  $\frac{a}{b} = \frac{c}{d}$  the numbers  $a$  and  $d$  are called the **extremes** and the numbers  $b$  and  $c$  are called the **means**.

If  $a:b = c:x$ , then  $x$  is called the **fourth proportional** to  $a, b$ , and  $c$ . If  $a:b = b:x$ , then  $x$  is called the **third proportional** to  $a$  and  $b$ . If  $a:x = x:b$ , then  $x$  is called a **mean proportional** to  $a$  and  $b$ .

Proportions were studied by the ancient Greek geometers. If two triangles are similar, then corresponding sides are in proportion. This may be written as follows: If the lengths of the sides of one triangle, all measured in terms of the same unit, are denoted by  $a, b$ , and  $c$ , and the lengths of the corresponding sides of the other (similar) triangle are denoted by  $a', b'$ , and  $c'$ , measured in terms of the same unit, then  $a:a' = b:b' = c:c'$ . The latter equality means that  $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$ .

The following theorems are found in Euclid's *Elements*, the great work on geometry written about 320 B.C. upon which geometry textbooks are based. The student should carry out the proofs. It is assumed throughout that values of the letters which would make any denominator zero are excluded.

**THEOREM I.** If  $\frac{a}{b} = \frac{c}{d}$ , then  $ad = bc$ . Or, in any proportion the product of the means equals the product of the extremes.

*Hint for proof:* Multiply both members by the L.C.D.  $bd$ .

**THEOREM II.** If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a}{c} = \frac{b}{d}$ . That is, the means of any proportion may be interchanged.



*Hint for proof:* Divide both members of  $ad = bc$  by  $cd$ .

In II, the second proportion is said to be obtained from the first by **alternation**.

**THEOREM III.** If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{b}{a} = \frac{d}{c}$ .

*Hint for proof:* Divide both members of  $ad = bc$  by  $ac$ .

In III, the second proportion is said to be obtained from the first by **inversion**.

**THEOREM IV.** If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a+b}{b} = \frac{c+d}{d}$ .

*Hint for proof:* Add 1 to both members of the given proportion and simplify.

In IV, the second proportion is said to be obtained from the first by **composition**.

**THEOREM V.** If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a-b}{b} = \frac{c-d}{d}$ .

*Hint for proof:* Subtract 1 from both members and simplify.

In V, the second proportion is said to be obtained from the first by **division**.

**THEOREM VI.** If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ .

*Hint for proof:* Divide the second proportion of IV by the second proportion of V and simplify.

In VI, the second proportion is said to be obtained from the first by **composition and division**.

*Example 1.* To find the fourth proportional to 2, 3, and 4 we write  $\frac{2}{3} = \frac{4}{x}$ . Solving, we get  $2x = 12$ , or  $x = 6$ . Check:  $\frac{2}{3} = \frac{4}{6}$  is correct.

*Example 2.* To find a mean proportional to 2 and 8 we write  $\frac{2}{x} = \frac{x}{8}$  or  $x^2 = 16$ , or  $x = \pm 4$ . Hence either +4 or -4 is a mean proportional to 2 and 8.

*Note 1.* Obviously, the fourth proportional to  $a$ ,  $b$ ,  $c$  satisfies the equation  $a/b = c/x$  or  $x = bc/a$ . Thus, to find the fourth proportional, we multiply the second number by the third and divide by the first.\*

\* This rule is known as the *Rule of Three* and was considered difficult in the Middle Ages, as indicated by the Mother Goose rhyme:

"Multiplication is vexation,  
Division is as bad;  
The Rule of Three doth puzzle me,  
And Practice drives me mad."



*Note 2.* In practical applications, care must be taken with the units in which measurements are given. In particular, in a proportion the two ratios must be expressed in *consistent units*. Thus a velocity of 3 feet per 2 seconds is a ratio  $\frac{3}{2}$  feet per second or 1.5  $\frac{\text{ft.}}{\text{sec.}}$ . It may also be expressed as 1 yard per 2 seconds or  $\frac{1}{2}$  yard per second or  $0.5 \frac{\text{yd.}}{\text{sec.}}$ . But it would be incorrect to write  $\frac{3}{2} = \frac{1}{2}$ . If the numerator and denominator of a ratio represent measurements expressed in the same units, the quotient or ratio is sometimes called a **pure or dimensionless number**, since it remains the same if the units are changed in a consistent way and therefore may be written without any unit. Thus 12 inches:1 inch =  $\frac{12}{1}$ ; if both numerator and denominator are expressed in yards, we get

$$\frac{1}{3} \text{ yd.} : \frac{1}{36} \text{ yd.} = \frac{\frac{1}{3} \text{ yd.}}{\frac{1}{36} \text{ yd.}} = \frac{36}{3} = \frac{12}{1}.$$

The ratio of two measurements of the same kind is most conveniently expressed as such a pure or dimensionless number.

*Example 3.* Express the ratio of 3 inches to 1 foot as a pure number.

*Solution.* Using the same units, the desired ratio is that of 3 inches to 12 inches, or 3:12, or  $\frac{1}{4}$ .

A symbol of the form  $a:b:c = 2:3:6$  is understood to be equivalent to the proportions  $a:b = 2:3$  and  $b:c = 3:6$ . A symbol  $a:b:c$  is called a **continued ratio**.

## EXERCISES

*Express each ratio as a fraction in simplest form:*

- |                           |             |                                  |                                 |
|---------------------------|-------------|----------------------------------|---------------------------------|
| 1. 3:9.                   | 2. 4.2:1.4. | 3. $1\frac{1}{2}:1\frac{1}{3}$ . | 4. $\frac{3}{4}:\frac{9}{16}$ . |
| 5. $3.4x^2:5.1x$ .        |             | 6. $5.7x^3y:9.5xy^2$ .           |                                 |
| 7. $(x^2 - 1):2(x - 1)$ . |             | 8. $(x^3 - 1):(x^2 - 1)$ .       |                                 |

*Express each ratio as a pure or dimensionless number in simplest form:*

- |                                                               |                                     |
|---------------------------------------------------------------|-------------------------------------|
| 9. 4 feet to 64 inches.                                       | 10. 6 inches to $\frac{1}{2}$ yard. |
| 11. 36 hours to 4 days.                                       | 12. 80 minutes to 3 hours.          |
| 13. 528 feet per second to 720 miles per hour.                |                                     |
| 14. 20 pounds per square inch to 1440 pounds per square foot. |                                     |







54. The volume  $V$  of a sphere is given by the formula  $V = \frac{4}{3}\pi r^3$  where  $r$  is the radius. Show that the volumes of two spheres are to each other as the cubes of their radii, that is,  $V_1:V_2 = r_1^3:r_2^3$ .

55. Carry out the proofs of theorems I–VI inclusive in the text above.

89. The terminology of variation. The terms discussed in this section are often used in scientific writing. When  $y$  is a linear function of  $x$  of the special form

$$(1) \quad y = kx,$$

where  $k$  is a constant, we say that  $y$  varies directly as  $x$ , or  $y$  is directly proportional to  $x$ . We call  $k$  the constant of proportionality or factor of proportionality. The statement “ $y$  varies directly as  $x$ ” is sometimes written symbolically as  $y \propto x$ , but for practical use the form (1) is preferable. The graph of such a function is an oblique straight line passing through the origin with slope  $k$  (Fig. 37).

*Example 1.* The circumference of a circle varies directly as the radius since  $C = 2\pi r$ . In this example the factor of proportionality is  $2\pi$ .

If sufficient information is given, it is often possible to determine the value of the factor of proportionality and write  $y$  as a definite function of  $x$ .

*Example 2.* Suppose  $y$  varies directly as  $x$  and  $y = 6$  when  $x = 3$ . (a) Write  $y$  as a function of  $x$ . (b) Find the value of  $y$  when  $x = 4$ .

*Solution.*  $y = kx$ . Hence  $6 = 3k$ , or  $k = 2$ . Therefore (a)  $y = 2x$ ; (b)  $y = 8$  when  $x = 4$ .

The statement  $y$  varies directly as the square of  $x$  means of course that  $y = kx^2$ . The graph of such a function is a parabola passing through the origin (Fig. 38).

*Example 3.* Suppose  $y$  varies directly as the square of  $x$  and  $y = 12$  when  $x = 2$ . (a) Write  $y$  as a function of  $x$ ; (b) find the value of  $y$  when  $x = 4$ .

*Solution.*  $y = kx^2$ . Hence  $12 = 4k$ , or  $k = 3$ . Therefore (a)  $y = 3x^2$ ; (b)  $y = 48$  when  $x = 4$ .

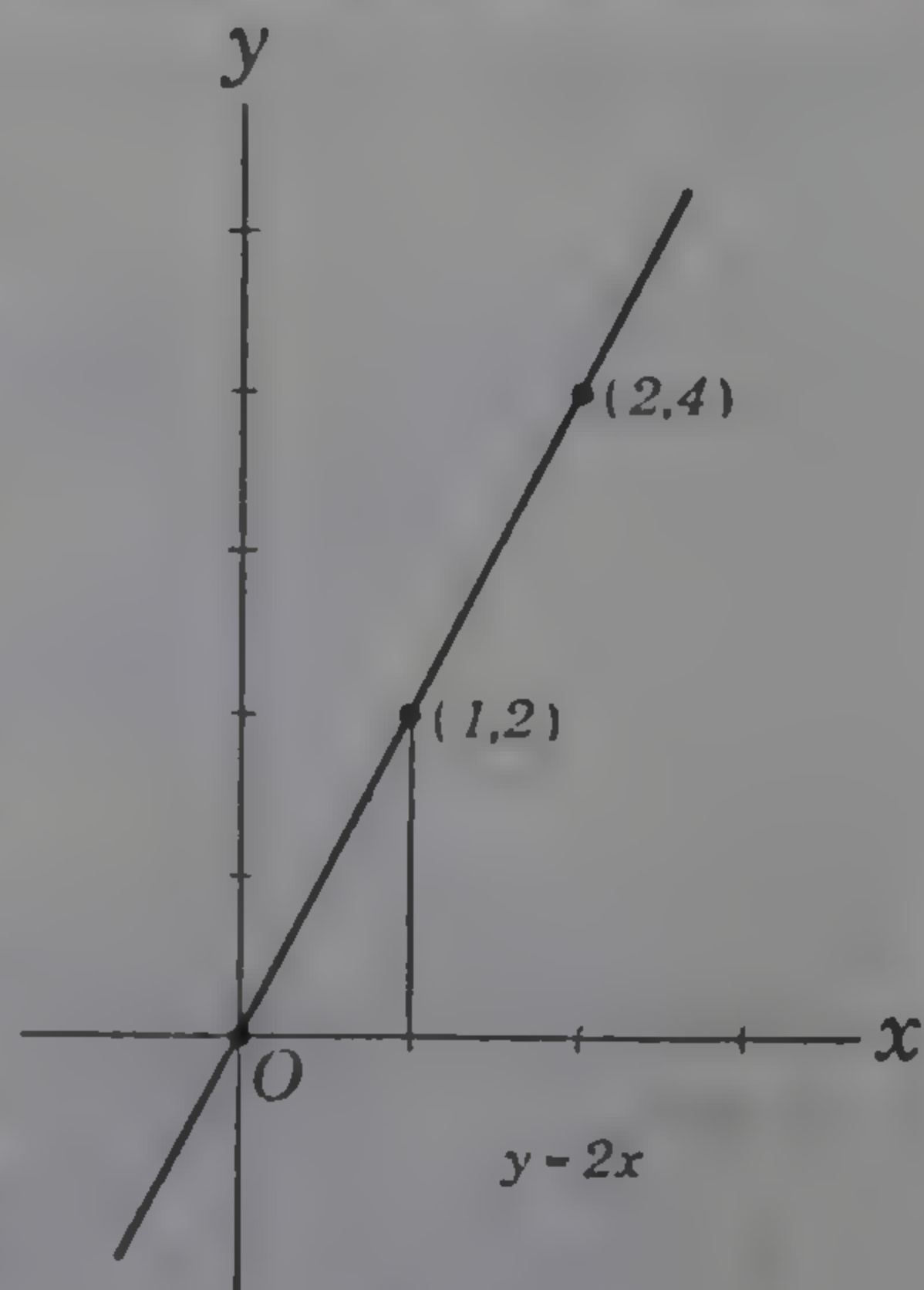


Fig. 37.



When  $y = \frac{k}{x}$ ,  $k$  being a constant, we say that  $y$  varies inversely as  $x$  or  $y$  is inversely proportional to  $x$ . The graph of such a function is a hyperbola (Fig. 39).

*Example 4.* If  $y$  varies inversely as  $x$  and  $y = 4$  when  $x = 3$ , (a) write  $y$  as a function of  $x$ ; (b) find the value of  $y$  when  $x = 6$ .

*Solution.*  $y = k/x$ . Hence  $4 = k/3$ , or  $k = 12$ . Therefore (a)  $y = 12/x$ ; (b)  $y = 2$  when  $x = 6$ .

The statement  $y$  varies inversely as the square of  $x$  means, of course, that  $y = k/x^2$ .

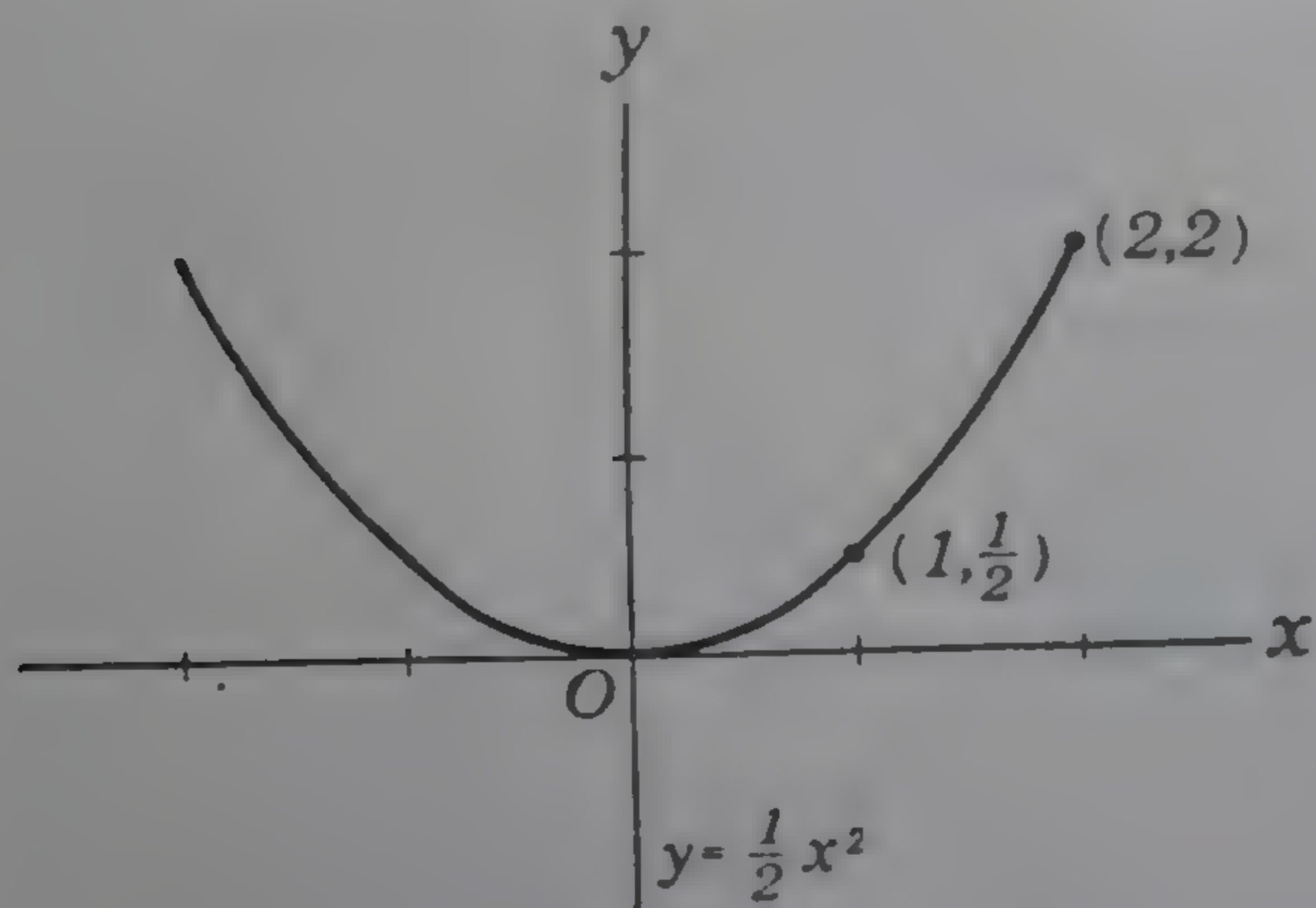


Fig. 38.

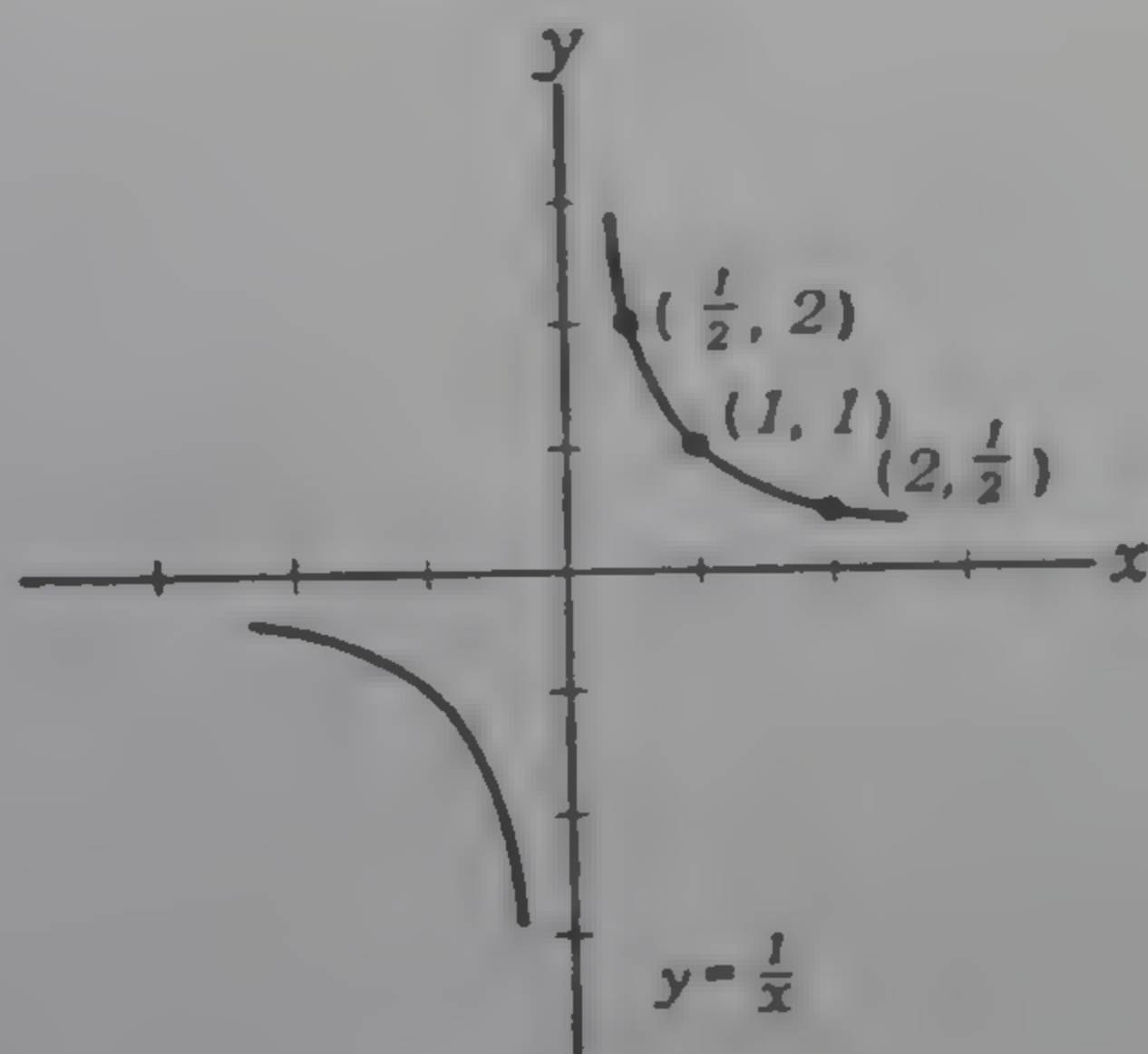


Fig. 39.

When  $y = kuv$ ,  $k$  being a constant, we say that  $y$  varies jointly as  $u$  and  $v$ . Similarly, we say that  $y$  varies jointly as  $u$  and  $v$  and inversely as  $w$  if  $y = \frac{kuv}{w}$ .

*Example 5.* Newton's law of gravitation says that the force  $F$  exerted by two particles on each other varies jointly as their masses  $m_1$  and  $m_2$  and inversely as the square of the distance  $d$  between them. In symbols,

$$F = \frac{km_1m_2}{d^2}.$$

The factor of proportionality  $k$  in this example is found to be approximately  $6.66 \times 10^{-8}$  when the units employed are centimeters, grams, and seconds; it is known as the **gravitational constant**.

The following theorems explain the use of the word "proportional" in direct variation, and "inversely proportional" in inverse variation.



**THEOREM 1.** *If  $y$  varies directly as  $x$ , and  $y = y_1$  when  $x = x_1$ , and  $y = y_2$  when  $x = x_2$ , then  $\frac{y_1}{y_2} = \frac{x_1}{x_2}$ .*

*Proof.* By hypothesis,  $y_1 = kx_1$  and  $y_2 = kx_2$ . Dividing the first equation by the second, we get  $\frac{y_1}{y_2} = \frac{kx_1}{kx_2}$ , or  $\frac{y_1}{y_2} = \frac{x_1}{x_2}$ .

**THEOREM 2.** *If  $y$  varies inversely as  $x$ , and  $y = y_1$  when  $x = x_1$ , and  $y = y_2$  when  $x = x_2$ , then  $\frac{y_1}{y_2} = \frac{x_2}{x_1}$ .*

*Proof.* By hypothesis,  $y_1 = k/x_1$  and  $y_2 = k/x_2$ . Dividing the first equation by the second, we have  $\frac{y_1}{y_2} = \frac{k/x_1}{k/x_2}$ , or  $\frac{y_1}{y_2} = \frac{x_2}{x_1}$ .

## EXERCISES

1. If  $y$  varies directly as  $x$ , and  $y = 21$  when  $x = 7$ , (a) express  $y$  as a function of  $x$ ; (b) find the value of  $y$  when  $x = 6$ ; (c) how is  $y$  affected if  $x$  is tripled?

2. If  $y$  varies inversely as  $x$  and  $y = 4$  when  $x = 3$ , (a) express  $y$  as a function of  $x$ ; (b) find the value of  $y$  when  $x = 6$ ; (c) how is  $y$  affected when  $x$  is doubled?

3. If  $y$  varies jointly as  $u$  and  $v$  and  $y = 10$  when  $u = 4$  and  $v = 5$ , (a) express  $y$  as a function of  $u$  and  $v$ ; (b) find the value of  $y$  when  $u = 2$  and  $v = 7$ .

4. If  $y$  varies directly as the square of  $u$  and inversely as the cube of  $v$ , and  $y = 2$  when  $u = 2$  and  $v = 2$ , (a) express  $y$  as a function of  $u$  and  $v$ ; (b) find the value of  $y$  when  $u = 4$  and  $v = 4$ .

5. Suppose that the maximum range of a projectile varies directly as the square of the initial velocity. If the maximum range is 15,000 feet when the initial velocity is 300 feet per second, (a) write the maximum range  $r$  as a function of the initial velocity  $v$ ; (b) find the maximum range when  $v = 500$  feet per second; (c) what is  $v$  when  $r = 60,000$  feet?

6. Boyle's law says that, for an enclosed gas at a constant temperature, the pressure  $p$  varies inversely as the volume  $v$ . If  $v = 600$  cubic inches when  $p = 24$  pounds per square inch, (a) express  $p$  as a function of  $v$ ; (b) find  $p$  when  $v = 720$  cubic inches; (c) find  $v$  when  $p = 100$  pounds per square inch.

7. The weight of a body is essentially the gravitational force exerted on it by the earth. Hence, as a special case of Newton's law of gravitation (example 5 above), we deduce that the weight  $w$  of a body varies inversely as the square of its distance  $d$  from the center of gravity of the earth. Assuming this to be true, and taking the earth to be a sphere of radius 4000 miles, how much would a girl weigh at a height of 100 miles above the surface of the earth if she weighs 100 pounds at the surface?

8. If a ball rolls down an inclined plane, the distance traversed varies as the square of the time, neglecting friction. If the ball rolls 8 feet in the first second, how far will it go in the first 4 seconds?



9. Neglecting air resistance, the distance a body falls from rest varies directly as the square of the time. If a body falls 64 feet in the first 2 seconds, how far will it have fallen in the first 6 seconds?

10. The surface area  $S$  of a sphere varies directly as the square of the radius  $r$ . If  $S = 16\pi$  square inches when  $r = 2$  inches, (a) find the formula for  $S$  in terms of  $r$ ; (b) find  $S$  when  $r = 6$  inches; (c) find  $r$  when  $S = 100\pi$  square inches.

11. The volume  $V$  of a right circular cone varies jointly as the altitude  $h$  and the square of the radius  $r$  of the base. If  $V = \pi$  cubic inches when  $r = 1$  inch and  $h = 3$  inches, (a) find the formula for  $V$  in terms of  $r$  and  $h$ ; (b) find  $V$  when  $r = 3$  inches and  $h = 4$  inches.

12. The illumination  $I$  from a source of light varies inversely as the square of the distance  $d$  from the source. If  $I = I_1$  when  $d = 16$  feet, how far from the source will the illumination be 4 times as much?

13. The period  $T$  of a simple pendulum (that is, the time required for a complete oscillation) varies directly as the square root of the length  $L$  of the pendulum. If  $T = 1.5$  seconds when  $L = 2$  feet, find  $T$  when  $L = 8$  feet.

14. According to the astronomer Kepler (1571–1630), the squares of the periods of the planets vary directly as the cubes of their mean distances from the sun. The period of the earth is 1 year. Taking the mean distance of the earth from the sun as the unit of distance, the mean distance of Jupiter is 5.2 units. Find the period of Jupiter to the nearest tenth of a year.

15. The volume  $V$  of a gas varies directly as the absolute temperature  $T$  and inversely as the pressure  $p$ . If a certain quantity of gas occupies 500 cubic feet at a pressure of 53 pounds per square foot and an absolute temperature of 500 degrees, what volume will it occupy at 600 degrees absolute temperature and pressure of 106 pounds per square foot?

16. The cost of labor varies jointly as the number of workers and the number of days they work. If 8 men working 9 days each are paid \$576, how many days will it take 6 men to earn \$960?

17. The current  $I$  in a wire varies directly as the electromotive force  $E$  and inversely as the resistance  $R$ . If  $I = 22$  amperes when  $E = 110$  volts and  $R = 5$  ohms, find  $I$  when  $E = 195$  volts and  $R = 15$  ohms.

18. The lift force on a monoplane wing varies jointly as the area of the wing and the square of the velocity (other factors being kept fixed). If the area is increased by 20% and the velocity decreased by 10%, show that the lift is decreased by approximately 3%.

19. Other conditions remaining the same, the thrust  $T$  of a propeller varies jointly as the fourth power of its diameter and the square of the number  $n$  of revolutions per second. Show that if the number of revolutions per second is doubled and the diameter is decreased by 50%, the thrust will be decreased by 75%.



## CHAPTER XIV

### Complex Numbers\*

90. The standard form of a complex number. It was proved, in section 16\*, that no negative number has a real square root. The symbol  $\sqrt{N}$  where  $N$  is a negative number is called a **pure imaginary number**. The symbol  $\sqrt{-1}$  is denoted by the letter  $i$  and is called the **imaginary unit**. Thus

$$(1) \quad i = \sqrt{-1} \text{ and } i^2 = -1.$$

We agree to express every pure imaginary number as a real multiple of  $i$ , as in the following examples.

$$\text{Example 1. } \sqrt{-9} = \sqrt{9 \cdot (-1)} = \sqrt{9} \sqrt{-1} = 3i.$$

$$\text{Example 2. } \sqrt{-2} = \sqrt{2(-1)} = \sqrt{2} \sqrt{-1} = \sqrt{2} \cdot i = i \sqrt{2}.$$

In examples like the latter the  $i$  is usually written to the left of the radical sign to prevent it from slipping under the radical sign by mistake.

Thus every pure imaginary number can be written as  $bi$ , where  $b$  is real and different from 0, and  $i = \sqrt{-1}$ .

Any number which can be written in the form

$$(2) \quad a + bi$$

where  $a$  and  $b$  are real and  $i = \sqrt{-1}$ , is called a **complex number**. When it is written in the form (2), it is said to be in **standard form**. This standard form of a complex number is often called the **rectangular form** for a reason which will appear below in section 92.

*Example 3.*  $3 + \sqrt{-4}$  is a complex number. In standard form it is written as  $3 + 2i$ .

When written in standard form (2), the number  $a$  is called the **real part** and  $bi$  the **imaginary part** of the complex number. Thus 3 is the real part of  $3 + 2i$  and  $2i$  is its imaginary part. A complex number  $a + bi$  for which  $b \neq 0$  is called **imaginary**. A complex number for which  $b = 0$  is called **real**.† Thus  $3 + 2i$  is imaginary,  $3 = 3 + 0i$  is real. The (real) number 0 may be written in standard form as  $0 + 0i$ .

\* The student should review section 16 at this time.

† See the classification of numbers in section 16.



Unless otherwise stated, complex numbers will be understood to be in standard form.

Two complex numbers  $a + bi$  and  $c + di$  are called equal if and only if  $a = c$  and  $b = d$ ; that is, their real parts must be equal and their imaginary parts must be equal. It follows that  $a + bi = 0$  if and only if  $a = 0$  and  $b = 0$ .

It will be understood in this chapter that all letters represent real numbers, except  $i$ , which always represents  $\sqrt{-1}$ , unless otherwise stated.

*Example 4.* Find  $x$  and  $y$  if  $2x + 3yi = 4 + 15i$ .

*Solution.*  $2x = 4$  and  $3y = 15$ . Hence  $x = 2$  and  $y = 5$ .

If  $i$  is raised to positive integral powers in succession, we obtain:

$$\begin{array}{l} i = \sqrt{-1} \\ i^2 = -1 \\ i^3 = i^2 i = -1i = -i \\ i^4 = i^2 i^2 = (-1)(-1) = +1 \\ i^5 = i^4 i = 1i = i \\ i^6 = i^4 i^2 = 1(-1) = -1 \end{array} \quad \left| \begin{array}{l} i^7 = i^4 i^3 = 1(-i) = -i \\ i^8 = i^4 i^4 = 1 \cdot 1 = 1 \\ i^9 = i^8 i = 1 \cdot i = i \\ i^{10} = i^8 i^2 = 1(-1) = -1 \\ i^{11} = i^8 i^3 = 1(-i) = -i \\ i^{12} = i^8 i^4 = 1 \cdot 1 = 1 \end{array} \right| \dots$$

Hence the successive positive integral powers of  $i$  recur in cycles of four:  $i, -1, -i, +1$ .

The complex numbers  $a + bi$  and  $a - bi$  are called **conjugates** of each other. Thus,  $2 + 3i$  and  $2 - 3i$  are conjugates of each other. To form the conjugate of a complex number in standard form, reverse the sign preceding the imaginary part.

*Note 1.* In electrical theory, the letter  $i$  is usually reserved for the current, and  $j$  is used for  $\sqrt{-1}$ . In mathematical books the notation used here is standard.

*Note 2.* The need for complex numbers is apparent from the fact that quadratic equations, even with real coefficients, often have complex imaginary roots.

## EXERCISES

1-12. Work exercises 1-12 of section 16.

*Simplify to either  $i, -1, -i$ , or  $1$ :*

13.  $i^7$ .      14.  $i^{10}$ .      15.  $i^{16}$ .      16.  $i^{21}$ .      17.  $i^{34}$ .      18.  $i^{17}$ .

*Write in standard form the conjugate of each number:*

19.  $3 + 4i$ .      20.  $2 + \sqrt{-9}$ .      21.  $5 - \sqrt{-4}$ .      22.  $3 - 6i$ .  
23.  $2i - 3$ .      24.  $5i + 2$ .      25.  $\sqrt{-8} - 1$ .      26.  $\sqrt{-24} + 3$ .



Find real values of  $x$  and  $y$  satisfying the given equation:

27.  $x + yi = 2 - 3i.$

28.  $x + yi = 3 - 2i.$

29.  $2x - 3yi = 8.$

30.  $3x + 2yi = 6i.$

31.  $(2x + 6) + (3y - 6)i = 0.$

32.  $(x - 3) + (2y + 10)i = 0.$

33–44. Do exercises 33–44 of section 16.

Find real values of  $x$  and  $y$  satisfying the given equation:

45.  $(x + y) + (x - y)i = 7 + i.$

46.  $(2x + 3y) + (3x + y)i = 4 + 13i.$

47.  $2x + xi - 4 = i - 2yi - 5y.$

48.  $4x + 2yi - 4i - 1 = -3y - xi.$

49.  $xi - 2i + 3x = 5 - 4y - 2yi.$

50.  $x + 4xi + 2i - 1 = 2yi - 2y.$

51. Show that if a complex number is equal to its conjugate, then it is a real number.

**91. Elementary operations with complex numbers.** It is advisable to put complex numbers in the standard form  $x + yi$  before performing any arithmetical operations with them. The sum of two complex numbers  $a + bi$  and  $c + di$  is defined as follows:

$$(1) \quad (a + bi) + (c + di) = (a + c) + (b + d)i.$$

For example,  $(2 + 3i) + (4 + 2i) = 6 + 5i.$

The difference  $(a + bi) - (c + di)$  is, as always, the complex number  $x + yi$  such that  $(c + di) + (x + yi) = a + bi.$  (Compare definition 1, section 5.) It follows that  $c + x = a$  and  $d + y = b$ , or

$$(2) \quad (a + bi) - (c + di) = (a - c) + (b - d)i.$$

For example,  $(5 + 6i) - (2 + 4i) = 3 + 2i.$

Thus addition and subtraction of complex numbers are performed without regard for the special meaning of the letter  $i$ .

Likewise, the product of two complex numbers is defined as the complex number obtained by multiplying them without regard for the special meaning of  $i$ , but to put the product into standard form it may be necessary to replace  $i^2$  by  $-1$ . Thus

$$(a + bi)(c + di) = ac + bci + adi + bdi^2,$$

or, in standard form,

$$(3) \quad (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$



For example,  $(2 + 3i)(4 + 5i) = 8 + 12i + 10i + 15i^2 = 8 + 22i + 15(-1) = -7 + 22i$ .

As before (see definition 3, section 5), the quotient of two complex numbers  $(a + bi) \div (c + di)$  is defined as the complex number  $x + yi$  such that  $(c + di)(x + yi) = a + bi$ . Thus

$$(cx - dy) + (cy + dx)i = a + bi$$

which implies that

$$\begin{cases} cx - dy = a, \\ dx + cy = b. \end{cases}$$

Solving this system of linear equations for  $x$  and  $y$  in terms of  $a$ ,  $b$ ,  $c$ , and  $d$ , we get

$$x = \frac{ac + bd}{c^2 + d^2} \text{ and } y = \frac{bc - ad}{c^2 + d^2}.$$

Hence we have the formula

$$(4) \quad \frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i.$$

In practice, this result can be obtained most conveniently by rationalizing the denominator of the left member of (4) just as was done with binomial quadratic surds in section 71. That is, to express the quotient of two complex numbers in standard form, we multiply numerator and denominator by the conjugate of the denominator. Thus

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \frac{c - di}{c - di} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

which is the same as the right member of (4).

For example,

$$\frac{2 + 3i}{4 - 5i} = \frac{2 + 3i}{4 - 5i} \frac{4 + 5i}{4 + 5i} = \frac{-7 + 22i}{41} = -\frac{7}{41} + \frac{22}{41}i.$$

It can be shown that complex numbers satisfy the usual laws of algebra. Compare sections 16 and 17.

## EXERCISES

Perform the indicated operations and express the result in standard form:

- |                                            |                             |                              |
|--------------------------------------------|-----------------------------|------------------------------|
| 1. $2i + 3i$ .                             | 2. $5i + \sqrt{-4}$ .       | 3. $\sqrt{-9} + \sqrt{-4}$ . |
| 4. $\sqrt{-16} + \sqrt{-25}$ .             | 5. $(2i)(3i)$ .             | 6. $2i\sqrt{-25}$ .          |
| 7. $\sqrt{-9}\sqrt{-4}$ .                  | 8. $\sqrt{-16}\sqrt{-25}$ . | 9. $2i(3i^2)$ .              |
| 10. $(2i)^3$ .                             | 11. $(3i)^2(2i)^3$ .        | 12. $\sqrt{-2}\sqrt{-18}$ .  |
| 13-32. Work exercises 13-32 of section 16. |                             |                              |



33.  $(3 + i\sqrt{2})(3 - i\sqrt{2})$       34.  $(5 + \sqrt{-8})(5 - \sqrt{-8})$   
 35.  $(3 + i\sqrt{2})(1 - \sqrt{-8})$       36.  $(2 + i\sqrt{3})(3 - \sqrt{-12})$   
 37.  $(4 - \sqrt{-18})(2 - \sqrt{-2})$       38.  $(-1 + i\sqrt{3})^2$   
 39.  $\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2$       40.  $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2$       41.  $\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2$   
 42.  $\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2$       43.  $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2$       44.  $\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2$   
 45.  $\frac{2 - 3i}{3 - 2i}$       46.  $\frac{1 - i}{1 + i}$       47.  $\frac{2 + i}{2 - 3i}$       48.  $\frac{3 + 2i}{1 - 2i}$   
 49.  $\frac{3 - 2i}{3 - 2i}$       50.  $\frac{3i}{2 - 5i}$       51.  $\frac{2i + 3}{4i}$       52.  $\frac{6 + 4i}{2i}$   
 53.  $\frac{1}{i}$       54.  $\frac{2}{3i}$       55.  $\frac{3}{2 + i}$       56.  $\frac{2 + i}{3i}$   
 57.  $\frac{3 + \sqrt{-4}}{2 - \sqrt{-9}}$       58.  $\frac{1 + \sqrt{-16}}{3 - \sqrt{-4}}$       59.  $\frac{1}{2 + \sqrt{-3}}$       60.  $\frac{2 + \sqrt{-3}}{3 - \sqrt{-3}}$   
 61.  $\frac{3 - \sqrt{-5}}{4 + 2\sqrt{-5}}$       62.  $\frac{2 + \sqrt{-12}}{3 - 3\sqrt{-3}}$       63.  $(2i - 5)^{-1}$   
 64.  $(1 + 2i)^{-1}$       65.  $(3 - \sqrt{-2})^{-1}$       66.  $(4 - \sqrt{-12})^{-1}$

Write the reciprocal of each number in standard form:

67.  $1 + 4i$       68.  $3 - 5i$       69.  $2i - 3$       70.  $2i$   
 71.  $-4i$       72.  $3 + \sqrt{-2}$       73.  $\sqrt{-3} + 2$       74.  $\sqrt{-5} - 3$

**92. Graphical representation of complex numbers.** Let us introduce rectangular coordinates into the plane. The complex number whose standard form is  $x + yi$  is usually represented graphically by the point whose rectangular coordinates are  $x$  and  $y$ . Thus (Fig. 40),  $3 + 2i$  is represented by the point whose rectangular coordinates are  $(3, 2)$ . This is why the standard form is also called the rectangular form of the complex number. It follows that all real numbers lie on the  $x$ -axis and all pure imaginary numbers on the  $y$ -axis. The complex numbers correspond to all points of the plane.

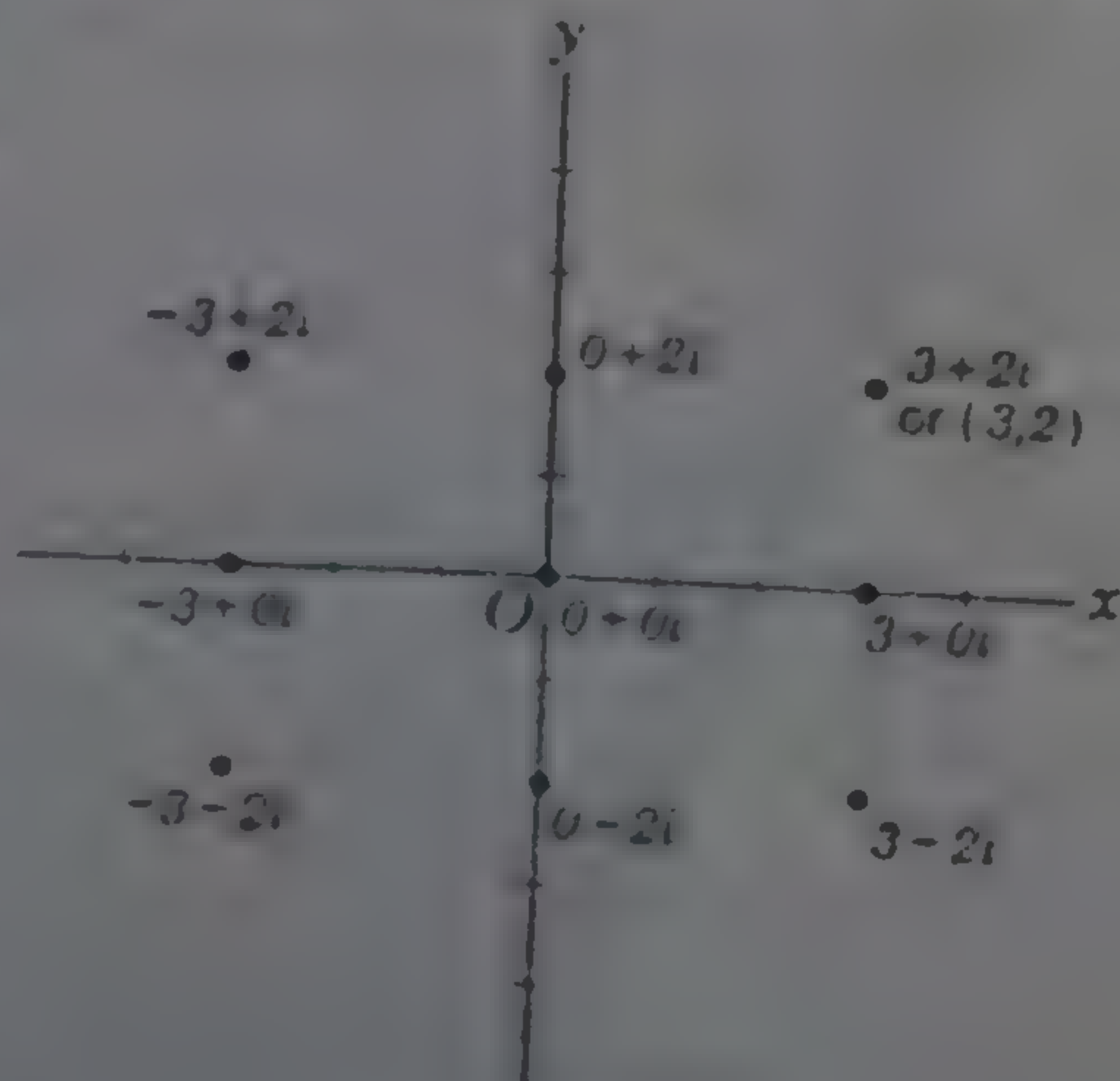


Fig. 40.



Another useful graphical representation of complex numbers is in terms of vectors, or directed line-segments. The number  $x + iy$  is represented (Fig. 41) by the vector pointing from the origin to the point with rectangular coordinates  $(x, y)$ . Thus  $3 + 2i$  is pictured by a vector (arrow) pointing from the origin to the point whose rectangular coordinates are  $(3, 2)$ . Vectors are useful for the representation of quantities involving direction as well as magnitude, such as velocities and forces.

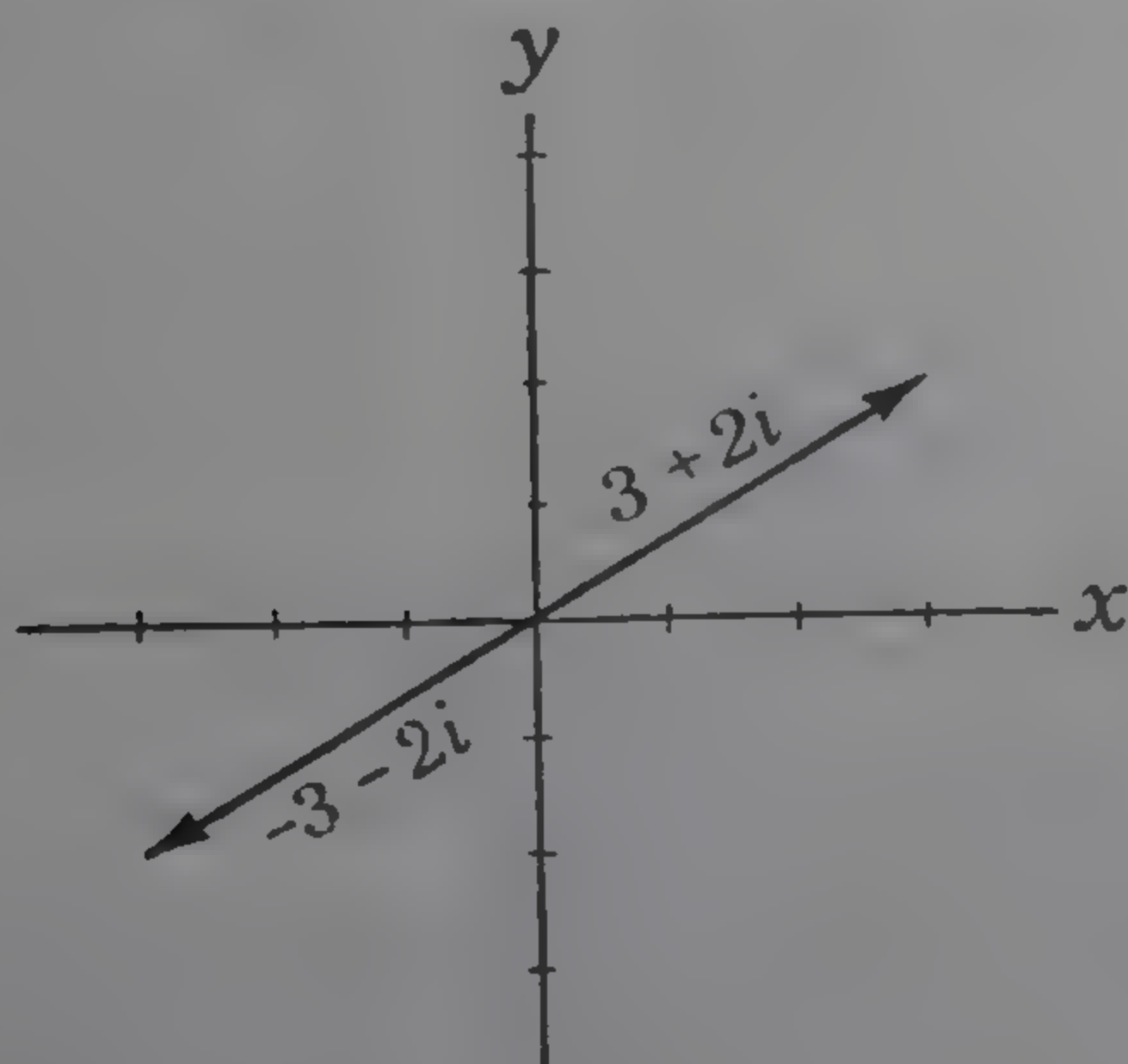


Fig. 41.

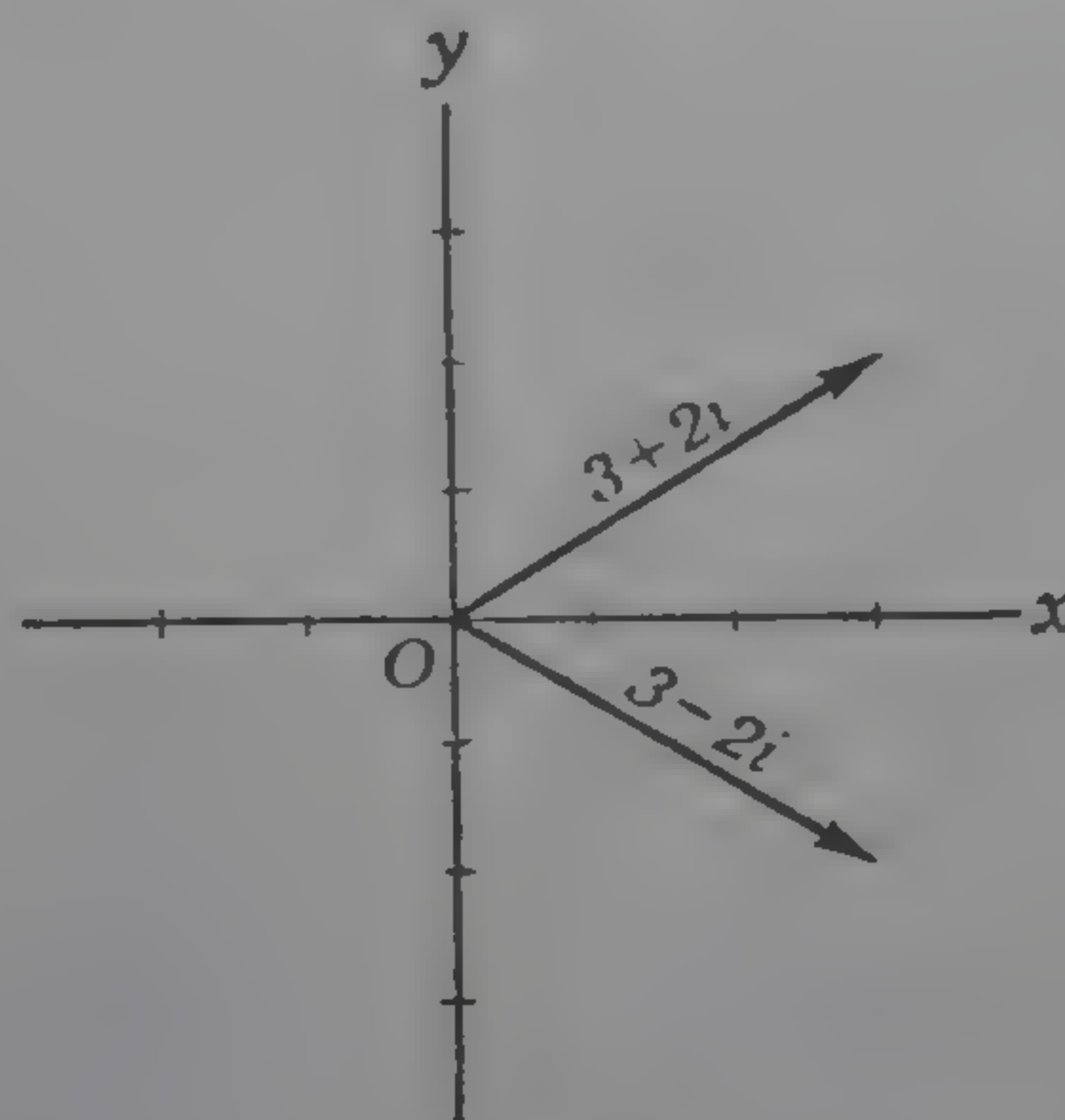


Fig. 42.

The negative of a complex number will be represented as a vector of the same length but opposite direction (Fig. 41). The conjugate of a complex number will be represented as a vector which is its reflection across the  $x$ -axis (Fig. 42).

## EXERCISES

*Represent each number graphically:*

- |               |                       |                |                               |
|---------------|-----------------------|----------------|-------------------------------|
| 1. $2 + 3i$ . | 2. $5 - 2i$ .         | 3. $-2 + 3i$ . | 4. $-3 - 4i$ .                |
| 5. $4$ .      | 6. $-4$ .             | 7. $4i$ .      | 8. $-4i$ .                    |
| 9. $0$ .      | 10. $3 + i\sqrt{2}$ . | 11. $2i - 3$ . | 12. $-\sqrt{3} - i\sqrt{2}$ . |

*On the same coordinate system, plot the given number, the negative of the given number, and the conjugate of the given number:*

- |                       |                       |                 |                                  |
|-----------------------|-----------------------|-----------------|----------------------------------|
| 13. $5 + 2i$ .        | 14. $5 - 3i$ .        | 15. $-3 + 5i$ . | 16. $-4 - i$ .                   |
| 17. $3$ .             | 18. $3i$ .            | 19. $-5$ .      | 20. $-5i$ .                      |
| 21. $3i - \sqrt{2}$ . | 22. $\sqrt{3} - 2i$ . | 23. $0$ .       | 24. $-\sqrt{5} - \frac{1}{2}i$ . |

93. Graphical significance of addition of complex numbers. The vector representing the sum of  $a + bi$  and  $c + di$  is the vector  $\vec{OP}$  along the diagonal of the parallelogram whose adjacent sides



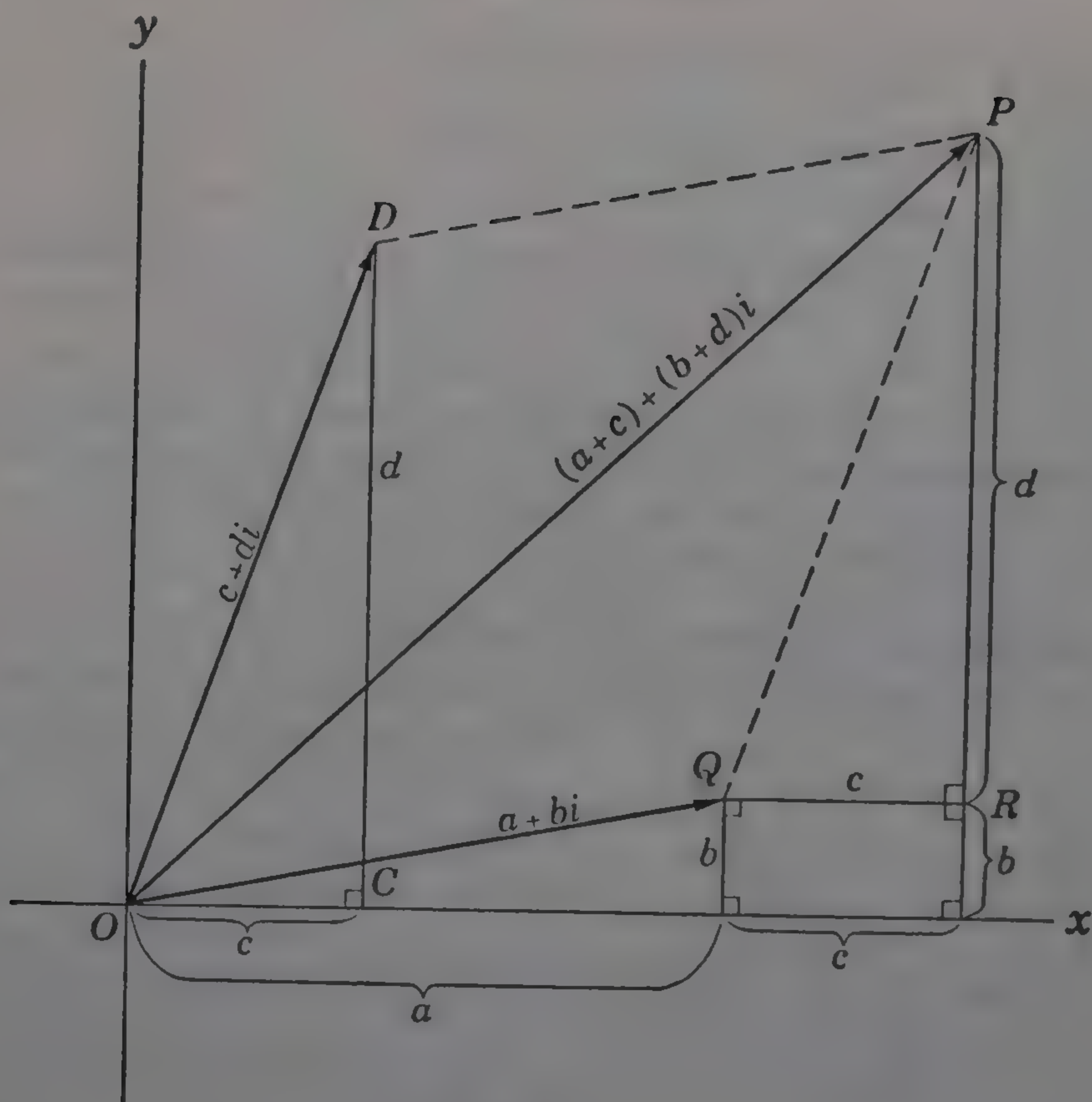


Fig. 43.

are the vectors representing  $a + bi$  and  $c + di$ , respectively (Fig. 43); for from Fig. 43 we see that the coordinates of  $P$  are  $(a + c, b + d)$ , since right triangles  $OCD$  and  $QRP$  are congruent. Hence  $\vec{OP}$  is the vector representing  $(a + c) + (b + d)i$ .

From the parallelogram law for the composition of vectors, we see that, if two vectors in a plane are represented by complex numbers, their resultant is represented by the sum of these complex numbers.

To subtract  $c + di$  from  $a + bi$  graphically, we add  $-c - di$  to  $a + bi$  graphically. Thus in Fig. 44, we subtract  $-2 + 3i$  from  $4 + 2i$  graphically by adding  $2 - 3i$  to  $4 + 2i$ , obtaining  $6 - i$ .

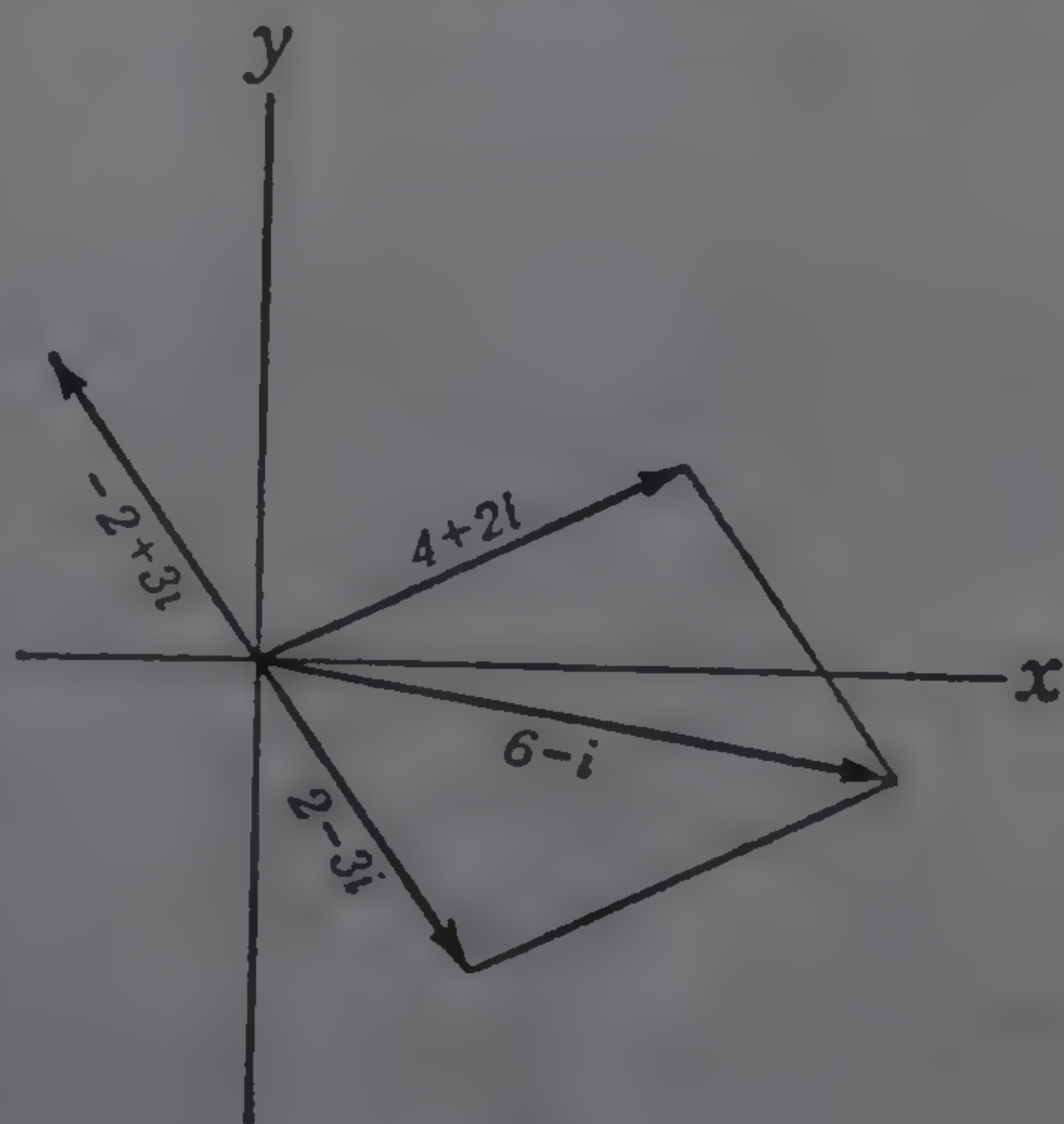


Fig. 44.



## EXERCISES

Construct each sum or difference graphically:

1.  $(2 + 5i) + (4 + i)$ .
2.  $(5 + i) + (1 - 3i)$ .
3.  $(-4 + i) + (-1 - 5i)$ .
4.  $(-4 - i) + (1 - 3i)$ .
5.  $(1 + 4i) - (-3 + 2i)$ .
6.  $(5 + 2i) - (-1 + 3i)$ .
7.  $(-4 + 2i) - (5 + 2i)$ .
8.  $(-2 - 6i) - (4 - 2i)$ .

9. Discuss the exceptional case where the given numbers  $a + bi$  and  $c + di$  are represented by vectors lying along the same straight line through the origin. Make up rules for constructing their sum and difference geometrically in this case.

**94. Review of trigonometric functions.** We review briefly\* some needed material concerning the trigonometric functions of a general angle. Recall that a general angle (of any number of degrees, positive, zero, or negative) represents a rotation, clockwise if negative, counterclockwise if positive. To define the trigonometric functions of a general angle  $\theta$ , we introduce rectangular coordinates into the plane in such a way that the vertex of the angle is at the origin and the initial ray along the positive side of the  $x$ -axis. (See Fig. 45.) We choose any point  $P$  we please on the terminal ray

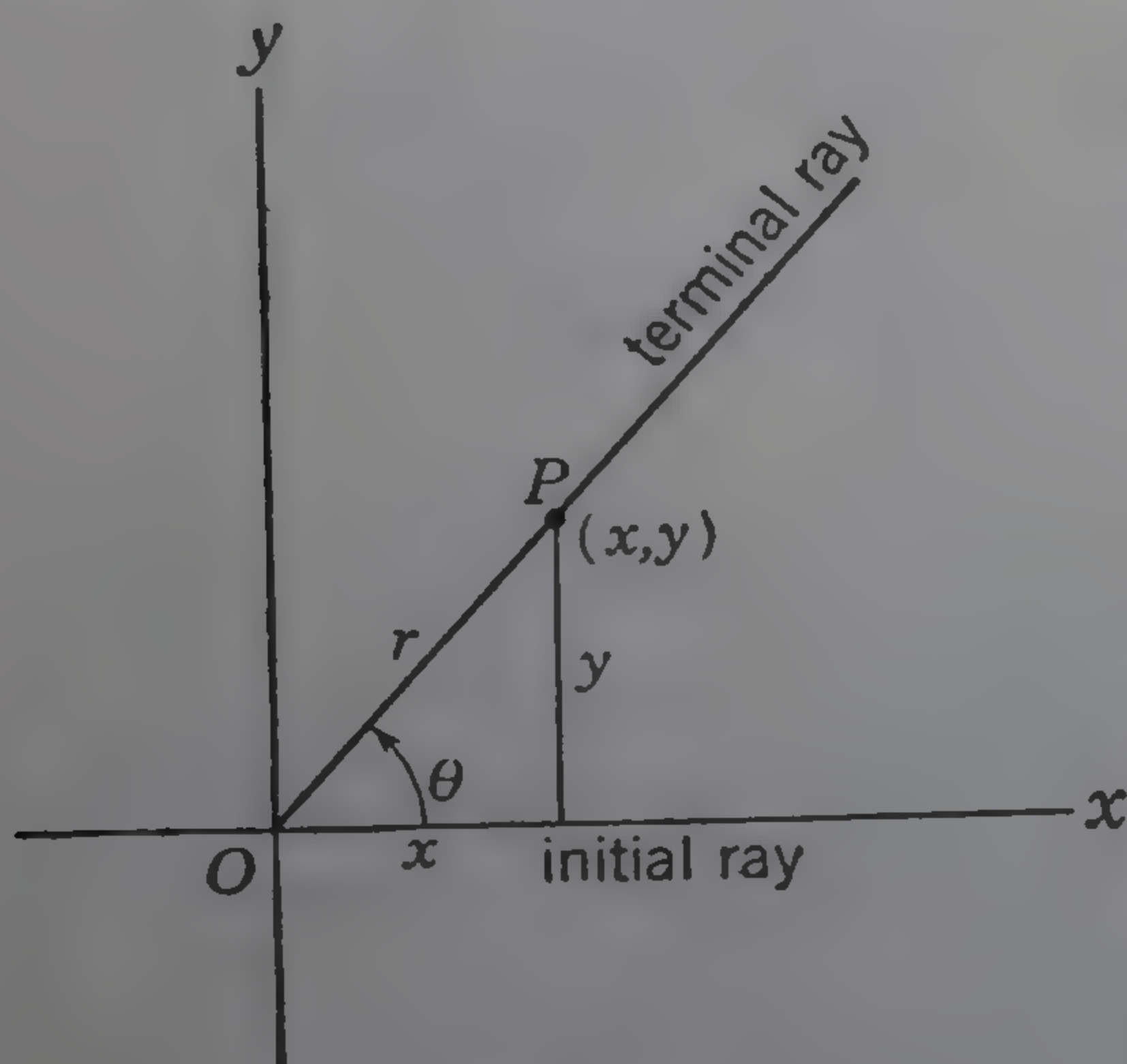


Fig. 45.

of the angle. If the coordinates of the point  $P$  are  $(x, y)$ , then its distance  $r$  from the origin is given by  $r = \sqrt{x^2 + y^2}$ . We then define

$$\sin \theta = y/r, \cos \theta = x/r, \tan \theta = y/x.$$

Recall that the sides of a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle are in the ratio  $1:\sqrt{3}:2$ , and that the sides of a  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle are in the ratio  $1:1:\sqrt{2}$ .

*Example 1.* To obtain the sine, cosine, and tangent of  $150^\circ$ , pick a point  $P$  on the terminal ray 2 units from the origin (Fig. 46).

\* For a more extended treatment see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, chapter XII, or textbooks on trigonometry.



Dropping a perpendicular from  $P$  to the  $x$ -axis, we form a so-called reference triangle, which is, in this example, a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle. Hence the coordinates of  $P$  are  $(-\sqrt{3}, 1)$ . Therefore  $\sin 150^\circ = 1/2$ ,  $\cos 150^\circ = -\sqrt{3}/2$ ,  $\tan 150^\circ = +1/-\sqrt{3} = -\sqrt{3}/3$ .

*Example 2.* To obtain the sine, cosine, and tangent of  $225^\circ$ , pick a point on the terminal ray with  $r = \sqrt{2}$  (Fig. 47). Drop a

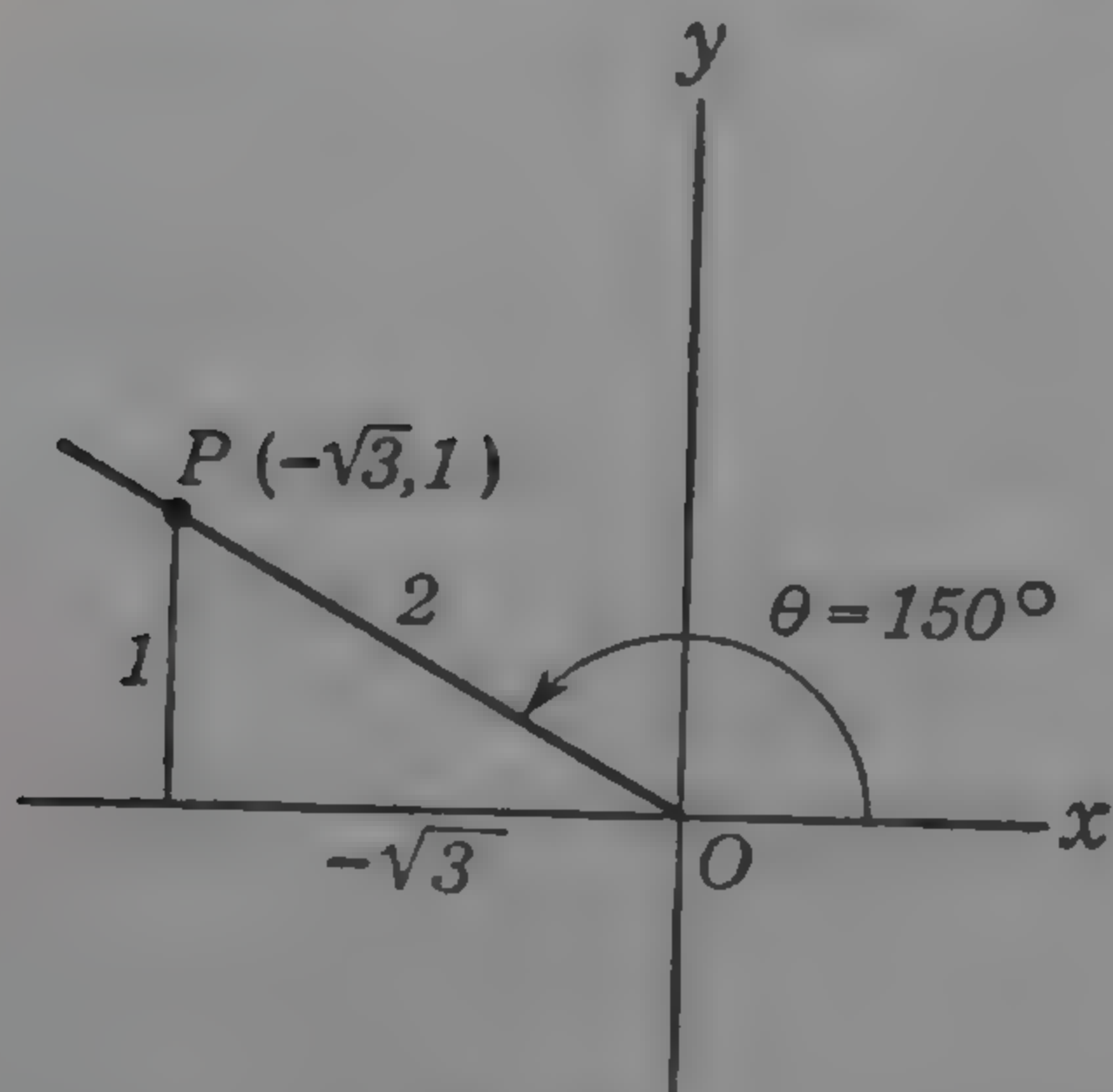


Fig. 46.

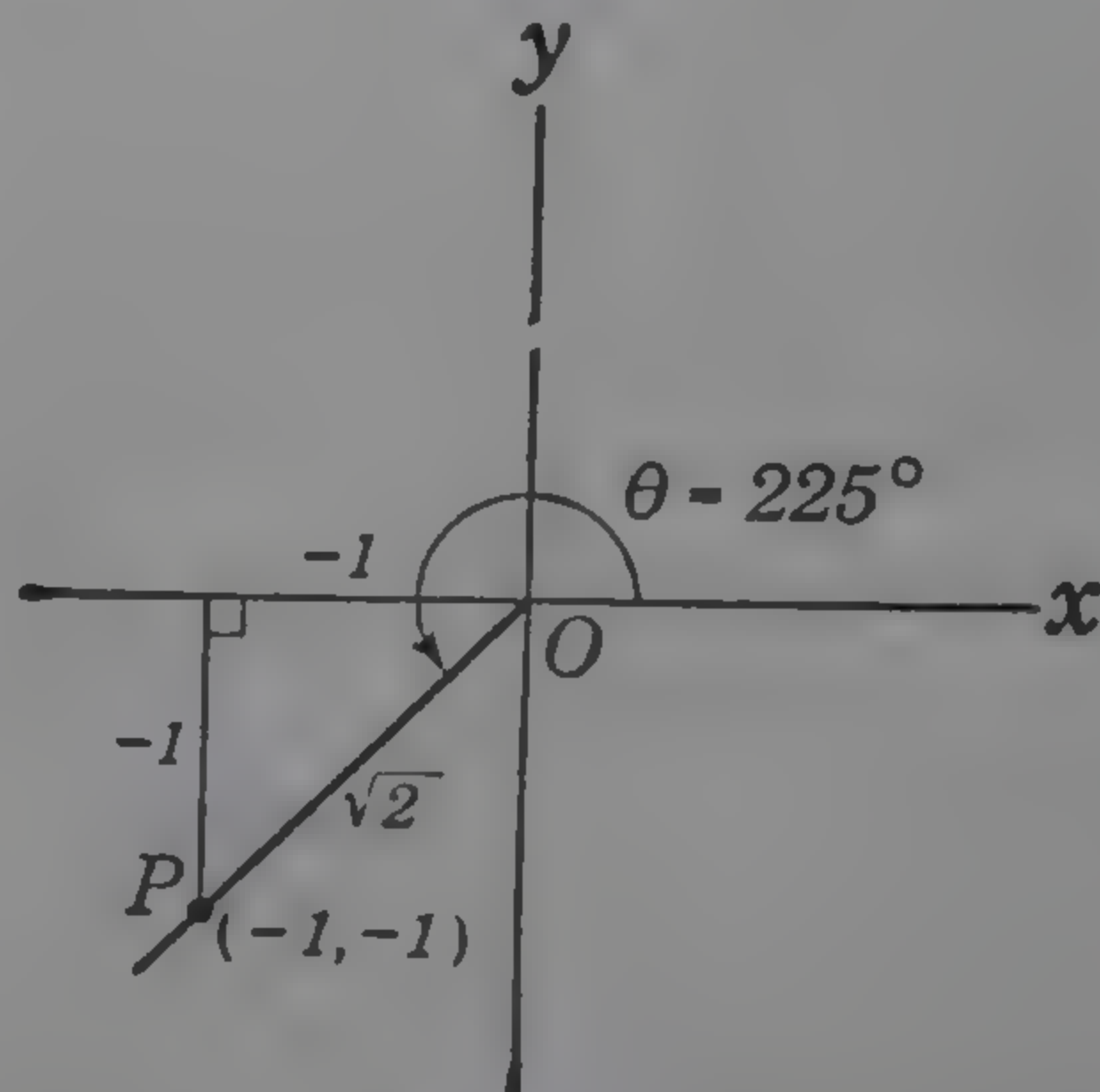


Fig. 47.

perpendicular to the  $x$ -axis to form the reference triangle which is a  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle. Hence  $P$  has the coordinates  $(-1, -1)$ , and  $\sin 225^\circ = -1/\sqrt{2} = -\sqrt{2}/2 = \cos 225^\circ$ ,  $\tan 225^\circ = -1/-1 = +1$ .

If the terminal ray of the angle  $\theta$  lies along one of the coordinate axes, there is no reference triangle, but we do not need one.

*Example 3.* To find the sine, cosine, and tangent of  $270^\circ$ , we pick a point  $P$  with coordinates  $(0, -1)$  on the terminal ray (Fig. 48). Then  $r = 1$  and  $\sin 270^\circ = -1/1 = -1$ ,  $\cos 270^\circ = 0/1 = 0$ ,  $\tan 270^\circ = -1/0$  which does not exist.

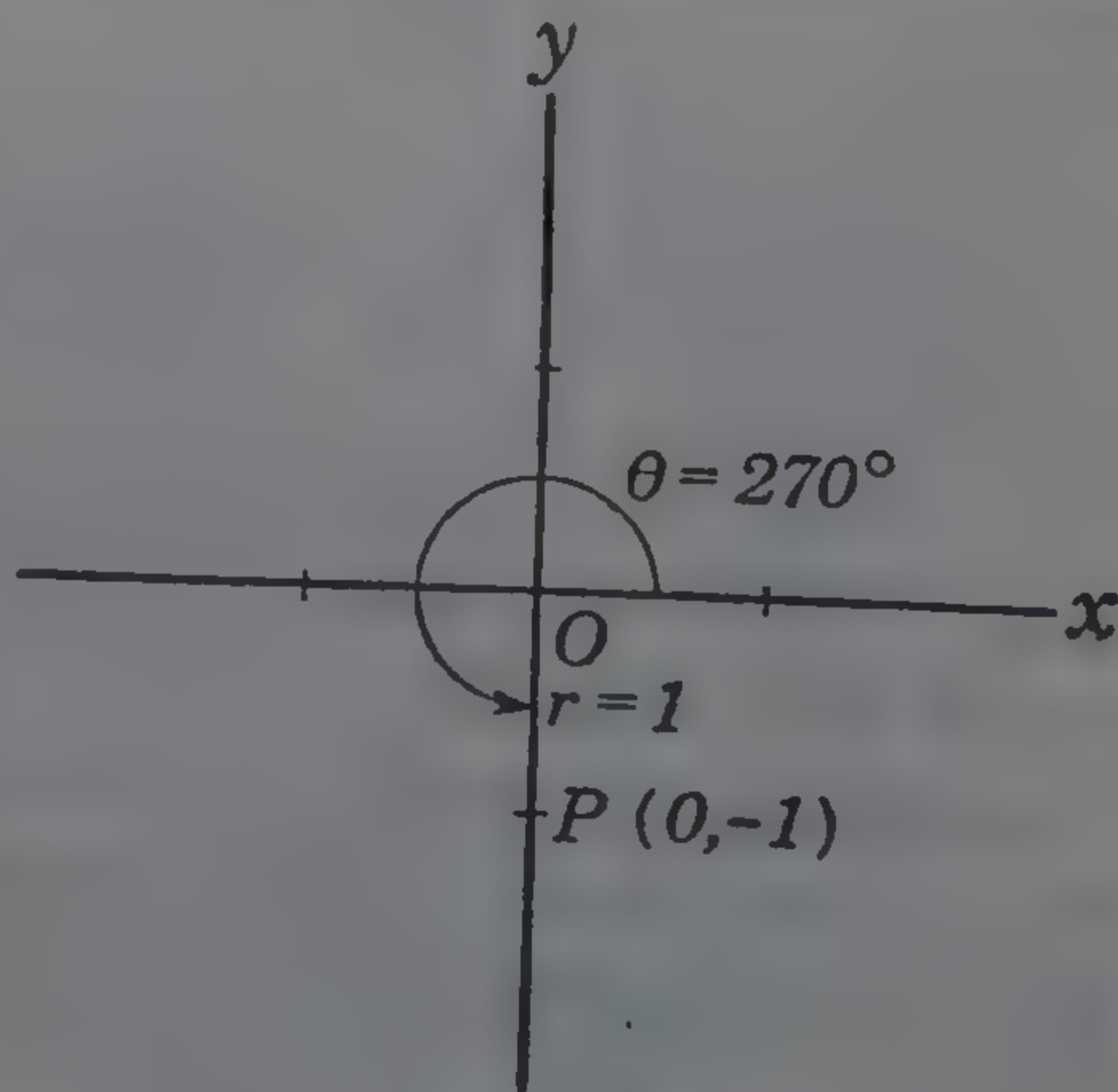


Fig. 48.

*The trigonometric functions of any angle which has a reference angle are the same as those of the reference angle except for sign. The sign may be determined by the quadrant in which the angle terminates (Fig. 49). The sine is positive in quadrants I and II, negative in III and IV. The cosine is positive in I and IV, negative in II and III. The tangent is positive in I and III, negative in II and IV.*



The trigonometric functions of certain angles are conveniently remembered as follows:

|     | $0^\circ$    | $30^\circ$   | $45^\circ$   | $60^\circ$   | $90^\circ$   |
|-----|--------------|--------------|--------------|--------------|--------------|
| sin | $\sqrt{0}/2$ | $\sqrt{1}/2$ | $\sqrt{2}/2$ | $\sqrt{3}/2$ | $\sqrt{4}/2$ |
| cos | $\sqrt{4}/2$ | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $\sqrt{1}/2$ | $\sqrt{0}/2$ |

The sin and cos of angles which are equal to  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , or have  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$  as reference angles, or angles which terminate along

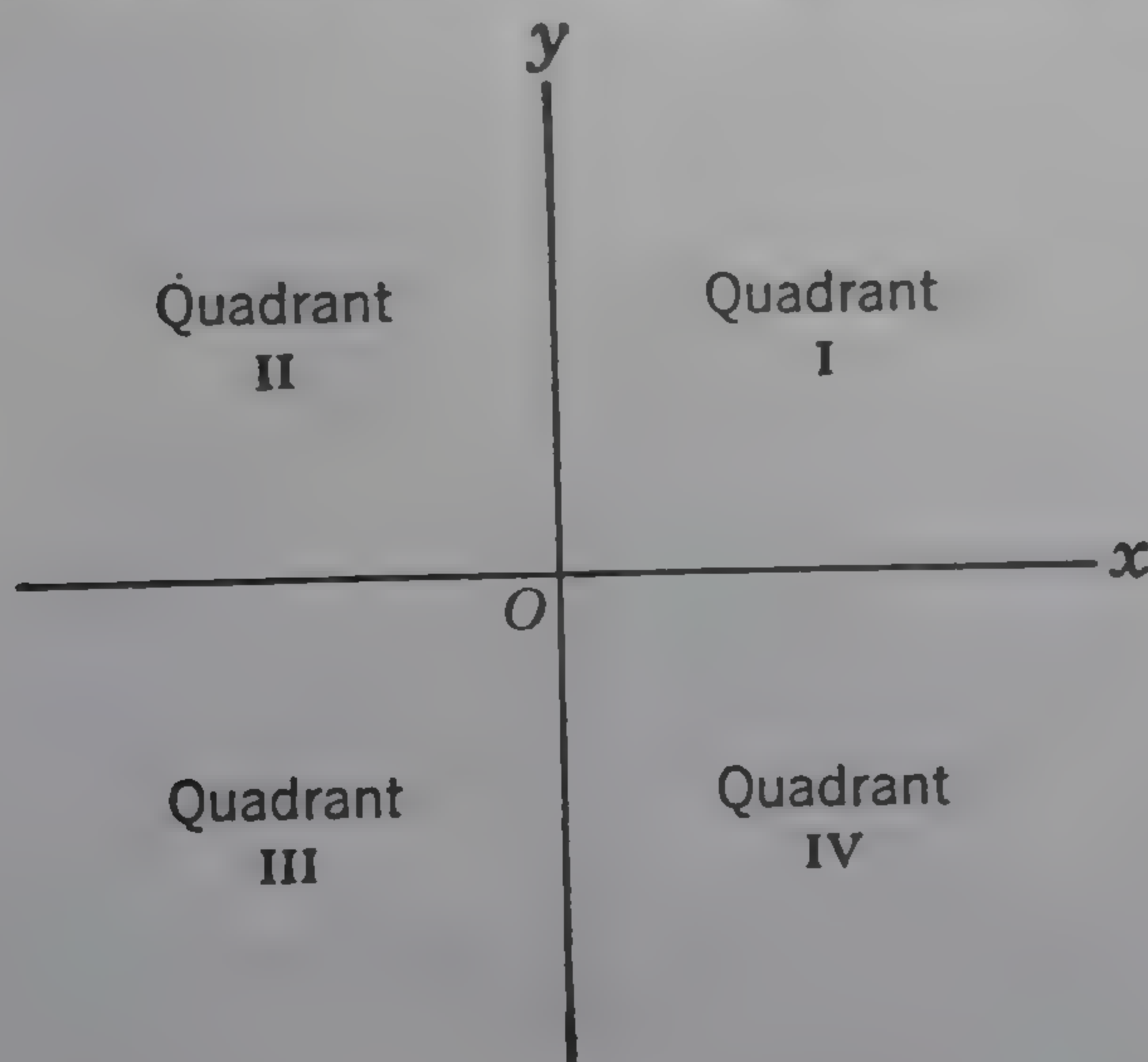


Fig. 49.

the axes, may be determined without tables as above. For other angles, we use table II, as follows.

*Example 4.* Find the sine and cosine of  $160^\circ$ . The reference angle is  $20^\circ$  and the angle  $160^\circ$  is in quadrant II. Hence  $\sin 160^\circ = \sin 20^\circ = 0.3420$  and  $\cos 160^\circ = -\cos 20^\circ = -0.9397$ .

Angles which differ by integral multiples of  $360^\circ$  have the same trigonometric functions, since they have the same terminal ray. For example, the trigonometric functions of  $750^\circ$ ,  $390^\circ$ , and  $30^\circ$  are the same.

## EXERCISES

Find, without tables, the values of the sine, cosine, and tangent of the following angles:

- |                  |                  |                  |                  |                   |
|------------------|------------------|------------------|------------------|-------------------|
| 1. $30^\circ$ .  | 2. $45^\circ$ .  | 3. $60^\circ$ .  | 4. $120^\circ$ . | 5. $135^\circ$ .  |
| 6. $150^\circ$ . | 7. $210^\circ$ . | 8. $225^\circ$ . | 9. $240^\circ$ . | 10. $300^\circ$ . |



- |                    |                   |                   |                   |                    |
|--------------------|-------------------|-------------------|-------------------|--------------------|
| 11. $315^\circ$ .  | 12. $330^\circ$ . | 13. $390^\circ$ . | 14. $405^\circ$ . | 15. $-210^\circ$ . |
| 16. $-750^\circ$ . | 17. $0^\circ$ .   | 18. $90^\circ$ .  | 19. $180^\circ$ . | 20. $270^\circ$ .  |
| 21. $360^\circ$ .  |                   | 22. $450^\circ$ . |                   |                    |

Find, using table II, the values of the sine and cosine of the following angles:

- |                    |                   |                   |                   |                    |
|--------------------|-------------------|-------------------|-------------------|--------------------|
| 23. $20^\circ$ .   | 24. $70^\circ$ .  | 25. $110^\circ$ . | 26. $160^\circ$ . | 27. $200^\circ$ .  |
| 28. $250^\circ$ .  | 29. $290^\circ$ . | 30. $340^\circ$ . | 31. $380^\circ$ . | 32. $-20^\circ$ .  |
| 33. $-130^\circ$ . | 34. $730^\circ$ . | 35. $516^\circ$ . | 36. $242^\circ$ . | 37. $-235^\circ$ . |
| 38. $472^\circ$ .  | 39. $605^\circ$ . |                   | 40. $700^\circ$ . |                    |

95. Trigonometric or polar form of a complex number. A point may be specified by its  $x$ - and  $y$ -coordinates, which give directions for reaching it from the origin. But it may be specified equally well by stipulating its direction from the origin and the distance  $r$  one must go in that direction. The direction may be specified by the angle  $\theta$  formed with the positive side of the  $x$ -axis (Fig. 50). The numbers  $(r, \theta)$  are called **polar coordinates** of the point  $P$ , to distinguish them from the rectangular coordinates  $(x, y)$ . The two systems of coordinates are related by the following equations, which are obvious from Fig. 50:

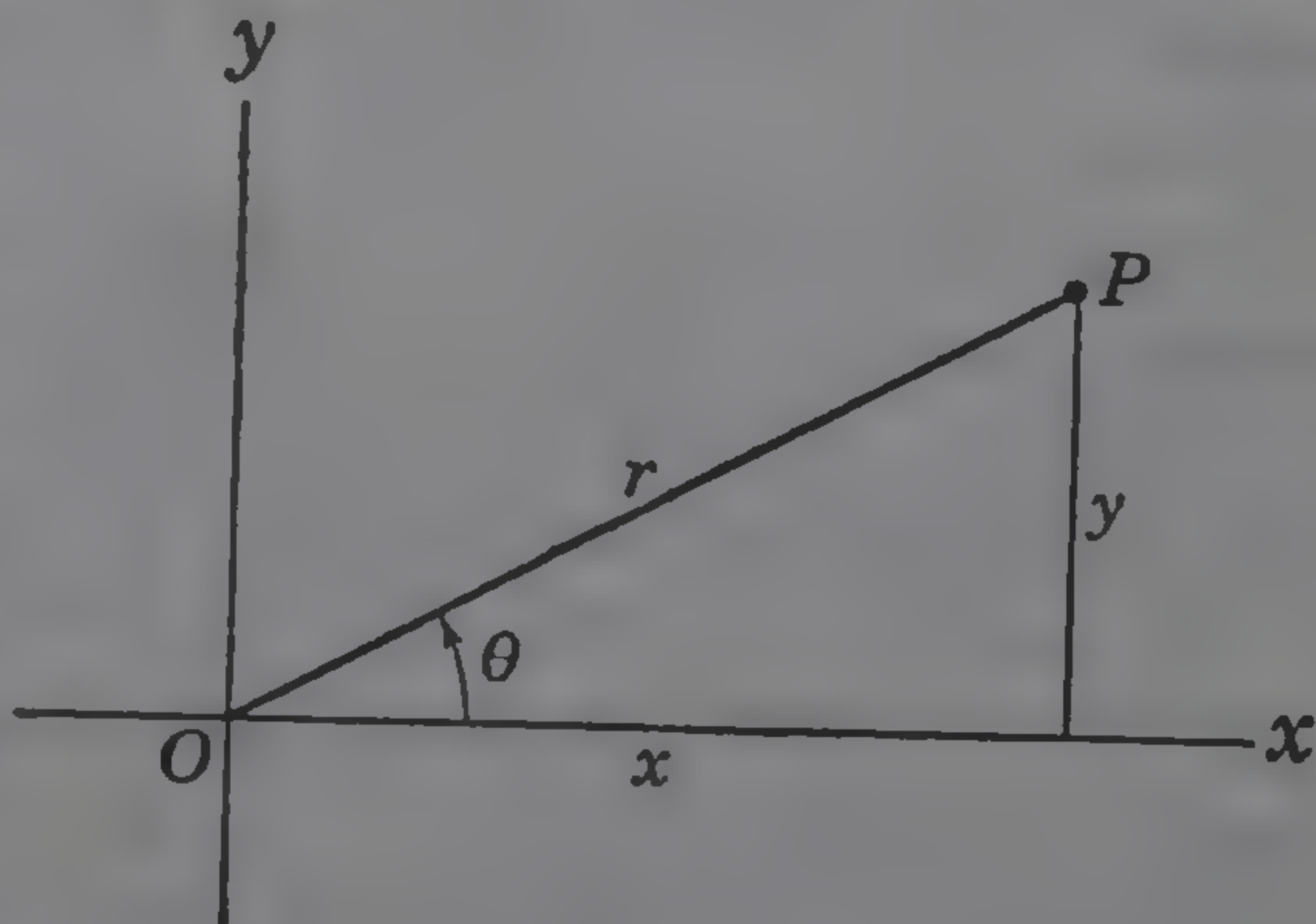


Fig. 50.

$$(1) \quad r = \sqrt{x^2 + y^2}, \sin \theta = y/r, \cos \theta = x/r.$$

$$(2) \quad x = r \cos \theta, y = r \sin \theta.$$

Equations (2) enable us to determine  $x$  and  $y$  when  $r$  and  $\theta$  are given. Equations (1) enable us to determine  $r$  and  $\theta$  when  $x$  and  $y$  are given, for knowledge of  $\sin \theta$  and  $\cos \theta$  determine\* the value of  $\theta$  if  $0^\circ \leq \theta < 360^\circ$ .

Using (2) we may write

$$x + yi = r \cos \theta + ir \sin \theta$$

\* Many books suggest the use of  $\tan \theta$  for this purpose. But  $\tan \theta$ , unassisted, does not determine the quadrant of  $\theta$ , since  $\tan(180^\circ + \theta) = \tan \theta$ ;  $\tan \theta$  may be used if the values of  $x$  and  $y$  are employed to determine the quadrant.



or

$$(3) \quad x + yi = r(\cos \theta + i \sin \theta).$$

The right member of (3) is called the **polar form** or **trigonometric form** of the complex number whose standard form is  $x + yi$ . The quantity  $r$ , which is always  $\geq 0$ , is called the **modulus** or **length** or **absolute value** of the complex number. The angle  $\theta$  is called the **amplitude** or **argument** or **angle** of the complex number. Note that the amplitude is not uniquely determined, for any integral multiple of  $360^\circ$  may be added without altering the sine and cosine. Hence the amplitude may be  $\theta + k \cdot 360^\circ$  where  $k = 0, \pm 1, \pm 2, \pm 3, \dots$ .

The polar form exhibits clearly the polar coordinates of the point  $P$ , while the standard or rectangular form exhibits the rectangular coordinates of  $P$ . Hence equations (1) enable us to convert the standard form into the polar form, and equations (2) enable us to convert the polar form into the standard form.

*Example 1.* Write  $-1 + i$  in polar form, using the smallest non-negative value of the amplitude.

*Solution.*  $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ ,  $\sin \theta = 1/\sqrt{2} = \sqrt{2}/2$ ,  $\cos \theta = -1/\sqrt{2} = -\sqrt{2}/2$ . Hence  $\theta = 135^\circ$ . Therefore the polar form of  $-1 + i = \sqrt{2}(\cos 135^\circ + i \sin 135^\circ)$ .

*Example 2.* Write  $2(\cos 300^\circ + i \sin 300^\circ)$  in standard form.

*Solution.*  $\cos 300^\circ = +1/2$ ,  $\sin 300^\circ = -\sqrt{3}/2$ . Hence  $2(\cos 300^\circ + i \sin 300^\circ) = 2\left(\frac{1}{2} + i\frac{-\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$ .

*Example 3.* Write the number  $-2$  in polar form, using the smallest non-negative value of the amplitude.

*Solution.* In standard form,  $-2 = -2 + 0i$ . Hence  $\theta = 180^\circ$  and  $r = 2$ . Thus  $-2 + 0i = 2(\cos 180^\circ + i \sin 180^\circ)$ .

*Example 4.* Write  $3(\cos 50^\circ + i \sin 50^\circ)$  in standard form.

*Solution.* From table II, we find  $\cos 50^\circ = 0.6428$  and  $\sin 50^\circ = 0.7660$ , approximately. Hence  $3(\cos 50^\circ + i \sin 50^\circ) = 3(0.6428 + 0.7660i) = 1.9284 + 2.2980i$ , approximately.

If it is desired to exhibit all possible values of the amplitude, we may replace  $\theta$  by  $\theta + k \cdot 360^\circ$  where  $k = 0, \pm 1, \pm 2, \dots$ . Thus in example 1 above we would write

$$-1 + i = \sqrt{2}(\cos [135^\circ + k \cdot 360^\circ] + i \sin [135^\circ + k \cdot 360^\circ])$$

where  $k$  may be any integer whatever.



## EXERCISES

Write each number in standard  $(x + yi)$  form, and plot:

- |                                                  |                                              |
|--------------------------------------------------|----------------------------------------------|
| 1. $2(\cos 30^\circ + i \sin 30^\circ)$ .        | 2. $4(\cos 60^\circ + i \sin 60^\circ)$ .    |
| 3. $6(\cos 45^\circ + i \sin 45^\circ)$ .        | 4. $8(\cos 120^\circ + i \sin 120^\circ)$ .  |
| 5. $1(\cos 135^\circ + i \sin 135^\circ)$ .      | 6. $3(\cos 150^\circ + i \sin 150^\circ)$ .  |
| 7. $3(\cos 90^\circ + i \sin 90^\circ)$ .        | 8. $2(\cos 270^\circ + i \sin 270^\circ)$ .  |
| 9. $4(\cos 180^\circ + i \sin 180^\circ)$ .      | 10. $2(\cos 210^\circ + i \sin 210^\circ)$ . |
| 11. $2(\cos 0^\circ + i \sin 0^\circ)$ .         | 12. $3(\cos 360^\circ + i \sin 360^\circ)$ . |
| 13. $4(\cos 225^\circ + i \sin 225^\circ)$ .     | 14. $6(\cos 240^\circ + i \sin 240^\circ)$ . |
| 15. $5(\cos [-90^\circ] + i \sin [-90^\circ])$ . | 16. $6(\cos 300^\circ + i \sin 300^\circ)$ . |
| 17. $2(\cos [-30^\circ] + i \sin [-30^\circ])$ . | 18. $4(\cos 315^\circ + i \sin 315^\circ)$ . |

Write each of the following in polar form, using the value of  $\theta$  lying in the interval  $0^\circ \leq \theta < 360^\circ$ , and plot:

- |                               |                                |                                 |
|-------------------------------|--------------------------------|---------------------------------|
| 19. $1 + i\sqrt{3}$ .         | 20. $2\sqrt{3} + 2i$ .         | 21. $4 + 4i$ .                  |
| 22. $1 + i$ .                 | 23. $-2 + 2i\sqrt{3}$ .        | 24. $-\sqrt{3} + i$ .           |
| 25. $-\sqrt{2} + i\sqrt{2}$ . | 26. $-\sqrt{3} - i$ .          | 27. $3i$ .                      |
| 28. $-2 - 2i\sqrt{3}$ .       | 29. $5$ .                      | 30. $-3\sqrt{2} - 3i\sqrt{2}$ . |
| 31. $-2$ .                    | 32. $2\sqrt{2} - 2i\sqrt{2}$ . | 33. $-3i$ .                     |
| 34. $3 - 3i\sqrt{3}$ .        | 35. $2\sqrt{3} - 2i$ .         | 36. $1$ .                       |
| 37. $-1$ .                    | 38. $i$ .                      | 39. $-8i$ .                     |
| 40. $-8$ .                    |                                |                                 |

Write each of the following in polar form, exhibiting all possible values of  $\theta$ , and plot:

- |                               |             |            |              |                      |
|-------------------------------|-------------|------------|--------------|----------------------|
| 41. $8$ .                     | 42. $-16$ . | 43. $8i$ . | 44. $-16i$ . | 45. $\sqrt{3} - i$ . |
| 46. $-\sqrt{2} + i\sqrt{2}$ . |             |            |              |                      |

Write each of the following in standard form, approximately, using table II, and plot:

- |                                              |                                              |
|----------------------------------------------|----------------------------------------------|
| 47. $2(\cos 20^\circ + i \sin 20^\circ)$ .   | 48. $3(\cos 170^\circ + i \sin 170^\circ)$ . |
| 49. $1(\cos 190^\circ + i \sin 190^\circ)$ . | 50. $2(\cos 350^\circ + i \sin 350^\circ)$ . |

51. Find an amplitude of (a) any positive real number; (b) any negative real number; (c) any pure imaginary number  $yi$  where  $y > 0$ ; (d) any pure imaginary number  $yi$  where  $y < 0$ .

96. Graphical meaning of multiplication and division of complex numbers. We have seen in section 93 that addition of complex numbers corresponds geometrically to the parallelogram law for the composition of vectors. To see what the geometric significance of multiplication of complex numbers is, we need the



polar form. Let any two numbers be written in their polar forms as

$$(1) \quad r(\cos \alpha + i \sin \alpha),$$

$$(2) \quad R(\cos \beta + i \sin \beta).$$

Multiplying them we get

$$(3) \quad rR([\cos \alpha \cos \beta - \sin \alpha \sin \beta] + i[\sin \alpha \cos \beta + \cos \alpha \sin \beta]).$$

In trigonometry, it is proved that the first bracket is equal to  $\cos [\alpha + \beta]$  while the second bracket is equal to  $\sin [\alpha + \beta]$ .

Hence the product of (1) and (2) may be written as

$$(4) \quad rR(\cos [\alpha + \beta] + i \sin [\alpha + \beta]).$$

But this is in polar form. Hence we conclude that the modulus of the product of two complex numbers is the product of their moduli, while the amplitude of the product is the sum of their amplitudes (Fig. 51).

*Example 1.* The product of  $2(\cos 20^\circ + i \sin 20^\circ)$  and  $3(\cos 70^\circ + i \sin 70^\circ)$  is equal to  $6(\cos 90^\circ + i \sin 90^\circ)$  or  $6i$ .

By definition, the quotient

of one complex number  $r_1(\cos \alpha + i \sin \alpha)$  by another  $r_2(\cos \beta + i \sin \beta)$  is the complex number  $R(\cos \theta + i \sin \theta)$  such that the second multiplied by the third equals the first. Hence

$$r_2 R(\cos [\beta + \theta] + i \sin [\beta + \theta]) = r_1(\cos \alpha + i \sin \alpha)$$

which is true if  $r_2 R = r_1$  and  $\beta + \theta = \alpha$ .

Hence we may write  $R = \frac{r_1}{r_2}$  and  $\theta = \alpha - \beta$ .

Therefore

$$(5) \quad \frac{r_1(\cos \alpha + i \sin \alpha)}{r_2(\cos \beta + i \sin \beta)} = \frac{r_1}{r_2} (\cos [\alpha - \beta] + i \sin [\alpha - \beta]).$$

That is, the modulus of the quotient of two complex numbers is the quotient of their moduli, and the amplitude of the quotient is the difference of their amplitudes.

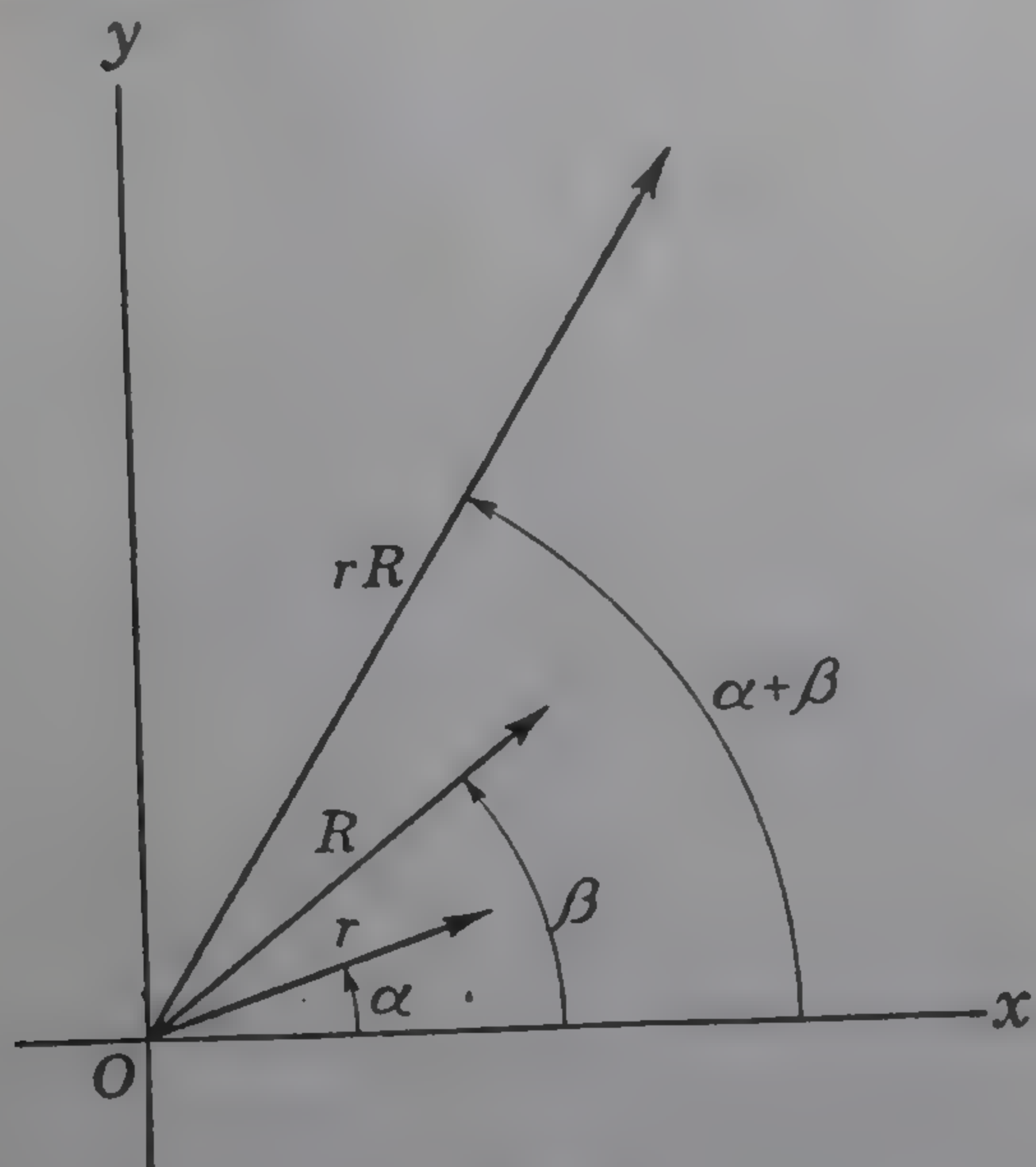


Fig. 51.



*Example 2.* 
$$\frac{6(\cos 50^\circ + i \sin 50^\circ)}{2(\cos 20^\circ + i \sin 20^\circ)} = 3(\cos 30^\circ + i \sin 30^\circ) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i.$$

## EXERCISES

(a) Perform the following multiplications and divisions by the above methods, leaving answers in polar form; (b) rewrite the answers in standard form, using table II if necessary:

1.  $3(\cos 10^\circ + i \sin 10^\circ) \cdot 2(\cos 35^\circ + i \sin 35^\circ).$
2.  $5(\cos 80^\circ + i \sin 80^\circ) \cdot 2(\cos 40^\circ + i \sin 40^\circ).$
3.  $4(\cos 110^\circ + i \sin 110^\circ) \cdot 2(\cos 70^\circ + i \sin 70^\circ).$
4.  $2(\cos 170^\circ + i \sin 170^\circ) \cdot 3(\cos 190^\circ + i \sin 190^\circ).$
5.  $6(\cos 110^\circ + i \sin 110^\circ) \div 2(\cos 20^\circ + i \sin 20^\circ).$
6.  $8(\cos 85^\circ + i \sin 85^\circ) \div 4(\cos 40^\circ + i \sin 40^\circ).$
7.  $10(\cos 155^\circ + i \sin 155^\circ) \div 2(\cos 35^\circ + i \sin 35^\circ).$
8.  $12(\cos 75^\circ + i \sin 75^\circ) \div 4(\cos 105^\circ + i \sin 105^\circ).$

**97. De Moivre's theorem.** Consider the square of a complex number

$$(1) \quad r(\cos \theta + i \sin \theta),$$

that is, the product of (1) by itself. By the preceding section we get

$$(2) \quad [r(\cos \theta + i \sin \theta)]^2 = r^2(\cos 2\theta + i \sin 2\theta).$$

Multiplying (2) by (1), we get

$$(3) \quad [r(\cos \theta + i \sin \theta)]^3 = r^3(\cos 3\theta + i \sin 3\theta),$$

and so on. This clearly suggests the following theorem.

**DE MOIVRE'S THEOREM:**  $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$

For positive integral powers of  $n$  this can be proved by mathematical induction (see chapter XIX). It can also be proved for other kinds of exponents but we shall not give any proof here.

*Example 1.*  $[2(\cos 30^\circ + i \sin 30^\circ)]^3 = 8(\cos 90^\circ + i \sin 90^\circ) = 8i.$

*Example 2.*  $(1 + i)^4 = [\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)]^4 = 4(\cos 180^\circ + i \sin 180^\circ) = -4.$

Abraham de Moivre (1667–1754) made important contributions to various branches of mathematics; he is especially famous for his work in the theory of probability.



## EXERCISES

Find the indicated powers by de Moivre's theorem, leaving answers in polar form except where the standard form can be found without use of tables:

1.  $[3(\cos 60^\circ + i \sin 60^\circ)]^3$ .
2.  $[2(\cos 120^\circ + i \sin 120^\circ)]^3$ .
3.  $[3(\cos 30^\circ + i \sin 30^\circ)]^3$ .
4.  $[3(\cos 45^\circ + i \sin 45^\circ)]^4$ .
5.  $[2(\cos 15^\circ + i \sin 15^\circ)]^5$ .
6.  $[2(\cos 15^\circ + i \sin 15^\circ)]^6$ .
7.  $[2(\cos 45^\circ + i \sin 45^\circ)]^6$ .
8.  $[2(\cos 30^\circ + i \sin 30^\circ)]^6$ .
9.  $[\frac{1}{2}(\cos 30^\circ + i \sin 30^\circ)]^4$ .
10.  $[\frac{1}{2}(\cos 72^\circ + i \sin 72^\circ)]^5$ .
11.  $(1 + i)^3$ .
12.  $(\sqrt{3} + i)^6$ .
13.  $(1 - i)^3$ .
14.  $(\sqrt{3} - i)^3$ .
15.  $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{12}$ .
16.  $(-\sqrt{3} + i)^3$ .
17.  $(\sqrt{2} + i\sqrt{2})^4$ .
18.  $(-\sqrt{3} - i)^6$ .
19.  $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^6$ .
20.  $(-\sqrt{2} + i\sqrt{2})^4$ .

98. The  $n$ th roots of a complex number. By means of de Moivre's theorem we can find the  $n$ th roots of any complex number.

*Example 1.* Find the cube roots of  $8i$ .

*Solution.* Expressing  $8i$  in polar form, we have  $8i = 8(\cos 90^\circ + i \sin 90^\circ)$ . But  $90^\circ$  can be replaced by  $90^\circ + k \cdot 360^\circ$  where  $k$  may be any integer, positive, negative, or zero. Hence we may write

$$(1) \quad 8i = 8(\cos [90^\circ + k \cdot 360^\circ] + i \sin [90^\circ + k \cdot 360^\circ]).$$

Suppose  $r(\cos \theta + i \sin \theta)$  is a cube root of  $8i$ . Then

$$(2) \quad [r(\cos \theta + i \sin \theta)]^3 = 8i$$

by definition. By de Moivre's theorem, the left member of (2) may be written as  $r^3(\cos 3\theta + i \sin 3\theta)$ . Hence, using (1) we get

$$(3) \quad r^3(\cos 3\theta + i \sin 3\theta) = 8(\cos [90^\circ + k \cdot 360^\circ] + i \sin [90^\circ + k \cdot 360^\circ]).$$

Therefore

$$(4) \quad r^3 = 8$$

and

$$(5) \quad 3\theta = 90^\circ + k \cdot 360^\circ.$$

From (4) we get  $r = \sqrt[3]{8} = 2$ , since  $r$  is necessarily a positive real number and therefore must be the principal cube root of 8. From

(5), we get  $\theta = 30^\circ + k \cdot 120^\circ$ . Hence,



$$(6) \quad r(\cos \theta + i \sin \theta) = 2(\cos [30^\circ + k \cdot 120^\circ] + i \sin [30^\circ + k \cdot 120^\circ]).$$

If  $k = 0$ , (6) yields  $2(\cos 30^\circ + i \sin 30^\circ) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$ .

If  $k = 1$ , (6) yields  $2(\cos 150^\circ + i \sin 150^\circ) = 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\sqrt{3} + i$ .

If  $k = 2$ , (6) yields  $2(\cos 270^\circ + i \sin 270^\circ) = 2(0 - 1i) = -2i$ .

Taking  $k = 3, 4, 5, \dots$  will yield only repetitions of these answers, since adding  $360^\circ$  does not alter the values of the trigono-

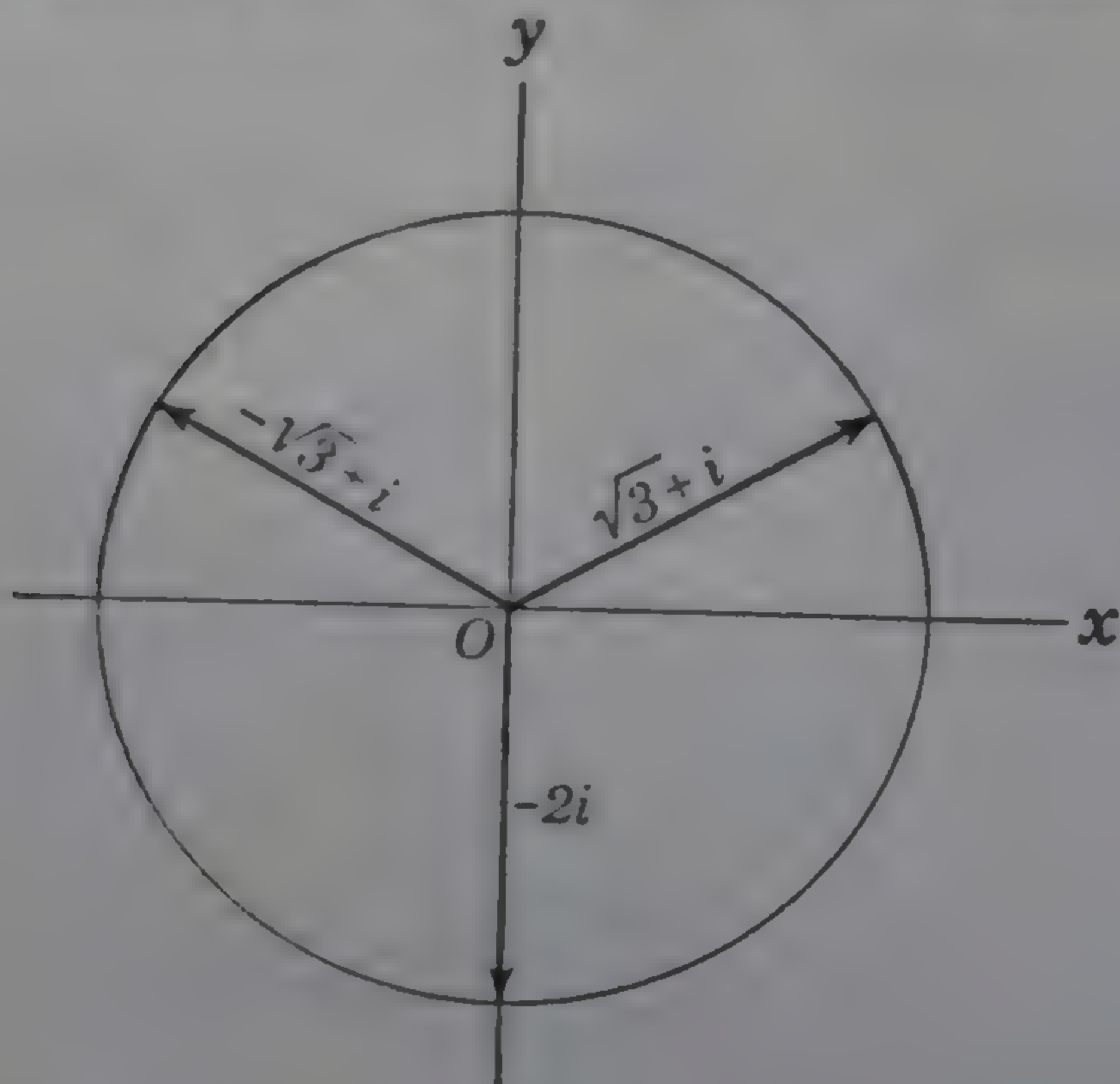


Fig. 52.

metric functions. Hence,  $8i$  has three distinct cube roots, namely  $\sqrt{3} + i$ ,  $-\sqrt{3} + i$ , and  $-2i$ . The student should cube each of them, as a check, and should obtain  $8i$  in each case.

Note that if the three cube roots of  $8i$  are plotted, the vectors which represent them graphically divide the circle of radius 2 with center at  $O$  into 3 equal sectors (Fig. 52).

In general, to obtain the  $n$ th roots of a given complex number, we first express the number in polar form

$$(7) \quad R(\cos \alpha + i \sin \alpha) = R(\cos [\alpha + k \cdot 360^\circ] + i \sin [\alpha + k \cdot 360^\circ]).$$

Let  $r(\cos \theta + i \sin \theta)$  be an  $n$ th root of (7). Then, by definition of  $n$ th root,



$$[r(\cos \theta + i \sin \theta)]^n = R(\cos [\alpha + k \cdot 360^\circ] + i \sin [\alpha + k \cdot 360^\circ]).$$

By de Moivre's theorem, this becomes

$$r^n(\cos n\theta + i \sin n\theta) = R(\cos [\alpha + k \cdot 360^\circ] + i \sin [\alpha + k \cdot 360^\circ]).$$

Hence

$$(8) \quad r^n = R, \text{ or } r = \sqrt[n]{R}$$

and

$$(9) \quad n\theta = \alpha + k \cdot 360^\circ, \text{ or } \theta = \frac{\alpha}{n} + k \cdot \frac{360^\circ}{n}.$$

Therefore the  $n$ th roots of  $R(\cos \alpha + i \sin \alpha)$  are given by the expression

$$(10) \quad \sqrt[n]{R} \left( \cos \left[ \frac{\alpha}{n} + k \frac{360^\circ}{n} \right] + i \sin \left[ \frac{\alpha}{n} + k \frac{360^\circ}{n} \right] \right).$$

It is clear that any successive  $n$  integral values of  $k$  will yield  $n$  different  $n$ th roots, while further integral values of  $k$  will yield repetitions. Hence we can get the  $n$  distinct  $n$ th roots by substituting  $k = 0, 1, 2, \dots, n - 1$  successively in (10). The vectors representing the  $n$  distinct  $n$ th roots should divide the circle of radius  $\sqrt[n]{R}$  with center at  $O$  into  $n$  equal sectors.

## EXERCISES

*Find the indicated roots, leaving answers in polar form except where the standard form can be found without tables:*

- |                                                          |                                         |
|----------------------------------------------------------|-----------------------------------------|
| 1. Square roots of $i$ .                                 | 2. Square roots of $-4i$ .              |
| 3. Cube roots of $27i$ .                                 | 4. Cube roots of $-8i$ .                |
| 5. Cube roots of $-64i$ .                                | 6. Cube roots of $i$ .                  |
| 7. Sixth roots of 1.                                     | 8. Sixth roots of $-1$ .                |
| 9. Cube roots of $-8$ .                                  | 10. Fourth roots of $-16$ .             |
| 11. Fifth roots of 32.                                   | 12. Cube roots of $4 + 4i\sqrt{3}$ .    |
| 13. Cube roots of $-4\sqrt{2} + 4i\sqrt{2}$ .            |                                         |
| 14. Cube roots of $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ . |                                         |
| 15. Cube roots of 1.                                     | 16. Cube roots of $-1$ .                |
| 17. Fourth roots of $-8 + 8i\sqrt{3}$ .                  | 18. Fourth roots of $-8 - 8i\sqrt{3}$ . |
| 19. Fourth roots of 16.                                  | 20. Fourth roots of $-81$ .             |
| 21. Square roots of $8 + 8i\sqrt{3}$ .                   | 22. Square roots of $-2 + 2i\sqrt{3}$ . |



23. (a) Show that if we let  $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ , one of the imaginary cube roots of 1, then the other imaginary cube root of 1 is  $\omega^2$ .

(b) Show that if  $z$  is any complex number and  $\sqrt[3]{z}$  denotes any one of its cube roots, then the other two cube roots of  $z$  are  $\omega \sqrt[3]{z}$  and  $\omega^2 \sqrt[3]{z}$ . (*Hint: consider the polar forms of  $\sqrt[3]{z}$  and  $\omega$  and the geometric meaning of multiplication of complex numbers.*)

99. **Solution of pure equations of  $n$ th degree.** By a pure equation of  $n$ th degree is meant a polynomial equation of  $n$ th degree which has no terms of degree lower than  $n$  except the constant term, that is, an equation which can be written in the form

$$(1) \quad x^n = A, \text{ or } x^n - A = 0,$$

where  $A$  is any (complex) constant. Clearly, any root of this equation is an  $n$ th root of  $A$ . Hence there are  $n$  different roots which may be found by the method of the preceding section.

*Example.* Solve the equation  $x^3 - 8i = 0$ .

*Solution.* Any root of this equation is, by definition, a cube root of  $8i$ , since  $x^3 = 8i$ . As in example 1 of the preceding section, we find that the roots are  $\sqrt{3} + i$ ,  $-\sqrt{3} + i$ , and  $-2i$ .

*Note.* Hitherto we have found imaginary roots of polynomial equations only if they were quadratic. In this section we have been able to find them no matter what the degree, provided the equation is of the special type (1). The further study of the solution of equations will be taken up in the next chapter.

## EXERCISES

*Find all the roots of each of the following equations, express them in standard form, and check:*

1.  $x^3 = 8$ .
2.  $x^4 = 16$ .
3.  $x^3 + 8i = 0$ .
4.  $x^6 = 64$ .
5.  $x^2 = 4i$ .
6.  $x^4 + 81 = 0$ .
7.  $x^3 - 27i = 0$ .
8.  $x^6 + 64 = 0$ .
9.  $x^4 + 8 - 8i\sqrt{3} = 0$ .
10.  $x^2 = -2 + 2i\sqrt{3}$ .



## CHAPTER XV

### Theory of Equations

100. The fundamental theorem of algebra. A polynomial equation in  $x$  or a rational integral equation in  $x$  of the  $n$ th degree is one that can be written in the form

$$(1) \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad (a_0 \neq 0)$$

where the coefficients  $a_0, a_1, \dots, a_n$  are constants and  $n$  is a positive integer. A root or solution of such an equation is a value of  $x$  which satisfies the equation; that is, a value such that the equation becomes a true statement when this value is substituted for  $x$ . To solve an equation\* means to find all its roots. Equations of degree 1, 2, 3, 4 are called **linear, quadratic, cubic, and quartic**, respectively. So far we have studied methods of solving equations of degree 1 and 2 only. In this chapter, we shall attack the problem of solving equations of degree higher than 2. In (1) the coefficient of the highest power,  $a_0$ , is called the **leading coefficient**, while  $a_n$  is called the **constant term**.

To begin with, let us review the development of the complex number system, and see why we need complex numbers and no others.† We shall be guided by our desire that every polynomial equation in  $x$  possess a root. Suppose, however, that our number system contained only natural numbers or positive integers, no other numbers having yet been invented. Then, while the linear equation  $x - 2 = 0$  would have the root 2, the linear equation

$$(1) \quad x + 2 = 0$$

would have no root in our system of numbers. Therefore we would invent negative integers in order to provide a root for equations like (1). The equation

$$(2) \quad 2x = 0$$

\* In this chapter the word *equation* will always mean *rational integral equation*.

† The student should review chapter I, especially sections 9-16, and familiarize himself thoroughly with the classification of numbers in section 16 without delay.



would still have no root among the positive or negative integers. Therefore we would invent the number zero. The linear equation

$$(3) \quad 2x - 1 = 0$$

would still have no root, even though our number system were now enlarged to include all integers, or whole numbers, since the only solution of (3) is  $1/2$ , which is not an integer. This would lead us to develop fractions or rational numbers, that is, numbers which can be expressed as quotients of integers. Even among the extensive system of all rational numbers, the simple quadratic equation

$$(4) \quad x^2 - 2 = 0$$

would have no root, since the only roots are  $\pm \sqrt{2}$ , which are not rational as we proved in section 13. Hence we invent the irrational numbers, which together with the rational numbers compose the system of real numbers. The entire system of real numbers still contains no root of the quadratic equation

$$(5) \quad x^2 + 4 = 0$$

as was proved in section 16. Hence we invent the pure imaginary numbers, such as  $\pm 2i$ . The quadratic equation

$$(6) \quad x^2 + 2x + 4 = 0$$

has no root among either the real or the pure imaginary numbers, since its roots  $-1 \pm i\sqrt{3}$  are neither real nor pure imaginary. Therefore we are led to invent the system of complex numbers.

So far, we have been impelled to enlarge our system of numbers again and again by considering only linear and quadratic equations. The student may well dread the possibility of having to invent still further kinds of numbers if we consider equations of degree higher than the second. That this will not be necessary was first proved by K. F. Gauss, a great German mathematician, in 1799, although it was suspected before. What Gauss proved is known as the fundamental theorem of algebra, and may be stated as follows.

**FUNDAMENTAL THEOREM OF ALGEBRA.** *Every polynomial equation of degree 1 or more has a root among the complex numbers (no matter how high its degree and no matter what complex numbers are its coefficients).*

For example, it is guaranteed that the equation  $x^{1945} + 1918x^{1598} + 1865x^{1812} + 1776x^{1492} + \frac{7+3i}{2}x^7 - \frac{\sqrt{3}}{2}x + 1 = 0$



has a root among the complex numbers. The proof of the fundamental theorem is too difficult to be given here.\* Unfortunately, it does not tell us how to find a root, but it does assure us of its existence. Therefore we do not find it necessary to invent more numbers in order to provide every polynomial equation with a root. The student should familiarize himself without delay with the facts about the various kinds of numbers discussed in sections 11–16 and with the classification of the complex numbers in section 16.

**101. The remainder theorem and the factor-theorem.** If we divide the polynomial  $F(x) = x^2 - 5x + 10$  by  $x - 2$  we get the quotient  $x - 3$  and the remainder 4:

$$\begin{array}{r} x - 3 \\ x - 2 \overline{) x^2 - 5x + 10} \\ \underline{x^2 - 2x} \phantom{+ 10} \\ - 3x + 10 \\ \underline{- 3x + 6} \\ 4. \end{array}$$

Also  $F(2) = 2^2 - 5 \cdot 2 + 10 = 4$ , the same as the remainder. That this is no mere coincidence is guaranteed by the following theorem:

**REMAINDER THEOREM.** *If a polynomial  $F(x)$  of degree 1 or more is divided by a divisor of the form  $x - r$ , then the remainder  $R = F(r)$ .*

*Proof.* By definition,† to divide  $F(x)$  by any polynomial  $D(x)$  means to find a quotient polynomial  $Q(x)$  and a remainder polynomial  $R(x)$  such that

$$(a) \quad F(x) = D(x) \cdot Q(x) + R(x)$$

and

(b) the degree of  $R(x)$  is less than the degree of  $D(x)$ .

Since our divisor  $x - r$  is of degree 1, the remainder must be of degree zero, that is, a constant. Hence we may write

$$(1) \quad F(x) = (x - r) \cdot Q(x) + R$$

where  $R$  is a constant. Since (1) is an identity, and therefore true for all values of  $x$ , it is true in particular when  $x = r$ . Sub-

\* See, for example, L. E. Dickson, *Elementary Theory of Equations*, Wiley, 1914, or L. Weisner, *Introduction to the Theory of Equations*, Macmillan, 1938, or G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, Macmillan, 1941.

† The student should review section 43 carefully at this point.



stituting  $x = r$  in (1) we obtain

$$F(r) = (r - r) \cdot Q(r) + R,$$

or

$$F(r) = 0 \cdot Q(r) + R,$$

or

$$F(r) = R,$$

which is what we had to prove.

If  $R = 0$ , then  $x - r$  is a factor\* of  $F(x)$ , (since  $F(x) = (x - r) \cdot Q(x)$ ), and conversely. Hence we have the following as an immediate consequence of the remainder theorem.

**THE FACTOR THEOREM.** *If  $x - r$  is a factor of  $F(x)$ , then  $r$  is a root of the equation  $F(x) = 0$ , and conversely.*

*Proof.* If  $x - r$  is a factor of  $F(x)$ , then the remainder  $R$  obtained by dividing  $F(x)$  by  $x - r$  will be zero. But then  $F(r) = 0$  which means that  $r$  is a root of the equation  $F(x) = 0$ . Conversely, if  $r$  is a root of  $F(x) = 0$ , then  $F(r) = 0$ . Hence the remainder  $R$  will be zero when  $F(x)$  is divided by  $x - r$ , or  $x - r$  is a factor of  $F(x)$ .

*Example 1.*  $x - 2$  is a factor of  $F(x) = x^2 - 5x + 6$ , and 2 is a root of  $x^2 - 5x + 6 = 0$ .

*Example 2.*  $x + 2$  is a factor of  $F(x) = x^2 + 5x + 6$  and  $-2$  is a root of  $x^2 + 5x + 6 = 0$ . Note that  $x + 2$  may be written as  $x - (-2)$  to put it in the form  $x - r$ ; here  $r = -2$ .

## EXERCISES

*Divide by long division (until a constant remainder is obtained) and then find the remainder again by means of the remainder theorem:*

1.  $(x^2 - 5x + 7) \div (x - 3)$ .
2.  $(3x^2 - x - 9) \div (x - 4)$ .
3.  $(x^3 - 2x^2 + x - 3) \div (x - 2)$ .
4.  $(x^3 - 5) \div (x - 1)$ .
5.  $(x^2 - 5x + 6) \div (x + 2)$ .
6.  $(x^2 - 5x + 6) \div (x - 2)$ .
7.  $(x^3 - 2x^2 + x - 3) \div (x + 2)$ .
8.  $(x^3 - 5) \div (x + 1)$ .

*By means of the factor theorem, decide whether or not each of the following statements is true:*

9.  $x - 2$  is a factor of  $x^3 - x^2 - 5x + 6$ .
10.  $x - 3$  is a factor of  $2x^3 - 6x^2 - 5x + 15$ .
11.  $x + 1$  is a factor of  $x^3 + 3x^2 + 3x + 1$ .
12.  $x + 1$  is a factor of  $x^3 + x^2 - x - 1$ .
13.  $x - 1$  is a factor of  $2x^3 - x^2 - 4x + 5$ .
14.  $x - 1$  is a factor of  $2x^3 - 3x^2 + 4x - 5$ .

\* No restriction on the kind of numbers occurring as coefficients in the factors will be made in this chapter. Unless otherwise specified, coefficients may be any complex numbers.



15.  $x + 2$  is a factor of  $3x^3 + 5x^2 - 7x - 10$ .

16.  $x + 3$  is a factor of  $x^3 + 27$ .

17.  $x - 3$  is a factor of  $x^3 + 27$ .

18.  $x + 2$  is a factor of  $x^4 + 16$ .

19.  $x + 2$  is a factor of  $x^4 - 16$ .

20.  $x - 2$  is a factor of  $x^4 - 16$ .

21. (a)  $x - y$  is a factor of  $x^n - y^n$  if  $n$  is any natural number.

(b)  $x - y$  is not a factor of  $x^n + y^n$  if  $n$  is any natural number.

(c)  $x + y$  is a factor of  $x^n - y^n$  if  $n$  is even.

(d)  $x + y$  is a factor of  $x^n + y^n$  if  $n$  is odd.

(e)  $x + y$  is not a factor of  $x^n - y^n$  if  $n$  is odd.

(f)  $x + y$  is not a factor of  $x^n + y^n$  if  $n$  is even.

22. Verify that  $x^2 + 1$  is a factor of  $F(x) = x^3 - 2x^2 + x - 2$ . May we conclude that therefore  $F(-1) = 0$ ?

23. Verify that  $2x - 1$  is a factor of  $F(x) = 2x^3 - x^2 + 2x - 1$ . May we conclude that therefore  $F(1) = 0$ ?

24. Find the value of  $k$  for which  $x - 3$  is a factor of  $kx^3 - 6x^2 + 2kx - 12$ .

25. Find the value of  $k$  for which  $x + 2$  is a factor of  $3x^3 + 2kx^2 - 4x - 8$ .

26. Show that  $x - 1$  is a factor of  $1776x^{1946} - 1492x^{1865} - 284$ .

**102. Synthetic division.** Because of the connection between roots of equations and factors of the form  $x - r$  of polynomials, we shall often have to divide polynomials by expressions of the form  $x - r$ . Whenever an operation has to be performed often, it is natural to try to arrange the work so as to involve the least possible waste motion. This will be done for division by  $x - r$ , as follows:

*Example 1.* Divide  $3x^3 - x^2 - 3x + 1$  by  $x - 2$ .

$$\begin{array}{r}
 \text{Solution.} \qquad \qquad \qquad 3x^2 + 5x + 7 \qquad \qquad \qquad = \text{quotient} \\
 \text{divisor} = x - 2 \overline{) 3x^3 - x^2 - 3x + 1} = \text{dividend} \\
 \qquad \qquad \qquad \underline{3x^3 - 6x^2} \qquad \qquad \qquad \\
 \qquad \qquad \qquad \qquad + 5x^2 - 3x + 1 \\
 \qquad \qquad \qquad \qquad \underline{5x^2 - 10x} \qquad \qquad \qquad \\
 \qquad \qquad \qquad \qquad \qquad + 7x + 1 \\
 \qquad \qquad \qquad \qquad \qquad \underline{7x - 14} \\
 \qquad \qquad \qquad \qquad \qquad \qquad \qquad 15 = \text{remainder.}
 \end{array}$$

Examination of the steps in the division process show that (since the coefficient of  $x$  in the divisor is 1) the numbers in heavy type are the coefficients of the quotient and the remainder in succession. The powers of  $x$  and the entire quotient may be omitted,



[illegible]

*Solution.*

$$\begin{array}{rrrr} -3 & \overline{) 2} & 1 & 0 & -4 \\ & & -6 & 15 & -45 \\ \hline & 2 & -5 & 15 & -49 \end{array}$$

*Example 3.* If  $f(x) = 2x^3 + x^2 - 4$ , find  $f(2)$ .



*Solution.*  $f(2)$  will be the remainder when  $f(x)$  is divided by  $x - 2$ . Hence  $f(2)$  will be the last number in the third line of synthetic division with 2 as multiplier:

$$\begin{array}{r|rrrr} 2 & 2 & 1 & 0 & -4 \\ & & 4 & 10 & 20 \\ \hline & 2 & 5 & 10 & 16 \end{array}$$

Thus,  $f(2) = 16$ .

*Example 4.* If  $f(x) = 2x^3 + x^2 - 4$ , find  $f(-3)$ .

*Solution.* The work done in example 2 shows that  $f(-3) = -49$ .

*Example 5.* Show that 2 is a root of the equation  $f(x) = 2x^3 + x^2 - 8x - 4 = 0$ .

*Solution.* By synthetic division

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -8 & -4 \\ & & 4 & 10 & 4 \\ \hline & 2 & 5 & 2 & 0 \end{array}$$

we find that  $f(2) = 0$ , and hence 2 is a root of  $f(x) = 0$ .

*Example 6.* Show that  $x - 2$  is a factor of  $f(x) = 2x^3 + x^2 - 8x - 4$ , and write another factor.

*Solution.* The work done in example 5 shows that  $x - 2$  is a factor since  $f(2) = 0$ . Another factor is the quotient polynomial  $2x^2 + 5x + 2$ ; that is,  $f(x) = (x - 2)(2x^2 + 5x + 2)$ .

## EXERCISES

Find quotient and remainder by synthetic division, and check:

1.  $(4x^3 - 2x + 5) \div (x - 2)$ .
2.  $(2x^3 - 3x^2 + 4) \div (x - 3)$ .
3.  $(x^3 + 4x^2 - 7) \div (x + 3)$ .
4.  $(x^4 - 3x^2 + 6) \div (x + 2)$ .
5.  $(2x^3 + 3x^2 + 4x + 5) \div (x - \frac{1}{2})$ .
6.  $(x^4 - 4x^2 + 3) \div (x - \frac{1}{3})$ .
7.  $(6x^4 - 10x^3 - 4x^2 + 3x + 3) \div (x + \frac{1}{3})$ .
8.  $(2x^4 + 6x^3 - 8x + 10) \div (x + \frac{1}{2})$ .

By means of synthetic division:

9. If  $f(x) = x^3 - 2x^2 + 3x + 4$ , find  $f(3)$  and  $f(-2)$ .
10. If  $f(x) = 2x^3 + 3x^2 - 7$ , find  $f(2)$  and  $f(-1)$ .
11. If  $F(x) = 6x^4 - 10x^3 - 4x^2 + 3x + 1$ , find  $F(\frac{1}{2})$  and  $F(-\frac{1}{3})$ .
12. If  $F(x) = 6x^3 - 4x^2 + 9x - 6$ , find  $F(-\frac{1}{2})$  and  $F(\frac{2}{3})$ .
13. If  $g(x) = -2x^3 + x^2 - 7$ , find  $g(0.2)$  and  $g(-1.3)$ .
14. If  $g(x) = -x^3 - 2x + 4$ , find  $g(1.2)$  and  $g(-2.3)$ .



*By means of synthetic division decide whether each statement is true or false; if true, write another factor:*

15.  $x - 2$  is a factor of  $x^3 - x^2 - 5x + 6$ .
16.  $x - 3$  is a factor of  $x^3 - 2x^2 + 2x - 15$ .
17.  $x + 2$  is a factor of  $x^3 - x^2 - 2x + 8$ .
18.  $x + 3$  is a factor of  $2x^3 + 7x^2 - 9$ .
19.  $x + 2$  is a factor of  $x^5 + 32$ .
20.  $x - 3$  is a factor of  $x^4 - 81$ .
21.  $x + 3$  is a factor of  $x^4 + 81$ .
22.  $x + r$  is a factor of  $x^5 + r^5$ .
23.  $x + r$  is a factor of  $x^6 + r^6$ .
24.  $x - r$  is a factor of  $x^6 - r^6$ .

*By synthetic division decide whether the statement is true or false:*

25. 2 is a root of  $2x^3 - 3x^2 - 4 = 0$ .
26.  $-3$  is a root of  $2x^4 + 6x^3 + x^2 + 3x - 6 = 0$ .
27.  $-1$  is a root of  $3x^3 - 4x^2 + 3x - 2 = 0$ .
28. 4 is a root of  $3x^3 + 10x^2 - 7x + 4 = 0$ .
29.  $1/2$  is a root of  $6x^4 - 3x^3 + 2x^2 + 3x - 2 = 0$ .
30.  $-2/3$  is a root of  $3x^3 - 4x^2 + 8x + 8 = 0$ .
31.  $\sqrt{2}$  is a root of  $x^4 + 2x^3 - 5x^2 - 4x + 6 = 0$ .
32.  $2i$  is a root of  $2x^3 - 3x^2 + 4x - 6 = 0$ .
33.  $1 + \sqrt{2}$  is a root of  $2x^4 - 4x^3 + x^2 - 6x - 3 = 0$ .
34.  $2 + 3i$  is a root of  $x^4 - 4x^3 + 12x^2 + 4x - 13 = 0$ .

*Using synthetic division find a value of  $k$  such that:*

35.  $-2$  is a root of  $3x^3 + 5x^2 + kx - 10 = 0$ .
36. 3 is a root of  $2x^3 + 3kx^2 - 5x + 15 = 0$ .

**103. The depressed equation.** Suppose we have somehow found out that the equation  $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$  of degree  $n \geq 1$  has the root  $r_1$ . Then  $x - r_1$  is a factor of  $f(x)$ , and we may write the equation as

$$(1) \quad f(x) = (x - r_1)Q_1(x) = 0$$

where the quotient  $Q_1(x)$  is of degree  $n - 1$  and begins with the same coefficient as  $f(x)$ , as can be seen from the process of synthetic division; that is,

$$Q_1(x) = a_0x^{n-1} + \cdots$$

Since the product of two factors can be zero when and only when one (or both) of the factors is itself zero, any root of  $f(x) = 0$



must satisfy either

$$(2) \quad x - r_1 = 0$$

or

$$(3) \quad Q_1(x) = 0.$$

The only root of  $x - r_1 = 0$  is  $r_1$ , which we knew about at the start. Hence any further root of (1) must be a root of (3), which is called the **depressed equation** because its degree is lower than that of the original equation by one. The depressed equation is often easier to solve than the original equation for this reason.

If  $r_1$  is known, the coefficients of the depressed equation may be found by synthetic division. If a depressed equation is quadratic, the equation may be solved completely.

*Example 1.* Given that 2 is a root of  $f(x) = x^3 - 4x^2 + x + 6 = 0$ , solve the equation.

*Solution.* Since 2 is a root of  $f(x) = 0$ ,  $x - 2$  is a factor of  $f(x)$ . By synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 1 & 6 \\ & & 2 & -4 & -6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

we find that  $f(x) = (x - 2)(x^2 - 2x - 3)$ . Hence the depressed equation is  $x^2 - 2x - 3 = 0$ , which has the roots 3 and  $-1$ . Hence the roots of the given equation are 2, 3,  $-1$ . The student may check by substituting in the original equation.

If more than one root is known, the depressed equation may be depressed again, as in the following example.

*Example 2.* Given that 2 and 3 are roots of  $f(x) = x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$ , solve the equation.

*Solution.* Since 2 is a root of  $f(x) = 0$ ,  $x - 2$  is a factor of  $f(x)$ . By synthetic division we find that  $f(x) = (x - 2)(x^3 - 3x^2 - x + 3)$ . But 3 is given to be a root of  $f(x) = 0$  and is therefore a root of the depressed equation  $Q_1(x) = x^3 - 3x^2 - x + 3 = 0$ . By synthetic division we find that  $Q_1(x) = (x - 3)(x^2 - 1) = 0$ . The remaining roots of  $f(x) = 0$  must be roots of  $x^2 - 1 = 0$ , or  $\pm 1$ . Hence the roots of the given equation are 2, 3, and  $\pm 1$ .

## EXERCISES

*Solve the equation:*

1.  $x^3 - 4x^2 + x + 6 = 0$ , given that 3 is a root.
2.  $x^3 - 7x + 6 = 0$ , given that 2 is a root.



3.  $x^3 + 4x^2 + 8x + 8 = 0$ , given that  $-2$  is a root.

4.  $2x^3 + 5x^2 - x - 1 = 0$ , given that  $1/2$  is a root.

5.  $2x^4 - 3x^3 - 20x^2 + 27x + 18 = 0$ , given that  $-3$  and  $-1/2$  are roots.

6.  $2x^4 + 11x^3 + 19x^2 + 11x + 2 = 0$ , given that  $-2$  and  $-1/2$  are roots.

7.  $3x^4 + 14x^3 - 4x^2 - 11x - 2 = 0$ , given that  $1$  and  $-2/3$  are roots.

8.  $6x^4 - 13x^3 + 31x^2 - 3x - 5 = 0$ , given that  $1/2$  and  $-1/3$  are roots.

9.  $6x^4 - 11x^3 + 15x^2 - 22x + 6 = 0$ , given that  $3/2$  and  $1/3$  are roots.

10.  $8x^4 + 10x^3 + 21x^2 + 30x - 9 = 0$ , given that  $1/4$  and  $-3/2$  are roots.

**104. Factorization of polynomials.** The number of roots of an equation. We shall prove several basic theorems.

**THEOREM 1.** *Every polynomial*

$$(1) \quad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \quad (a_0 \neq 0)$$

*of degree 1 or more can be expressed as the product of  $n$  linear factors of the form*

$$(2) \quad f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

*Proof.* By the fundamental theorem of algebra, the equation  $f(x) = 0$  has a root  $r_1$ . Then  $x - r_1$  is a factor of  $f(x)$ . Therefore

$$(3) \quad f(x) = (x - r_1) \cdot Q_1(x)$$

where

$$Q_1(x) = a_0x^{n-1} + \cdots$$

may be found by division. If  $n - 1 > 0$ , the equation  $Q_1(x) = 0$  must also have a root  $r_2$  by the fundamental theorem of algebra. Therefore  $x - r_2$  is a factor of  $Q_1(x)$  and we may write

$$(4) \quad Q_1(x) = (x - r_2)Q_2(x)$$

where  $Q_2(x) = a_0x^{n-2} + \cdots$ . Substituting (4) in (3) we have

$$(5) \quad f(x) = (x - r_1)(x - r_2) \cdot Q_2(x).$$

This process may be performed  $n$  times until the quotient  $Q_n(x) = a_0$  of degree 0 is reached. Hence

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n) \cdot a_0$$

which was to be proved.



The numbers  $r_1, r_2, \dots, r_n$  may or may not be all different from each other. By the factor theorem, each of them is a root of the equation  $f(x) = 0$ .

**THEOREM 2.** No number  $r$  different from all the numbers  $r_1, r_2, \dots, r_n$  found in theorem 1 can be a root of the equation  $f(x) = 0$ .

*Proof.* By hypothesis  $r - r_1, r - r_2, \dots, r - r_n$  are all different from zero. Hence, substituting  $r$  for  $x$  in (2), we find

$$f(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n) \neq 0,$$

which proves the theorem.

The following is an immediate corollary.

**COROLLARY 1.** Any equation of degree  $n \geq 1$  has at most  $n$  distinct roots.

**COROLLARY 2.** If two polynomials

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \\ \text{and} \quad g(x) &= b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n, \end{aligned}$$

of degrees not greater than  $n$ , are equal for more than  $n$  distinct values of  $x$ , then

$$(6) \quad a_0 = b_0, a_1 = b_1, \dots, a_n = b_n,$$

and the two polynomials are identically equal.

*Proof.* Consider the equation

$$(7) \quad f(x) - g(x) = (a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_{n-1} - b_{n-1})x + (a_n - b_n) = 0.$$

If any of the coefficients  $a_0 - b_0, a_1 - b_1, \dots, a_{n-1} - b_{n-1}$  were different from zero, (7) would be an equation of degree  $d$ , where  $1 \leq d \leq n$ , with more than  $n$  distinct roots, contrary to corollary 1. Hence  $a_0 = b_0, a_1 = b_1, \dots, a_{n-1} = b_{n-1}$ . But then the equation (7) reads  $a_n - b_n = 0$ , or  $a_n = b_n$ . This completes the proof.

**THEOREM 3.** The factorization (2) is unique. That is, the polynomial (1) can be expressed as a product of factors of the form (2) in one and only one way, apart from the order in which the factors are written down.

This theorem will be proved in the next section.

A number  $r$  is a root of  $f(x) = 0$  of multiplicity  $k$  if the factor  $(x - r)$  occurs exactly  $k$  times in the factorization (2). Roots of



multiplicity 1, 2, and 3 are called **simple**, **double**, and **triple roots**, respectively.

*Example 1.* In the equation  $f(x) = (x - 3)(x - 4)(x - 4)(x - 4)(x + 7)(x + 7) = 0$  of degree six, the number 3 is a simple root, 4 is a triple root, and  $-7$  is a double root.

From theorems 1, 2, and 3 we have the following immediate consequence.

**THEOREM 4.** *Every equation of degree  $n \geq 1$  has exactly  $n$  roots provided a root of multiplicity  $k$  is counted as  $k$  roots.*

Thus the equation of example 1 has the six roots: 3, 4, 4, 4,  $-7$ ,  $-7$ .

To form an equation with given roots, we have only to multiply together the appropriate factors, as in the following example.

*Example 2.* Form an equation with the roots  $\frac{1}{2}$ ,  $-\frac{3}{5}$ , and 1, having integral coefficients.

*Solution.* An equation having the desired roots is

$$(x - \frac{1}{2})(x + \frac{3}{5})(x - 1) = 0.$$

Obviously the equation obtained by multiplying these factors together will have fractional coefficients, which may then be cleared away by multiplying both sides by the L.C.D. However, it is less trouble to introduce the L.C.D., namely 10, as the value of  $a_0$  and to write  $10(x - \frac{1}{2})(x + \frac{3}{5})(x - 1) = 0$  or

$$2(x - \frac{1}{2}) \cdot 5(x + \frac{3}{5}) \cdot (x - 1) = 0,$$

or

$$(2x - 1)(5x + 3)(x - 1) = 0.$$

Multiplying the latter factors, we get the desired equation  $10x^3 - 9x^2 - 4x + 3 = 0$ .

*Note.* The idea that an equation of degree  $n$  has  $n$  roots is a fairly modern one. The great Greek algebraist Diophantus (3rd century A.D.) was able to find solutions of quadratic equations but was content with one root. He rejected negative and irrational roots as absurd; and, even when both roots are positive, he took only one of them. Even in the 16th century, great mathematicians such as Cardan and Vieta rejected negative and imaginary roots, although irrational roots were admitted, and Stifel asserted that except for the case of a quadratic with two positive roots, no equation has more than one root. In the 17th century, mathematicians such as Descartes began to have a better grasp of the subject.



## EXERCISES

*Solve each equation without multiplying the factors:*

1.  $(x - 3)(2x - 1)(x + 3) = 0$ .
2.  $(x - 2)(x + 5)(3x + 2) = 0$ .
3.  $(x - 1)(x - 1)(x^2 + 2x + 4) = 0$ .
4.  $(x^2 + 4x + 4)(x^2 + x + 1) = 0$ .
5.  $3(x + 2)(2x + 1)(2x - 3) = 0$ .
6.  $2(x + \frac{1}{2})(x - 3)(4x + 3) = 0$ .
7.  $x(x + 1)(x - 2)(2x - 5) = 0$ .
8.  $3x(x + 2)(2x + 5)(5x - 2) = 0$ .

*Form an equation with integral coefficients having the given numbers, and no others, as roots:*

- |                                                                                   |                                  |                                  |
|-----------------------------------------------------------------------------------|----------------------------------|----------------------------------|
| 9. 1, 2, -3.                                                                      | 10. 2, -1, -4.                   | 11. $1/2, 3, -2$ .               |
| 12. $2/3, -1, -3$ .                                                               | 13. 2, 2, 0.                     | 14. 3, 3, -1, -1.                |
| 15. $1/2, 1/2, -3, 0$ .                                                           | 16. $3/2, 1/3, 0$ .              | 17. $\pm \sqrt{3}, 0, 0$ .       |
| 18. $\pm \sqrt{2}, 3/4, 0$ .                                                      | 19. $(1 \pm \sqrt{3}), \pm 2i$ . | 20. $(3 \pm \sqrt{2}), \pm 3i$ . |
| 21. $1 \pm 2i, 2 \pm \sqrt{3}$ .                                                  | 22. $2 \pm i, 1 \pm i\sqrt{2}$ . | 23. $1/2, 2/3, \pm 2i$ .         |
| 24. $3/2, -1/3, 1 \pm i\sqrt{2}$ .                                                |                                  |                                  |
| 25. Form a cubic equation with 2 as a triple root.                                |                                  |                                  |
| 26. Form a cubic equation with 3 as a double root and -2 as a simple root.        |                                  |                                  |
| 27. Form a quartic equation with 1 as a double root and 0 and -2 as simple roots. |                                  |                                  |
| 28. Form a quartic equation with -2 as a triple root and 3 as a simple root.      |                                  |                                  |

*Solve each of the following equations:*

29.  $x^4 - 6x^3 + 10x^2 - 6x + 9 = 0$  given that 3 is a double root.
30.  $x^4 + 3x^3 + 4x^2 + 3x + 1 = 0$  given that -1 is a double root.
31.  $4x^4 + x^2 - 3x + 1 = 0$  given that  $1/2$  is a double root.
32.  $9x^4 + 24x^3 + 49x^2 + 26x + 4 = 0$  given that  $-1/3$  is a double root.
33. Form a cubic equation with  $\pm 2$  and  $i$  as simple roots.
34. Form a cubic equation with 2, -3, and  $1 + i$  as simple roots.
35. Form a quartic equation one of whose roots is 2 and whose other three roots are the roots of  $x^3 + x^2 + x + 2 = 0$ .

**\*105. Proof of the uniqueness of factorization.** We shall prove theorem 3 of the preceding section.

\* This section may be omitted upon first reading without disturbing the continuity of the chapter. It is included for the general reasons explained in the preface, and in particular because the definition of multiplicity of a root is no more than a pious wish without it.



*Proof.* Suppose that there are  $d$  distinct numbers among the  $n$  numbers  $r_1, r_2, \dots, r_n$ . Let these  $d$  distinct numbers be  $\rho_1, \rho_2, \dots, \rho_d$  where  $\rho_1$  occurs  $\alpha_1$  times,  $\rho_2$  occurs  $\alpha_2$  times, and so on. Then (2) of section 104 may be written as

$$(1) \quad f(x) = a_0(x - \rho_1)^{\alpha_1}(x - \rho_2)^{\alpha_2} \cdots (x - \rho_d)^{\alpha_d},$$

where  $\alpha_1 + \alpha_2 + \cdots + \alpha_d = n$ . By theorem 2, section 104, no number  $r$  different from  $\rho_1, \rho_2, \dots, \rho_d$  can be a root of  $f(x) = 0$ , and hence no expression of the form  $x - r$  other than those appearing in (1) can be a factor of  $f(x)$ . But it is still conceivable that the factors in (1) might also occur with different multiplicities; that is, that, besides (1), we might have

$$(2) \quad f(x) = a_0(x - \rho_1)^{\beta_1}(x - \rho_2)^{\beta_2} \cdots (x - \rho_d)^{\beta_d},$$

where  $\beta_1 + \beta_2 + \cdots + \beta_d = n$  but not all the equalities

$$(3) \quad \alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_d = \beta_d$$

hold. Suppose that  $\alpha_1 \neq \beta_1$  and in particular that  $\alpha_1 > \beta_1$ . Then, from (1) we have  $f(x) = (x - \rho_1)^{\alpha_1}F(x)$ , and from (2) we have  $f(x) = (x - \rho_1)^{\beta_1}G(x)$  where  $G(\rho_1) \neq 0$  since  $x - \rho_1$  is not a factor of  $G(x)$ . Then

$$(x - \rho_1)^{\alpha_1}F(x) = (x - \rho_1)^{\beta_1}G(x)$$

is an identity. Dividing both sides by  $(x - \rho_1)^{\beta_1}$  we have

$$(4) \quad (x - \rho_1)^{\alpha_1 - \beta_1}F(x) = G(x),$$

for all values of  $x$  except  $x = \rho_1$  since division by zero is excluded. But (4) is therefore an identity by corollary 2, section 104, because it is true for all values of  $x$  except  $x = \rho_1$ , hence for infinitely many values of  $x$ , and hence certainly for more than  $n$  values of  $x$ . But  $G(x)$  is not divisible by  $(x - \rho_1)$ , and therefore the left member of (4) cannot have  $(x - \rho_1)$  as a factor. Therefore  $\alpha_1 - \beta_1 = 0$ , or  $\alpha_1 = \beta_1$ . The argument can be repeated to show that all the equalities (3) must hold. This completes the proof.

**106. Imaginary roots of equations with real coefficients.\*** The familiar formula for the roots of a quadratic equation shows clearly that if the quadratic has real coefficients and has an imaginary root, then the conjugate of this root is the other root. This result is generalized to apply to all equations with real coefficients no matter what the degree, by the following theorem.

\* The student should review carefully the classification of the number system given in section 16.



**THEOREM 1.** *If all the coefficients of the equation  $F(x) = 0$  are real and  $a + bi$  ( $b \neq 0$ ) is an imaginary root, then its conjugate  $a - bi$  is also a root.*

*Proof.* To show that  $a - bi$  is a root of  $F(x) = 0$  it will suffice to show that  $x - (a - bi)$  is a factor of  $F(x)$ . This will be done by showing that

$$(1) \quad D(x) = [x - (a - bi)][x - (a + bi)]$$

is a factor of  $F(x)$ . Multiplying out the right member of (1), we find that

$$(2) \quad D(x) = x^2 - 2ax + a^2 + b^2,$$

which has real coefficients. Dividing\*  $F(x)$  by  $D(x)$  we find a quotient  $Q(x)$ , and remainder  $R(x)$  of degree not greater than 1. Writing  $R(x) = cx + d$  we have

$$(3) \quad F(x) = D(x) \cdot Q(x) + cx + d.$$

Note that in the division process the coefficients of the dividend and divisor are combined only by multiplication, division, addition, and subtraction. Since the coefficients of  $F(x)$  and  $D(x)$  are all real, this process can yield nothing but real coefficients in the quotient and remainder. In particular,  $c$  and  $d$  are real. By hypothesis,  $a + bi$  is a root of  $F(x) = 0$ . Hence substituting  $a + bi$  in (3), we obtain

$$(4) \quad F(a + bi) = D(a + bi) \cdot Q(a + bi) + ca + cbi + d = 0.$$

From (1) we see that  $D(a + bi) = 0$ . Hence (4) implies that

$$ca + cbi + d = 0.$$

This implies that

$$(5) \quad ca + d = 0$$

and

$$(6) \quad cb = 0.$$

Since  $b \neq 0$  by hypothesis, (6) implies that  $c = 0$ . Substituting this in (5), we find  $d = 0$ . Hence  $R(x) = 0$  and  $D(x)$  is a factor of  $F(x)$ . This completes the proof.

**COROLLARY 1.** *Every polynomial  $f(x)$  of degree  $n \geq 1$  with real coefficients can be expressed as a product of linear factors and quadratic factors having negative discriminants, all with real coefficients.*

*Proof.* We have  $f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n)$ . Any root of  $f(x) = 0$  that is real provides a linear factor with real

\* See the definition of division in section 43.



coefficients. Any root that is imaginary can be paired with its conjugate root. The product of the factors arising from such a pair is of the form

$$(x - [a + bi])(x - [a - bi]) = x^2 - 2ax + a^2 + b^2,$$

which provides a quadratic factor with real coefficients and negative discriminant  $-4b^2$ . The constant  $a_0$  can be absorbed into any of these factors. This completes the proof.

The quadratic formula shows similarly that if a quadratic with rational coefficients has a root  $a + \sqrt{b}$  where  $a$  and  $b$  are rational but  $\sqrt{b}$  is irrational, then  $a - \sqrt{b}$  is also a root. This result can also be extended to equations of any degree by the following theorem:

**THEOREM 2.** *If an equation with rational coefficients has a binomial surd root of the form  $a + \sqrt{b}$  (or  $a - \sqrt{b}$ ) where  $a$  and  $b$  are rational but  $\sqrt{b}$  is irrational, then the conjugate surd  $a - \sqrt{b}$  (or  $a + \sqrt{b}$ ) is also a root.*

The proof is similar to that of theorem 1 and is left to the reader.

## EXERCISES

*If all the coefficients of  $f(x) = 0$  are real numbers and the given number is a root, write another number which must be a root of  $f(x) = 0$ :*

1.  $3 + 2i$ .      2.  $-2 + 3i$ .      3.  $-3 - 3i$ .      4.  $3 - 4i$ .      5.  $3i$ .
6.  $-2i$ .

7. Form a cubic equation with real coefficients having  $1 + 2i$  and 5 as roots.

8. Form a cubic equation with integral coefficients having  $2 - 3i$  and  $1/2$  as roots.

9. Form a quartic equation with integral coefficients having  $1 + i$  as a simple root and  $-1/2$  as a double root.

10. Form a quartic equation with real coefficients having 3 as a double root and  $1 - 2i$  as a simple root.

11. Solve  $x^4 - 2x^3 + 3x^2 - 2x + 2 = 0$ , given that  $1 + i$  is a root.

12. Solve  $x^4 - 2x^3 + 2x^2 + 6x - 15 = 0$ , given that  $1 - 2i$  is a root.

13. Solve  $x^4 + 3x^3 + 2x^2 + 3x + 1 = 0$ , given that  $i$  is a root.

14. Solve  $x^4 - 4x^3 + 5x^2 - 16x + 4 = 0$ , given that  $-2i$  is a root.

15. Solve  $x^6 + 3x^5 + 3x^4 + 6x^3 + 3x^2 + 3x + 1 = 0$ , given that  $i$  is a double root.

16. Prove that a cubic equation with real coefficients has either three real roots, or one real and two conjugate imaginary roots.



17. Prove that a quartic equation with real coefficients has either 4 real roots, or 4 imaginary roots, or 2 real and 2 imaginary roots.

18. State and prove a theorem similar to those of exercises 16 and 17 for equations of degree: (a) five; (b) six.

19. Prove that an equation of odd degree with real coefficients has at least one real root.

20. Verify that the equation  $x - (1 + i) = 0$  has the root  $1 + i$ , but the conjugate  $1 - i$  is not a root. Does this contradict theorem 1? Explain.

*Decide whether each of the following statements is true or false and explain:*

21. If  $1 + i$  is a root of  $x^3 - (6 + i)x^2 + (11 + 5i)x - (6 + 6i) = 0$ , then  $1 - i$  is also a root.

22. If  $i$  is a root of  $x^3 - \frac{\sqrt{3} + 1}{2}x^2 + x - \frac{\sqrt{3} + 1}{2} = 0$ , then  $-i$  is also a root.

23. Form a cubic equation with rational coefficients having  $2 + \sqrt{3}$  and 1 as roots.

24. Form a cubic equation with integral coefficients having  $1 - \sqrt{2}$  and  $1/2$  as roots.

25. Form a quartic equation with rational coefficients having  $1 + \sqrt{3}$ , 2, and 0 as roots.

26. Form a quartic equation with integral coefficients having  $\frac{1}{2} - \frac{\sqrt{3}}{2}$  as a root and 0 as a double root.

27. Solve  $x^4 - 4x^3 + 2x^2 - 4x + 1 = 0$ , given that  $2 + \sqrt{3}$  is a root.

28. Solve  $x^4 - 3x^2 - 6x - 2 = 0$ , given that  $1 - \sqrt{2}$  is a root.

**107. Rational roots of equations with integral coefficients.** In section 99, we learned how to find imaginary roots of a certain simple type of equation of the  $n$ th degree, namely the pure equation. Aside from this case, we shall find imaginary roots only when they arise from a quadratic depressed equation, because, except for certain advanced applications, the real\* roots are what are generally needed for practical purposes. We are now ready to attack the problem of finding the real roots of an equation of any degree. Recall that real numbers are either rational or irrational, and that a rational number is one which can be expressed as a quotient or ratio of two integers. Hence all rational numbers are included in the following scheme:

\* The student should review carefully the classification of number system in section 16.



$$\begin{array}{c}
0, \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{3}{1}, \pm \frac{4}{1}, \dots \\
\pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \pm \frac{4}{2}, \dots \\
\pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{3}{3}, \pm \frac{4}{3}, \dots \\
\pm \frac{1}{4}, \pm \frac{2}{4}, \pm \frac{3}{4}, \pm \frac{4}{4}, \dots \\
\vdots \\
\vdots \\
\vdots
\end{array}$$

In this section, we give a method for finding the rational roots of equations with integral coefficients. Whether any particular rational number  $r$  is a root of a given equation  $f(x) = 0$  or not can be decided by substituting it for  $x$  in the equation; this work is most conveniently done by synthetic division, the last number in the third line of synthetic division being  $f(r)$ . The number  $r$  is a root if and only if  $f(r) = 0$ . Hence we might try to find all rational roots of the given equation simply by trying all rational numbers one at a time. (Compare section 13.) But since there are infinitely many rational numbers, this is a gloomy prospect. In fact, if we tried in this way to find the rational roots of an equation like  $x^2 - 2 = 0$  which has no rational roots, we would waste eternity on this one problem without reaching any conclusion. This is considered to be too long an assignment. The following theorem has the virtue of cutting the number of trials from infinitely many to a limited number, thus making the method of trial possible in practice.

**THEOREM.** *If the coefficients of the equation*

$$(1) \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0 \quad (a_0 \neq 0)$$

*are all integers, and if  $p/q$  is a rational root, reduced to lowest terms, then*

- (a) *the numerator  $p$  must be a factor of the constant term  $a_n$ , and*
- (b) *the denominator  $q$  must be a factor of the leading coefficient  $a_0$ .*

*Proof.* Since  $p/q$  is a root, it must satisfy (1). Hence

$$a_0 \frac{p^n}{q^n} + a_1 \frac{p^{n-1}}{q^{n-1}} + a_2 \frac{p^{n-2}}{q^{n-2}} + \dots + a_{n-1} \frac{p}{q} + a_n = 0.$$

Multiplying both sides by the L.C.D.  $q^n$ , we have

$$(2) \quad a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \dots + a_{n-1}pq^{n-1} + a_nq^n = 0.$$



Transposing the last term and taking the common factor  $p$  out of what remains in the left member, we have

$$(3) \quad p[a_0p^{n-1} + a_1p^{n-2}q + a_2p^{n-3}q^2 + \cdots + a_{n-1}q^{n-1}] = -a_nq^n.$$

Since all the letters  $a_0, a_1, \dots, a_n, p$  and  $q$  represent integers, both members of (3) and the quantity in brackets are integers, because the sum, product, and difference of integers are always integers again. The left member of (3) has  $p$  as a factor. Therefore  $p$  is a factor of the right member  $-a_nq^n$ . But  $p$  has no factor in common with  $q$ , except  $\pm 1$ , since  $p/q$  was reduced to lowest terms. Hence  $p$  has no factor in common with  $q^n$  (compare theorem A, section 13). By theorem 2, section 6,  $p$  must be a factor of  $a_n$ . This proves (a).

Transposing the first term of (2) and factoring  $q$  out of what remains in the left member, we get

$$(4) \quad q[a_1p^{n-1} + a_2p^{n-2}q + \cdots + a_{n-1}pq^{n-2} + a_nq^{n-1}] = -a_0p^n.$$

An argument, exactly like that above, now proves (b). The student should write out the details.

*Example 1.* Find the rational roots of  $4x^4 - 4x^3 + 13x^2 - 12x + 3 = 0$ , and if possible solve the equation (completely).

*Solution.* By the above theorem we find that the only:

possible numerators are  $\pm 1, \pm 3$  (the factors of 3);

possible denominators are  $\pm 1, \pm 2, \pm 4$  (the factors of 4);

possible rational roots are  $\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}$ .

Trying the numbers in the preceding line in order, we find that  $\frac{1}{2}$  is a root:

$$\begin{array}{r} \frac{1}{2} | 4 \quad -4 \quad 13 \quad -12 \quad 3 \\ \quad \quad 2 \quad -1 \quad 6 \quad -3 \\ \hline 4 \quad -2 \quad 12 \quad -6 \quad 0. \end{array}$$

Any further root must be a root of the depressed equation  $4x^3 - 2x^2 + 12x - 6 = 0$ , or, dividing both members by 2,  $2x^3 - x^2 + 6x - 3 = 0$ . The only possible rational roots of the latter equation are  $\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$ . Trying these in order, omitting those which failed before, we find  $\frac{1}{2}$  is a root again; thus:

$$\begin{array}{r} \frac{1}{2} | 2 \quad -1 \quad 6 \quad -3 \\ \quad \quad 1 \quad 0 \quad 3 \\ \hline 2 \quad 0 \quad 6 \quad 0. \end{array}$$



Any further root is a root of the depressed equation  $2x^2 + 6 = 0$ , or  $x^2 + 3 = 0$ . The roots of this are  $\pm i\sqrt{3}$ . Hence the given equation has  $\frac{1}{2}$  as a double root, and  $\pm i\sqrt{3}$  as simple roots.

*Always make a complete list of possible numerators, possible denominators, and possible rational roots, as in example 1. One cannot be sure of finding by elimination what is sought unless one is careful not to omit any possibilities.*

*Note 1.* Our theorem cut the number of possible rational roots to be tried from infinitely many to twelve in this example.

*Note 2.* The work of example 1 shows also that the polynomial  $4x^4 - 4x^3 + 13x^2 - 12x + 3$  can be expressed as  $4(x - \frac{1}{2})(x - \frac{1}{2})(x - i\sqrt{3})(x + i\sqrt{3})$ , by section 104, theorem 1. If we wish to express this polynomial as a product of factors with integral coefficients, we may write it as  $2(x - \frac{1}{2}) \cdot 2(x - \frac{1}{2}) \cdot (x^2 + 3)$  or, finally,  $(2x - 1)(2x - 1)(x^2 + 3)$ .

**COROLLARY 1.** *If an equation has integral coefficients, and if the coefficient of the highest power of  $x$  is  $\pm 1$ , then every rational root is an integer and a factor of the constant term.*

*Proof.* Immediate consequence of theorem. Left to reader.

*Example 2.* Solve  $x^3 + 3x^2 + 4x + 12 = 0$ , finding rational roots first.

*Solution.* The only possible rational roots are  $\pm 1, 2, 3, 4, 6, 12$ . Trying these in turn, we find that  $-3$  is a root, thus:

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 4 & 12 \\ & & -3 & 0 & -12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

Further roots must be roots of the depressed equation  $x^2 + 4 = 0$  or  $x = \pm 2i$ . Hence  $-3, \pm 2i$  are the roots of the given equation.

*Example 3.* Factor the polynomial  $x^3 - 2x^2 - x + 2$ .

*Solution.* We find that the rational roots of the equation  $x^3 - 2x^2 - x + 2 = 0$  are 2 and  $\pm 1$ . Hence  $x^3 - 2x^2 - x + 2 = (x - 2)(x - 1)(x + 1)$ .

*Note 3.* The method of theorem 1 can be used for equations with rational coefficients also, as follows.

*Example 4.* Find the rational roots of  $x^3 - 4x^2 - \frac{1}{2}x + 2 = 0$ .

*Solution.* It is now incorrect to assert that a rational root must have a numerator which is a factor of 2 and a denominator which is a factor of 1, because not every coefficient of the given equation is an integer. *But every coefficient will become integer*



if we multiply both members by the L.C.D. Thus our equation is equivalent to the equation  $2x^3 - 8x^2 - x + 4 = 0$ , and we can assert that a rational root must have a numerator which is a factor of 4 and a denominator which is a factor of 2. Hence the only possible rational roots are  $\pm 1, 2, 4, \frac{1}{2}$ . We find that 4 is a root, thus:

$$\begin{array}{r|rrrr} 4 & 2 & -8 & -1 & +4 \\ & & 8 & 0 & -4 \\ \hline & 2 & 0 & -1 & 0. \end{array}$$

The remaining roots are therefore  $\pm \sqrt{2}/2$ .

*Note 4.* Some theorems designed to shorten the labor by diminishing the number of necessary trials still further will be presented in sections 110 and 111. The instructor may delay the more tedious exercises until these are taken up if he wishes.

## EXERCISES

*Find all rational roots, and, if a quadratic depressed equation is obtained, solve the equation (completely):*

1.  $x^3 - 2x^2 - 5x + 6 = 0$ .
2.  $x^3 - 4x^2 + x + 6 = 0$ .
3.  $x^3 - 5x^2 - 8x + 12 = 0$ .
4.  $x^3 + 3x^2 - 4x - 12 = 0$ .
5.  $x^4 - x^3 - 19x^2 - 11x + 30 = 0$ .
6.  $x^4 + 3x^3 - 12x^2 - 13x - 15 = 0$ .
7.  $9x^4 - 3x^3 + 7x^2 - 3x - 2 = 0$ .
8.  $3x^3 - 2x^2 + 15x - 10 = 0$ .
9.  $4x^4 - 7x^2 - 5x - 1 = 0$ .
10.  $3x^3 - 7x^2 - 3x + 2 = 0$ .
11.  $6x^4 + 2x^3 + 7x^2 + x + 2 = 0$ .
12.  $3x^4 + 2x^2 + 5 = 0$ .
13.  $x^4 - 6x^3 + 10x^2 - 6x + 9 = 0$ .
14.  $x^4 - 8x^3 + 24x^2 - 32x + 16 = 0$ .
15.  $x^4 - 2x^3 + x^2 - 2x = 0$ .
16.  $2x^4 - 5x^3 - 10x^2 + 3x = 0$ .
17.  $2x^4 + 5x^3 + 6x^2 + 2x = 0$ .
18.  $x^4 - 4x^3 + x^2 + 6x = 0$ .
19.  $x^4 - 3x^3 + \frac{13}{4}x^2 - 3x + \frac{9}{4} = 0$ .
20.  $x^4 - x^3 - \frac{11}{4}x^2 + 3x - \frac{3}{4} = 0$ .
21.  $x^4 - \frac{1}{2}x^3 + x^2 - 2x - 12 = 0$ .
22.  $x^4 - \frac{7}{2}x^3 - 7x - 4 = 0$ .
23.  $x^3 - \frac{5}{2}x^2 - 5x + \frac{3}{2} = 0$ .
24.  $9x^4 - \frac{27}{2}x^3 + 11x^2 - \frac{27}{2}x + 2 = 0$ .

*Factor each of the following polynomials completely (that is, into linear factors):*

25.  $x^3 + 3x^2 - 10x - 24$ .
26.  $x^3 - 5x^2 + 2x + 8$ .
27.  $2x^3 - x^2 - 8x + 4$ .
28.  $x^3 - 4x^2 + x + 6$ .



29.  $2x^3 - 5x^2 - 10x + 3$ .      30.  $2x^3 + 5x^2 + 5x + 3$ .  
 31.  $4x^4 + 12x^3 + 17x^2 + 10x + 2$ .      32.  $x^4 - 4x^3 + x^2 - 6x + 36$ .

Express each of the following as a product of linear and quadratic factors having real coefficients:

33.  $2x^3 - x^2 + 8x - 4$ .      34.  $3x^3 + 2x^2 + 27x + 18$ .  
 35.  $x^4 - 3x^3 + x^2 + 4$ .      36.  $9x^4 + 12x^3 + 7x^2 - 10x + 2$ .  
 37.  $6x^4 + 13x^3 + 24x^2 - 8$ .  
 38.  $9x^4 + 30x^3 + 82x^2 + 80x + 24$ .

Decide whether each of the following statements is true or false and explain:

39. If  $p/q$  is a rational root, reduced to lowest terms, of  $x^3 + \frac{7}{2}x^2 - \frac{5}{2}x + 2 = 0$ , then  $p$  must be a factor of 2 and  $q$  must be a factor of 1.  
 40. Every rational root of  $x^4 - \frac{37}{8}x^3 - 2x^2 + \frac{37}{3}x - 2 = 0$  must be an integer and a factor of 2.

108. Graphical solution of equations. The real roots, whether rational or irrational, of an equation  $f(x) = 0$  with real coefficients, can be estimated approximately from the graph of the function  $y = f(x)$ , since they are the  $x$ -coordinates of the points where the graph meets the  $x$ -axis. (The student should review section 37.) The following fact is useful (see section 37):

If  $f(x)$  is a polynomial with real coefficients, and  $f(a)$  and  $f(b)$  have opposite signs,  $a$  and  $b$  being two different real numbers, then the equation  $f(x) = 0$  has at least one root between  $a$  and  $b$ .

Example. Solve graphically  $2x^3 + 3x^2 - 5x - 6 = 0$ .

Solution. From the function  $y = 2x^3 + 3x^2 - 5x - 6$ , we get the table

| $x$ | -3  | -2 | -1 | 0  | 1  | 2  |
|-----|-----|----|----|----|----|----|
| $y$ | -18 | 0  | 0  | -6 | -6 | 12 |

and the graph in Fig. 53. From the graph we estimate the roots to be  $-2$ ,  $-1$ , and  $3/2$ .

It can be proved that at a multiple root, the graph is tangent to the  $x$ -axis.

## EXERCISES

Plot the graphs of and estimate graphically the real roots of the equations in exercises 1-20, section 107, and of:

21.  $x^4 + x^3 - 2x^2 - 4 = 0$ .      22.  $x^4 + 2x^3 - 6x - 9 = 0$ .  
 23.  $x^5 + x^3 - 2x^2 - 2 = 0$ .  
 24.  $x^4 + x^3 - 6x^2 - 8x - 16 = 0$ .



25. If  $f(a)$  and  $f(b)$  are both of the same sign, may we conclude that there exists no root between  $a$  and  $b$ ?

26. If  $f(a)$  and  $f(b)$  are of opposite sign, may we conclude that there is only one root between  $a$  and  $b$ ?

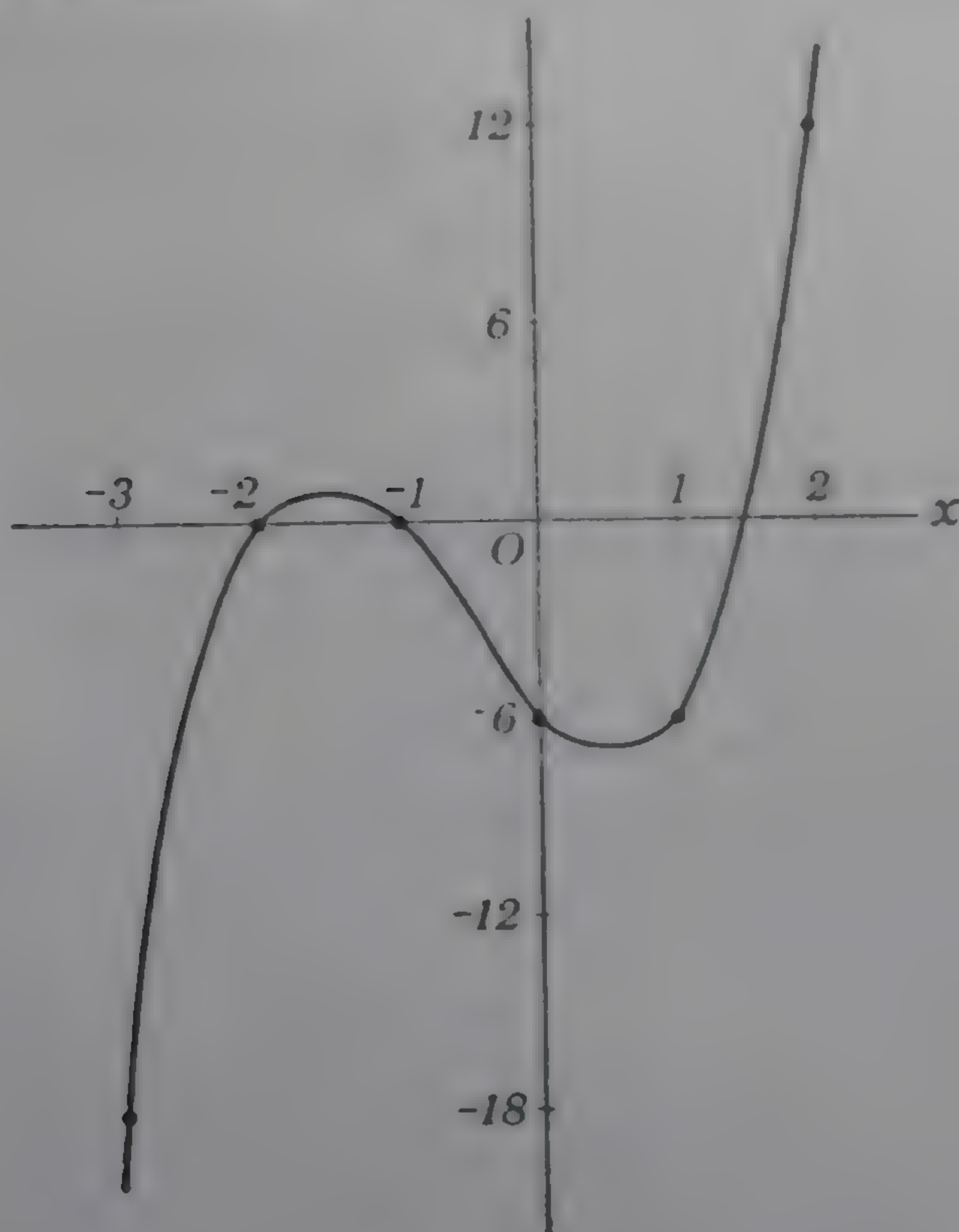


Fig. 53.

109. Transformation reversing the signs of the roots. The roots  $r_1, r_2, \dots, r_n$  of the equation

$$(1) \quad f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = 0$$

or

$$(2) \quad f(x) = a_0(x - r_1)(x - r_2) \dots (x - r_n) = 0$$

may be obtained by setting each of the linear factors in (2) equal to zero. Clearly the equation  $f(-x) = 0$ , or

$$(3) \quad f(-x) = a_0(-x - r_1)(-x - r_2) \dots (-x - r_n) = 0$$

has the roots  $-r_1, -r_2, \dots, -r_n$ , as may be seen by setting the linear factors of (3) equal to zero. Hence the equation  $f(-x) = 0$  has as its roots the roots of  $f(x) = 0$  each with its sign reversed.

From (1) we see that

$$(4) \quad f(-x) = a_0(-x)^n + a_1(-x)^{n-1} + \dots + a_n.$$



But  $(-x)^k = x^k$  if  $k$  is even, and  $(-x)^k = -x^k$  if  $k$  is odd. Hence to form the function  $f(-x)$  we have only to reverse the signs of the odd-powered terms.

*Example.* As seen in section 108,

$$f(x) = 2x^3 + 3x^2 - 5x - 6 = 0$$

has the roots  $-2, -1, 3/2$ . Hence

$$f(-x) = -2x^3 + 3x^2 + 5x - 6 = 0$$

has the roots  $+2, +1, -3/2$ .

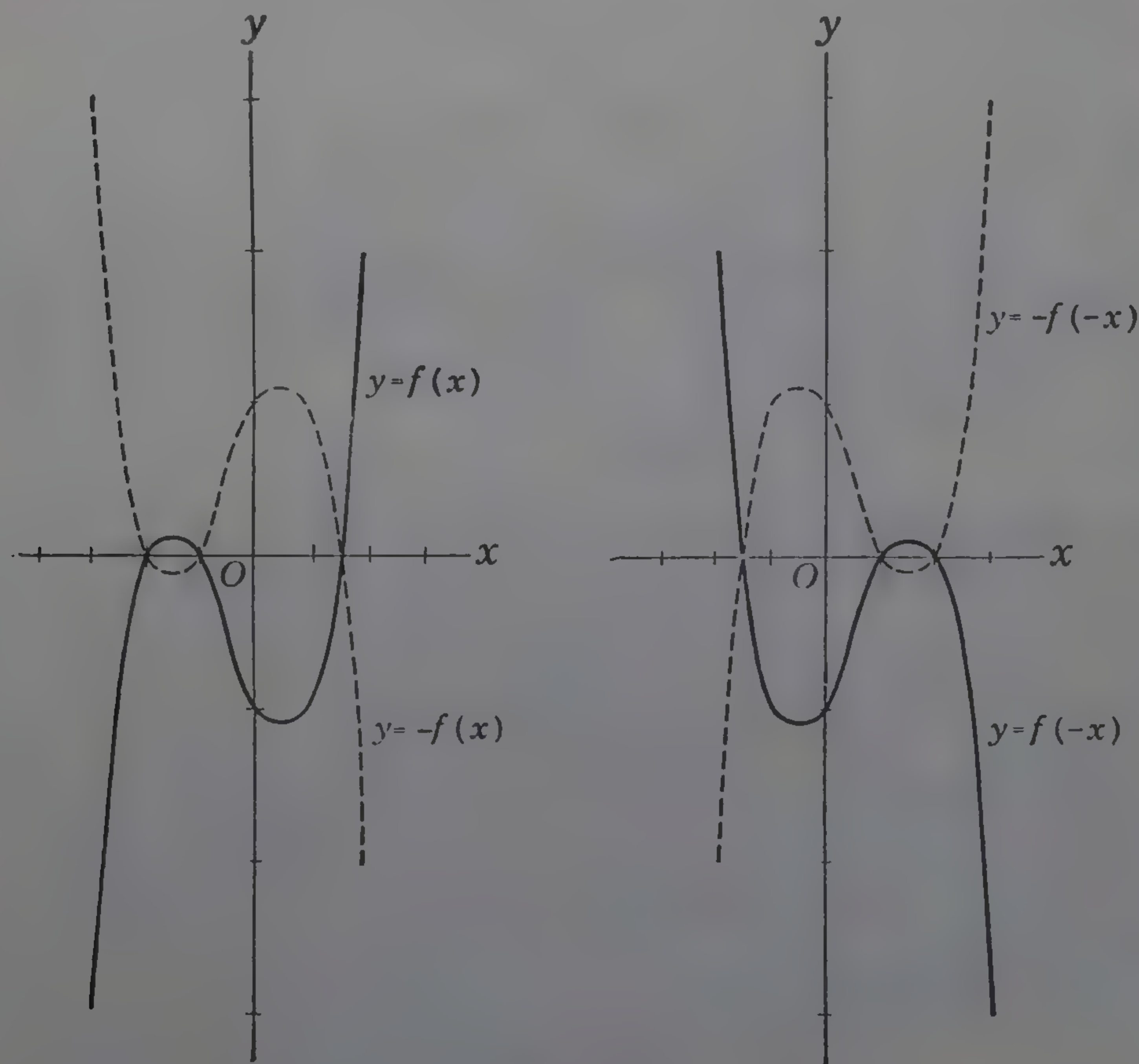


Fig. 54.

*Note.*  $f(-x) = 0$  must be distinguished from  $-f(x) = 0$  which is obtained merely by multiplying both members of  $f(x) = 0$  by  $-1$  and consequently has exactly the same roots as  $f(x) = 0$ . See Fig. 54.

It follows that if we wish to avoid negative multipliers in synthetic division, we may find the negative roots of  $f(x) = 0$  by finding the positive roots of  $f(-x) = 0$  and then reversing their signs.



## EXERCISES

Without solving, write an equation whose roots are those of the given equation, each with its sign reversed:

1.  $x^3 - 2x^2 - 5x + 6 = 0.$

2.  $x^3 - 5x^2 - 8x + 12 = 0.$

3.  $x^3 - 4x^2 + x + 6 = 0.$

4.  $x^3 + 3x^2 - 4x - 12 = 0.$

5.  $9x^4 - 3x^3 + 7x^2 - 3x - 2 = 0.$

6.  $4x^4 - 7x^3 - 5x - 1 = 0.$

7.  $3x^4 + 2x^3 + 5 = 0.$

8.  $3x^3 - 7x^2 - 3x + 2 = 0.$

Find the negative rational roots of the given equations by means of the positive roots of  $f(-x) = 0$ :

9.  $x^3 + 2x^2 - x - 2 = 0.$

10.  $x^3 + 3x^2 - 10x - 24 = 0.$

11.  $2x^3 - x^2 - 8x + 4 = 0.$

12.  $x^3 + 6x^2 + 11x + 6 = 0.$

**110.** Upper and lower bounds for the roots. The method of trial for finding rational roots (section 107) cuts the number of possible trials from infinitely many to a limited number. But if the coefficients are large, the number of trials may be unpleasantly large. For example, in the equation

$$(1) \quad x^4 + x^3 + 70x^2 - 2x - 144 = 0$$

the possible rational roots are the factors of 144 or  $\pm 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144$ , or thirty factors in all. If we tried them all, we would find that none of them satisfy the equation, which therefore has no rational roots. This work can be cut down by the following theorem:

**THEOREM.** If  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$  has real coefficients and has a positive leading coefficient  $a_n$ , and, if we divide synthetically, using a positive multiplier  $k$ , and obtain a third line containing no negative numbers, then no root of  $f(x) = 0$  can be greater than  $k$ .

*Proof.* Dividing  $f(x)$  by  $x - k$  we get

$$f(x) = (x - k)Q(x) + R.$$

By hypothesis the third line of synthetic division contains no negative numbers; that is, the coefficients of  $Q(x)$  and  $R$  are not negative. Consider any value  $x = m$  greater than  $k$ . Then  $m - k$  is positive; and since  $k$  is positive, so is  $m$ . Then substituting  $m$  in  $Q(x)$ , which has no negative coefficients, it is clear that  $Q(m)$  is not negative. Also either  $Q(m)$  or  $R$  must be positive, since they cannot both be zero. Hence  $f(m)$  is positive and not zero. This completes the proof.



Note that any equation with real coefficients can be written with  $a_0$  positive. (If it were negative, we could multiply both members by  $-1$  without altering the roots.)

A number  $k$  such that no root is greater than  $k$  is called an **upper bound** for the roots. A **lower bound** for the negative roots can be found by finding an upper bound for the roots of  $f(-x) = 0$  and reversing its sign.

*Example.* In the equation (1) above, we find that 2 is an upper bound for the roots, since no number in the third line of synthetic division is negative:

$$\begin{array}{r|rrrrr} 2 & 1 & 1 & 70 & -2 & -144 \\ & & 2 & 6 & 152 & 300 \\ \hline & 1 & 3 & 76 & 150 & 156. \end{array}$$

Therefore we need try no number greater than 2.

Here  $f(-x) = x^4 - x^3 + 70x^2 + 2x - 144 = 0$  has 2 as an upper bound also, as can be seen from the synthetic division:

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & 70 & 2 & -144 \\ & & 2 & 2 & 144 & 292 \\ \hline & 1 & 1 & 72 & 146 & 148. \end{array}$$

Hence  $-2$  is a lower bound for the negative roots of  $f(x) = 0$ . That is, we need try no number less than  $-2$ .

## EXERCISES

*Find upper and lower bounds for the roots:*

1.  $x^4 + x^3 - 2x - 4 = 0$ .
2.  $x^4 + 2x^3 - 6x - 9 = 0$ .
3.  $x^5 + x^3 - 2x^2 - 2 = 0$ .
4.  $x^4 + x^3 - 6x^2 - 8x - 16 = 0$ .

*Find all rational roots, using upper and lower bounds for the roots to cut down the number of trials:*

5.  $x^4 - x^3 + 30x^2 - 32x - 64 = 0$ .
6.  $x^4 + x^3 + 34x^2 + 36x - 72 = 0$ .
7.  $x^4 + x^3 + 32x^2 - 4x - 144 = 0$ .
8.  $x^4 + 2x^3 + 33x^2 + 72x - 108 = 0$ .
9.  $x^4 - 2x^3 + 29x^2 - 64x - 96 = 0$ .
10.  $x^4 + 17x^2 - 30x - 144 = 0$ .
11.  $x^4 + x^3 + 20x^2 + 8x + 96 = 0$ .
12.  $x^4 + x^3 + 10x^2 - 8x - 144 = 0$ .
13.  $x^4 + 2x^3 + 15x^2 - 6x - 54 = 0$ .



$$14. x^4 + 2x^3 + 6x^2 - 12x - 72 = 0.$$

$$15. x^6 + x^5 + 20x^4 + 4x^3 + 59x^2 - 5x - 80 = 0.$$

$$16. x^4 + x^5 + 24x^4 - 3x^3 - 85x^2 - 4x - 108 = 0.$$

**111. Descartes' rule of signs.** Any information we can obtain concerning the roots of an equation before trying to solve it is likely to save us some labor. In this section we shall obtain some information concerning the number of positive roots and the number of negative roots we may expect to find.

When the terms of a polynomial with real coefficients are written in order of descending powers, we say that a **variation in sign** occurs if two successive terms have opposite signs, missing terms being ignored. For example, the polynomial  $2x^3 - 4x^2 + x - 5$  has three variations in sign, since its coefficients taken in succession have the signs  $+$   $-$   $+$   $-$ . The polynomial  $x^3 - 2x^2 - 3x^2 - 4x + 5$  has two variations in sign. The polynomial  $x^3 + x + 5$  has no variations in sign. We shall prove the following theorem:

**THEOREM 1.** (*Descartes' rule of signs.*) If the coefficients of  $f(x) = 0$  are real, (a) the number of positive roots is not greater than the number of variations in sign; (b) the number of negative roots is not greater than the number of variations in sign of the polynomial  $f(-x)$ .

*Example 1.* The number of positive roots of  $f(x) = 2x^3 - 4x^2 + x - 5 = 0$  is three or less. There are no negative roots, since  $f(-x) = -2x^3 - 4x^2 - x - 5$  has no variations in sign. That zero is not a root can be seen by inspection. Hence there are either three positive roots, or one positive root and two conjugate imaginary roots. (Why can there not be two positive and one imaginary?) As a result, we are saved the trouble of searching for negative roots.

*Example 2.* Show that  $f(x) = x^4 - 10x + 1 = 0$  has exactly two imaginary roots.

*Solution.* By Descartes' rule, there are two or fewer positive roots, no negative roots, and zero is not a root. Hence there are either two imaginary and two positive roots, or four imaginary and no positive roots. The latter possibility is excluded by plotting the graph, for, since  $f(0) = 1$  and  $f(1) = -8$ , there must be a real root between 0 and 1. Hence there must be two positive roots and two imaginary roots. This completes the proof.

*Example 3.* The number of positive roots of  $x^4 - x^3 - x^2 - 7 = 0$  is one or none. The number of negative roots is one or none.



Hence there are either two real roots and two imaginaries, or else four imaginaries.

Thus if one positive root in example 3 had already been located, we would not waste time looking for another. Similarly, if one negative root were already located, we would not look for more.

These remarks should make clear the practical value of Descartes' rule.

Before proving theorem 1, we need the following.

PRELIMINARY THEOREM. *If  $r$  is a positive root of the equation  $f(x) = 0$  with real coefficients, then the quotient  $Q(x)$  obtained by dividing  $f(x)$  by  $x - r$  has at least one less variation in sign than  $f(x)$  has.*

*Outline of proof.* We write the equation  $f(x) = 0$  so that the leading coefficient  $a_0$  is positive; that is, if necessary, we multiply both members by  $-1$ , which has no effect on the roots or on the number of variations in sign. Now consider the process of synthetic division with  $r$  as multiplier. For example, consider the synthetic division of  $x^8 + x^6 + 2x^5 - 4x^4 - 81x^2 + x + 2$  by  $x - 2$ :

|   |  |   |   |  |   |   |   |   |   |   |    |   |   |    |    |    |   |    |    |   |    |   |   |   |   |   |   |  |    |
|---|--|---|---|--|---|---|---|---|---|---|----|---|---|----|----|----|---|----|----|---|----|---|---|---|---|---|---|--|----|
| 2 |  | + | 1 |  | 0 |   | + | 1 |   | + | 2  |   | - | 4  |    | 0  |   | -  | 81 |   | +  | 1 |   | + | 2 |   |   |  |    |
|   |  |   |   |  | 2 |   |   | 4 |   |   | 10 |   |   | 24 |    | 40 |   |    | 80 |   |    | - | 2 |   | - | 2 |   |  |    |
|   |  |   |   |  | + | 1 |   | + | 2 |   | +  | 5 |   | +  | 12 |    | + | 20 |    | + | 40 |   | - | 1 |   | - | 1 |  | 0. |

It is clear from this example, and true in general, that the leading coefficient in  $Q(x)$  has to be positive and that the coefficients of  $Q(x)$  will remain positive at least until the first variation in sign of  $f(x)$  and possibly longer. When a variation of  $Q(x)$  takes place, from  $+$  to  $-$ , the succeeding coefficients of  $Q(x)$  remain negative at least until the next variation in sign of  $f(x)$  and possibly longer. But since the last number in the third line of synthetic division must be zero, the last coefficient of  $Q(x)$  must have the opposite sign from that of the last coefficient in  $f(x)$ . Hence the largest number of variations  $Q(x)$  can have is one less than the number of variations of  $f(x)$ , and it may have fewer than that.

*Proof of theorem 1.* (a) Let  $r_1, r_2, \dots, r_k$  be the positive roots of  $f(x) = 0$ . Dividing  $f(x)$  by  $x - r_1$  we get a quotient  $Q_1(x)$  which has at least one less variation than  $f(x)$ . Dividing  $Q_1(x)$  by  $x - r_2$  we get a quotient  $Q_2(x)$  which has at least two less variations than  $f(x)$ . And so on until we have  $f(x) = (x - r_1)(x - r_2) \dots (x - r_k)Q_k(x)$  where  $Q_k(x)$  has at least  $k$  less variations than  $f(x)$ . But  $Q_k(x)$  cannot have fewer than no variations at all.



Hence  $f(x)$  has at least as many variations as the number  $k$  of its positive roots.

(b) Part (b) follows at once from the fact that the positive roots of  $f(-x) = 0$  correspond to the negative roots of  $f(x) = 0$ .

We get an even more useful result than theorem 1 by extending Descartes' rule as follows.

**THEOREM 2.** *If the number of positive roots of the equation  $f(x) = 0$ , with real coefficients, is fewer than the number of variations in sign, then it is fewer by an even number.*

*Example 4.* From this theorem we see that the equation in example 3 above *must* have one positive and one negative root, since the number of positive roots cannot be one less than the number of variations in sign.

*Outline of proof of theorem 2.* By corollary 1, section 106,  $f(x)$  can be expressed as a product of linear and quadratic factors, with real coefficients. The quadratic ones arise from pairs of conjugate imaginary roots. They are of the form  $(x - [a + bi])(x - [a - bi]) = x^2 - 2ax + a^2 + b^2$  or  $x^2 + px + q$  where  $q = a^2 + b^2$  is necessarily positive. The linear ones arise from real roots, positive, negative, or zero. Let  $r_1, r_2, \dots, r_k$  be the positive roots,  $-\rho_1, -\rho_2, \dots, -\rho_h$  the negative ones. And let there be  $j$  pairs of conjugate imaginary roots, and  $g$  zero roots. Then

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_k) \cdot (x + \rho_1)(x + \rho_2) \cdots (x + \rho_h) \cdot (x^2 + p_1x + q_1)(x^2 + p_2x + q_2) \cdots (x^2 + p_jx + q_j) \cdot x^g.$$

A little experimentation will indicate that the product of factors of the form  $F(x) = (x - r_1) \cdots (x - r_k)$ , arising from the positive roots, will have alternating signs. The product of all the other factors will be a polynomial  $G(x)$  whose first and last coefficients are surely positive. Now  $F(x)$  has  $k$  variations and its first and last coefficients will have the same sign if  $k$  is even and opposite signs if  $k$  is odd. Hence  $f(x) = a_0 \cdot F(x) \cdot G(x)$  will be such that its first and last coefficients have the same sign if  $k$  is even and opposite signs if  $k$  is odd. But the number of variations in sign of a polynomial  $f(x)$  is even if its first and last coefficients have the same sign and odd if the first and last coefficients have opposite signs. Hence the number  $V$  of variations in sign of  $f(x)$  is even if the number  $k$  of positive roots is even and odd if the number  $k$  of positive roots is odd. Since the difference between two even numbers is even, and the difference between two odd



numbers is even, the number  $V - k$  is always even. This completes the proof.

*Note.* The method of theorem 1, section 107, can be used to prove that numbers such as  $\sqrt{2}$ ,  $\sqrt{3}$ ,  $\sqrt{6}$ ,  $\sqrt[3]{2}$ , etc., are not rational, as in the following example:

*Example 5.* Prove that  $\sqrt{2}$  is not rational.

*Solution.* Let  $x = \sqrt{2}$ . Then  $x^2 = 2$ , or  $x^2 - 2 = 0$ . The only possible rational roots of this equation are  $\pm 1, 2$ . Synthetic division will show that none of these are roots. Hence the equation  $x^2 - 2 = 0$  has no rational roots. But it has the roots  $\pm \sqrt{2}$ . Hence these numbers are not rational.

René Descartes (French, 1596–1650), to whom theorem 1 is credited, was a great mathematician and philosopher. Perhaps his most valuable contribution to human knowledge was his work in analytic geometry, of which he was the principal founder.

## EXERCISES

Without solving, obtain whatever information you can concerning the number of positive, negative, and imaginary roots (a) by means of Descartes' rule (theorem 1) and previous theorems; (b) by means of the extended form of Descartes' rule (theorem 2):

- |                                  |                                       |
|----------------------------------|---------------------------------------|
| 1. $x^3 + 5x - 4 = 0$ .          | 2. $2x^3 + 3x + 1 = 0$ .              |
| 3. $x^4 - 2x^3 + 3x^2 + 5 = 0$ . | 4. $x^3 - 3x^2 + 4x - 5 = 0$ .        |
| 5. $x^4 + x^2 - 3 = 0$ .         | 6. $x^3 + 2x^2 + 3x = -5$ .           |
| 7. $x^4 + 2x^2 + 4 = 0$ .        | 8. $x^6 + 2x^4 + 3x^2 + 7 = 0$ .      |
| 9. $x^4 + 2x^2 - 2x - 3 = 0$ .   | 10. $x^5 + 2x^3 + 3x = 5$ .           |
| 11. $x^3 - 2x^2 + x - 3 = 0$ .   | 12. $x^4 + 2x^3 - x^2 - 3x + 5 = 0$ . |
| 13. $x^4 - x^2 + 5 = 0$ .        | 14. $x^6 + x^4 - 2x^2 - 5 = 0$ .      |
| 15. $x^5 + 32 = 0$ .             | 16. $x^7 - 1 = 0$ .                   |
| 17. $x^6 - 1 = 0$ .              | 18. $x^4 - 3 = 0$ .                   |
| 19. $x^3 + x = 0$ .              | 20. $x^5 + 2x^3 + 3x = 0$ .           |

Prove that each equation has exactly two real and two imaginary roots:

21.  $x^4 - 10x^3 + 1 = 0$ . (Hint: consider the graph.)  
 22.  $2x^4 - 15x + 3 = 0$ .

Prove that each equation has exactly two real and four imaginary roots:

23.  $x^6 - 14x^3 + x^2 + 3 = 0$ .      24.  $2x^6 + x^4 + 10x + 3 = 0$ .

Prove that each equation has exactly two irrational roots:

25.  $x^4 + x^2 - 6 = 0$ .      26.  $x^4 + x^3 - 2x^2 - 4 = 0$ .



*Find all rational roots, using Descartes' rule to cut down the number of trials:*

27.  $x^6 - x^5 - x^4 - x^3 + 30x^2 - 32x - 64 = 0.$

28.  $x^6 + 2x^5 - 2x^4 + 2x^3 + 33x^2 + 72x - 108 = 0.$

29. Work the exercises of section 107, using the theorems of the present section to cut down the number of trials.

*Show by the method of example 5 that each of the following numbers is not rational:*

30.  $\sqrt{3}.$       31.  $\sqrt{6}.$       32.  $\sqrt{8}.$       33.  $\sqrt[3]{2}.$       34.  $\sqrt[3]{3}.$

35.  $\sqrt[3]{6}.$       36.  $\sqrt[4]{6}.$       37.  $\sqrt[5]{8}.$

38. The lengths of the sides of a rectangular box, measured in inches, are three consecutive integers. If the volume is 120 cubic inches, find the sides.

39. After a slice 1 centimeter thick is cut off from one side of a cube, the volume which remains is 48 cubic centimeters. Find the side of the original cube.

40. An open box is to be made from a rectangular sheet of tin 10 inches by 14 inches by cutting equal squares from the corners and folding up the sides. Find one possible length of the side of the cutout square if the volume of the box is to be 120 cubic inches. How many solutions are there?

41. A rectangular box has the dimensions 3 inches, 4 inches, 5 inches. If the length, width, and height were increased by the same amount, the volume would be doubled. Find the amount by which the dimensions are to be increased.

42. An open box is to be made from a square sheet of tin 18 inches on each side by cutting out equal squares from the corners and folding up the sides. How long should the edge of the cutout square be if the volume of the box is to be 432 cubic inches?

**112. Real roots by successive approximations.** In section 107 a method was given for finding the rational roots of polynomial equations with rational coefficients exactly. For practical purposes, an exact answer is seldom required. Rather, what is needed is a sufficiently good approximation. For example, in finding the distance of an enemy battleship from a shore battery we would not be interested in tenths of an inch. In drilling cylinders for an airplane engine we would, but there we would not care about millionths of an inch. Thus, for practical purposes it is sufficient if we provide a method which will yield as many decimal places of the real roots of an equation as we happen to want, no matter whether the roots are rational or irrational. Such a method for



equations with real coefficients will now be explained by an example.

*Example.* Find the positive root of  $f(x) = x^4 + x^3 - x^2 - 3x - 6 = 0$ , correct to the nearest hundredth.

*Solution.* Substituting successive integers, we find that  $f(0)$  and  $f(1)$  are negative, while  $f(2)$  is positive. (The substitution is most conveniently done by synthetic division.) That is,

1

1

-1

-3

-6

1

2

1

-2

1

2

1

-2

-8

= f(1)

2

1

-1

-3

-6

2

6

10

14

1

3

5

7

8

= f(2).

By section 108, the change of sign indicates that the graph has crossed the  $x$ -axis; that is, the positive root  $r$  is between 1 and 2. Hence  $r = 1. \dots$

To find the tenths place, we split the interval from 1 to 2 into tenths and try them in succession. We find that  $f(1.1)$ ,  $f(1.2)$ ,  $f(1.3)$ ,  $f(1.4)$ ,  $f(1.5)$ ,  $f(1.6)$ , and  $f(1.7)$  are negative, while  $f(1.8)$  is positive. That is,

1.7

1

-1

-3

-6

1.7

4.59

6.103

5.2751

1

2.7

3.59

3.103

-0.7249

= f(1.7)

1.8

1

-1

-3

-6

1.8

5.04

7.272

7.6896

1

2.8

4.04

4.272

1.6896

= f(1.8).

Therefore, the positive root is between 1.7 and 1.8, or  $r = 1.7 \dots$

To find the hundredths place, we split the interval between 1.7 and 1.8 into hundredths and try them in succession. We find that  $f(1.71)$ ,  $f(1.72)$ , and  $f(1.73)$  are negative, while  $f(1.74)$  is positive. That is,

1.73

1

-1

-3

-6

1.73

4.7229

6.440617

5.95226741

1

2.73

3.7229

3.440617

-0.04773259

= f(1.73)

1.74

1

-1

-3

-6

1.74

4.7676

6.555624

6.18678576

1

2.74

3.7676

3.555624

0.18678576

= f(1.74).

Hence the root  $r$  lies between 1.73 and 1.74, or  $r = 1.73 \dots$



The nearest hundredth can be determined by finding the thousandths place; if it is 5 or more, we correct the hundredths place upwards by 1, and if it is less than 5 we leave the hundredths place as it is. But it is unnecessary to do this much work if one thinks of the graph. To correct to the nearest hundredth, we have only to try the 5 in the thousandths place: if  $f(x)$  has the same sign as the lower hundredth, we must correct upward, if  $f(x)$  has the same sign as the upper hundredth, we leave the hundredths place as it is (Fig. 55). Thus

|       |   |       |          |             |                |
|-------|---|-------|----------|-------------|----------------|
| 1.735 | 1 | 1     | -1       | -3          | -6             |
|       |   | 1.735 | 4.745225 | 6.497965375 | 6.068969925625 |
|       | 1 | 2.735 | 3.745225 | 3.497965375 | 0.068969925625 |
|       |   |       |          |             | = $f(1.735)$   |

Since  $f(1.735)$  has the same sign as  $f(1.74)$ , the crossing (Fig. 55) took place to the left of the middle point, and is nearer 1.73 than 1.74. Hence  $r = 1.73$  to the nearest hundredth.

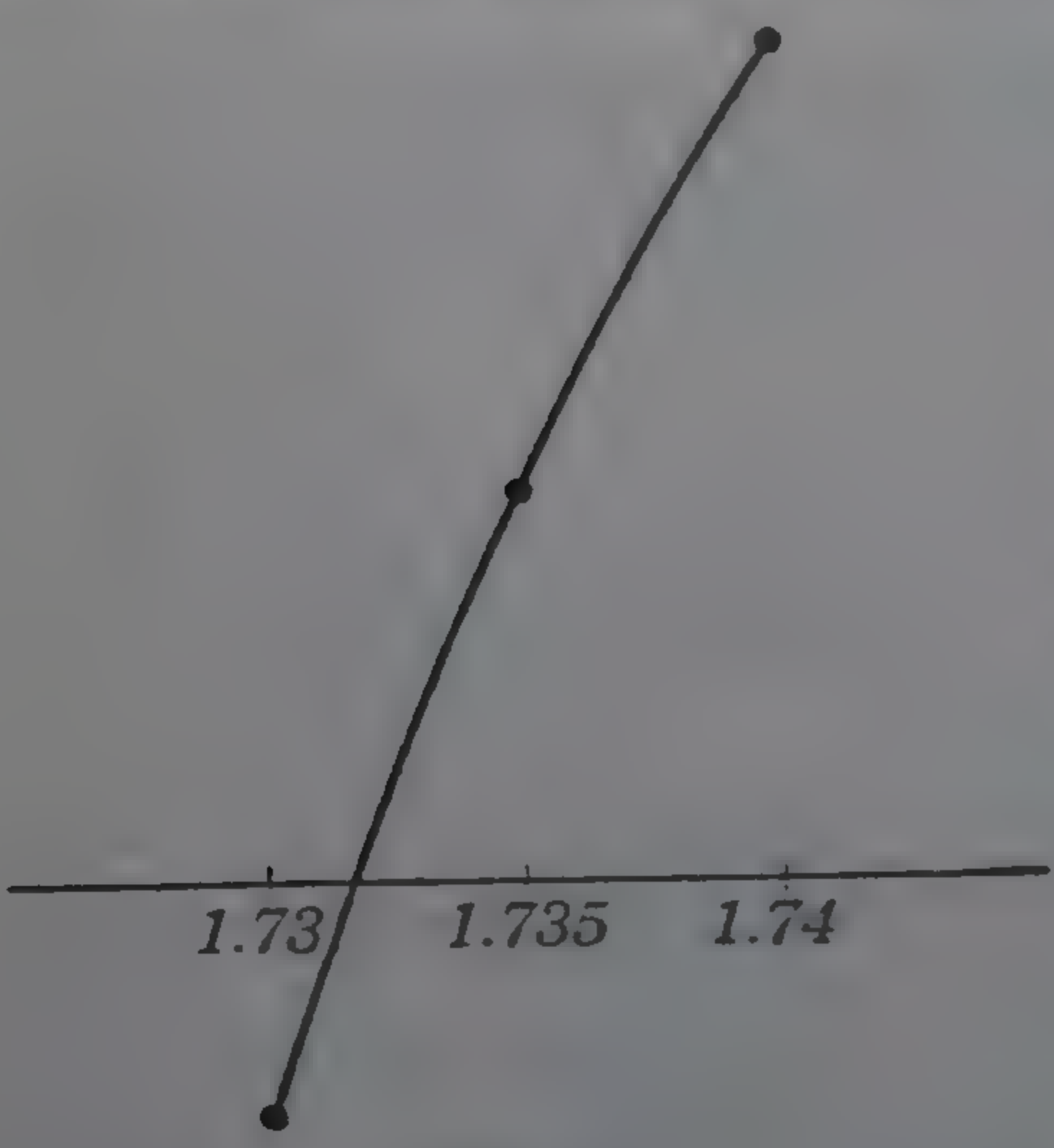


Fig. 55.

To find a negative root of  $f(x) = 0$ , we may use this process to find the corresponding positive root of  $f(-x) = 0$  and then reverse the sign.

*Note 1.* The process of successive approximations is much like that of running down a base runner who tried to steal second base. We crowd the root  $r$  between narrowing bounds, first between successive integers, then between successive tenths, hundredths, thousandths,

and so on. In the case of an irrational number or a rational number which happens to have an endless decimal expression, we never tag it exactly, but we approximate it more closely at each stage.

*Note 2.* This process may be used for any equation  $f(x) = 0$  where  $f(x)$  is a continuous function, and is not restricted to polynomial equations, except that for functions  $f(x)$  which are not polynomials the substitutions have to be done directly instead of by synthetic division.

*Note 3.* Unless a calculating machine is available for the multiplications involved in the synthetic divisions, this process



becomes tedious if it has to be carried beyond one or two decimal places. The next section will explain one way to avoid some of the labor. A further laborsaving device will be discussed in section 115.

## EXERCISES

*Approximate to the nearest hundredth:*

1. The positive root of  $x^4 + x^3 + x^2 - 2x - 6 = 0$ .
2. The root of  $x^4 - 11x^2 + 15x - 2 = 0$  lying between 0 and 1.
3. The positive root of  $x^3 + 3x - 5 = 0$ .
4. The positive root of  $x^4 + x^3 - 3x^2 - 5x - 10 = 0$ .
5. The largest real root of  $x^3 + x^2 - 9x - 7 = 0$ .
6. The negative root of  $x^4 - x^3 - 11x^2 + 12x - 12 = 0$ .
7. The negative root of  $x^4 - 2x^3 - 3x^2 - 2x - 4 = 0$ .
8.  $\sqrt[3]{3}$ . (Hint:  $\sqrt[3]{3}$  is the positive root of  $x^3 - 3 = 0$ .)
9.  $\sqrt[3]{4}$ .
10.  $\sqrt[3]{-5}$ .

**\*113. Linear interpolation.** If a root  $r$  of an equation  $f(x) = 0$  has been computed, by the method of the preceding section, say, up to a certain decimal place, an approximation of the next place may be obtained as follows.

Suppose that, in the example of the preceding section, we have found that  $f(1) = -8$  and  $f(2) = 8$ . We want to find the value of  $r$  between 1 and 2 for which  $f(r) = 0$ . If we assume that the change in  $f(x)$  is proportional to the change in  $x$ , we may say that the total change in  $f(x)$  is  $f(2) - f(1) = 16$ , while the change in  $x$  is  $2 - 1 = 1$ ; but the change in  $f(x)$  from  $-8$  to 0 is 8, while the change in  $x$  from 1 to  $r$  is called  $d$ . Then our assumption amounts to the proportion

$$(1) \quad \frac{d}{1} = \frac{8}{16} \text{ or } d = 0.5.$$

Thus we would conclude that  $r = 1 + 0.5 = 1.5$  approximately.

The calculation may be arranged as follows:

$$1 \left[ \begin{array}{c|c} x & f(x) \\ \hline 1 & -8 \\ r = 1 + d & 0 \\ 2 & 8 \end{array} \right] \begin{array}{c} 8 \\ 16 \end{array}$$

Therefore  $\frac{d}{1} = \frac{8}{16}$ , or  $d = 0.5$ . Hence  $r = 1.5$ .

\* This section may be omitted if Horner's method is to be taken up.



Note that this is not accurate, since the more careful calculation of the preceding section shows that, as far as the tenths place is concerned,  $r = 1.7 \dots$ . But the present calculation would serve a purpose if only to save us the trouble of calculating  $f(1.1)$ ,  $f(1.2)$ ,  $f(1.3)$ ,  $f(1.4)$  in the more accurate computation of the preceding section.

The proportion (1) rewritten in the form  $8/d = 16/1$  amounts to the assumption that the graph between the points  $(1, -8)$  and  $(2, 8)$  is a straight line of slope 16. That is, we are replacing the actual curve by the chord (straight line segment) joining these two points. For this reason, this process is called **linear interpolation** or **interpolation by proportional parts**. The accuracy of the result will thus depend on how closely the chord approximates the curve. Hence, the accuracy is likely to improve as the distance between the values of  $x$  diminishes. Thus, suppose we have found in the example of the preceding section that  $f(1.7) = -0.7249$  and  $f(1.8) = 1.6896$ . Then

$$0.1 \left[ d \left[ \begin{array}{c|c} x & f(x) \\ \hline 1.7 & -0.7249 \\ r = 1.7 + d & 0 \\ 1.8 & 1.6896 \end{array} \right] 0.7249 \right] 2.4145$$

and  $\frac{d}{0.1} = \frac{0.7249}{2.4145} = 0.30$ , approximately.

Hence  $d = 0.03$ , approximately,

and  $r = 1.73$ , approximately,

which is accurate as far as the hundredths place is concerned.

The process of linear interpolation can also be done graphically if desired. Thus in the preceding calculation we could have plotted on a graph the points  $(1.7, -0.7249)$  and  $(1.8, 1.6896)$  as closely as possible, joined them by a straight line, and read from the graph the  $x$ -coordinate of the point where the straight line crosses the  $x$ -axis.

## EXERCISES

**1-10.** Work the exercises 1-10 of the preceding section, using the method of linear interpolation to find the digit in the hundredths place.

*Approximate to the nearest thousandth, using linear interpolation to find the digit in the thousandths place:*



11. The positive root of  $x^4 + 2x^3 - 3x^2 + 2x - 4 = 0$ .

12. The negative root of  $x^4 + 2x^3 - 3x^2 + 2x - 4 = 0$ .

13.  $\sqrt[3]{6}$ .      14.  $\sqrt[3]{7}$ .      15.  $\sqrt[3]{9}$ .      16.  $\sqrt[3]{-10}$ .

**114. Transformations of equations.** It is often useful to transform an equation into a new equation whose roots have a certain relation to the roots of the given equation. One such transformation was taken up in section 109.

(A) *Transformation diminishing all the roots by a specified constant  $h$ .* From the form

$$(1) \quad f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0$$

of the given equation

$$(2) \quad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

it is clear that substituting  $x = X + h$  will yield the desired equation; for then the roots of

$$f(X + h) = a_0(X + h - r_1)(X + h - r_2) \cdots (X + h - r_n) = 0$$

are obtained by setting each factor equal to zero, thus obtaining the required roots  $X = r_1 - h$ ,  $X = r_2 - h$ ,  $\cdots$ ,  $X = r_n - h$ . Hence we could obtain the desired equation from (2) by writing

$$(3) \quad f(X + h) = a_0(X + h)^n + a_1(X + h)^{n-1} + \cdots + a_n = 0,$$

expanding each power by multiplication, and collecting like terms, thus obtaining the desired equation

$$(4) \quad F(X) = A_0X^n + A_1X^{n-1} + \cdots + A_n = 0.$$

This process is tedious and we will calculate the coefficients of the desired equation (4) by a more efficient method, as follows.

Since  $x = X + h$ , and  $f(x) = f(X + h) = F(X)$ , we may write (4) as

$$f(x) = A_0(x - h)^n + A_1(x - h)^{n-1} + \cdots + A_{n-1}(x - h) + A_n = 0.$$

Dividing the left member by  $x - h$ , we therefore must get the remainder  $R_1 = A_n$ . Dividing the resulting quotient

$$A_0(x - h)^{n-1} + A_1(x - h)^{n-2} + \cdots + A_{n-1}$$

by  $x - h$ , we get the remainder  $R_2 = A_{n-1}$ , and so on until finally  $A_0 = a_0$ , as can be seen from (3). The successive remainders and  $a_0$  are the coefficients we seek in reverse order. Writing  $x$  instead of  $X$ , we have the desired equation



$$(5) \quad a_0x^n + R_nx^{n-1} + R_{n-1}x^{n-2} + \cdots + R_2x + R_1 = 0.$$

The divisions may be arranged synthetically, as in the following example.

*Example 1.* Find the equation whose roots are the roots of  $x^3 - 2x^2 - 5x + 6 = 0$ , each diminished by 2.

*Solution.* Dividing synthetically by  $x - 2$  repeatedly, we get

$$\begin{array}{r} 2 \overline{) 1 \quad -2 \quad -5 \quad 6} \\ \quad 2 \quad 0 \quad -10 \\ \hline 1 \quad 0 \quad -5 \quad -4 = R_1 = A_3 \\ \quad 2 \quad 4 \\ \hline 1 \quad 2 \quad -1 = R_2 = A_2 \\ \quad 2 \\ \hline a_0 = 1 \quad 4 = R_3 = A_1. \end{array}$$

The desired equation is, according to (5),  $x^3 + 4x^2 - x - 4 = 0$ .

Solving both equations as a check, we find the original equation has the roots 1, 3, -2, while the new equation has the roots -1, 1, -4, as required.

*Example 2.* Find the equation whose roots are the roots of  $x^3 - 2x^2 - 5x + 6 = 0$ , each increased by 2.

*Solution.* Increasing by 2 is equivalent to diminishing by -2. Hence, from the calculation

$$\begin{array}{r} -2 \overline{) 1 \quad -2 \quad -5 \quad 6} \\ \quad -2 \quad 8 \quad -6 \\ \hline 1 \quad -4 \quad 3 \quad 0 \\ \quad -2 \quad 12 \\ \hline 1 \quad -6 \quad 15 \\ \quad -2 \\ \hline 1 \quad -8, \end{array}$$

we find the equation  $x^3 - 8x^2 + 15x = 0$ . This equation has the roots 3, 5, 0, as required.

(B) *Transformation multiplying all the roots by a constant k.*

Substituting  $x = \frac{X}{k}$  in (1) will clearly produce an equation whose roots are  $X = kr_1$ ,  $X = kr_2$ ,  $\cdots$ ,  $X = kr_n$  as may be seen by setting each of the factors of

$$f\left(\frac{X}{k}\right) = a_0 \left(\frac{X}{k} - r_1\right) \left(\frac{X}{k} - r_2\right) \cdots \left(\frac{X}{k} - r_n\right) = 0$$



equal to zero. From (2) we see that

$$f\left(\frac{X}{k}\right) = a_0 \left(\frac{X}{k}\right)^n + a_1 \left(\frac{X}{k}\right)^{n-1} + a_2 \left(\frac{X}{k}\right)^{n-2} + \cdots \\ + a_{n-1} \left(\frac{X}{k}\right) + a_n = 0.$$

Multiplying by the L.C.D.  $k^n$ , we find the desired equation

$$F(X) = a_0 X^n + a_1 k X^{n-1} + a_2 k^2 X^{n-2} + \cdots \\ + a_{n-1} k^{n-1} X + a_n k^n = 0.$$

Hence the coefficients of the desired equation are obtained by multiplying each coefficient  $a_m$  ( $m = 0, 1, 2, \cdots, n$ ) of the given equation by  $k^m$ .

*Example 3.* Write an equation whose roots are those of  $x^3 - 2x^2 - 5x + 6 = 0$ , each multiplied by 2.

*Solution.* The desired equation is

$$X^3 - 2 \cdot 2X^2 - 5 \cdot 2^2X + 6 \cdot 2^3 = 0$$

or, writing  $x$  instead of  $X$ ,

$$x^3 - 4x^2 - 20x + 48 = 0.$$

Since the original equation has the roots 1, 3,  $-2$ , the new equation should have the roots 2, 6,  $-4$ . The student should check.

*Note.* The transformation of section 109 is a special case of (B), above, taking  $k = -1$ .

## EXERCISES

Write an equation whose roots are those of the given equation, each decreased by the given number  $h$ , and check:

1.  $x^3 - 3x^2 - 4x + 12 = 0$ ;  $h = 1$ .
2.  $x^3 - 3x + 2 = 0$ ;  $h = 3$ .
3.  $x^3 - 4x^2 + x + 6 = 0$ ;  $h = 2$ .
4.  $4x^4 + 12x^3 - 17x^2 - 3x + 4 = 0$ ;  $h = -1$ .
5.  $2x^3 - 5x^2 - 4x + 3 = 0$ ;  $h = -2$ .
6.  $x^3 - 6x^2 + 11x - 6 = 0$ ;  $h = -3$ .
7.  $x^3 - 2x^2 - x + 2 = 0$ ;  $h = 0.2$ .
8.  $x^3 - x^2 - 4x + 4 = 0$ ;  $h = -0.1$ .
9.  $x^3 - 4x^2 + x + 6 = 0$ ;  $h = 0.02$ .
10.  $x^4 + 7.8x^3 + 21.44x^2 + 21.922x - 0.7249 = 0$ ;  $h = 0.03$ .

Write an equation whose roots are those of the given equation each multiplied by the given number  $k$ :



- 11.  $x^3 - 3x^2 - 4x + 12 = 0; k = 2.$
- 12.  $x^3 - 3x + 2 = 0; k = 3.$
- 13.  $x^3 - 4x^2 + x + 6 = 0; k = 10.$
- 14.  $4x^4 + 12x^3 - 17x^2 - 3x + 4 = 0; k = 10.$
- 15.  $x^4 + 7.8x^3 + 21.44x^2 + 21.922x - 0.7249 = 0; k = 10.$
- 16.  $x^4 + 2.3x^3 + 1.43x^2 - 6.347x - 1.5782 = 0; k = 10.$

**115. Horner's method.** The most tedious feature of the method of section 112 is the fact that the synthetic divisions with multipliers of two, three, or more digits require numerous multiplications which, in the absence of a calculating machine, demand much labor. This labor can be somewhat diminished by a method published by W. G. Horner in 1819,\* the advantage of which is that all multipliers in synthetic divisions will consist of one digit only, so that all multiplications can be done mentally. We shall explain the method by reworking the example of section 112.

*Example.* Find the positive root of  $f(x) = x^4 + x^3 - x^2 - 3x - 6 = 0$ , correct to the nearest hundredth.

*Solution.* Substituting successive integers, we locate the positive root  $r$  between 1 and 2, since  $f(1) = -8$  and  $f(2) = 8$ .

Now instead of substituting 1.1, 1.2, 1.3, . . . , which would involve 2-digit multipliers, we apply the transformation (1) of section 114, forming a new equation whose roots are equal to those of the original equation each diminished by 1, thus:

|          |   |    |    |    |
|----------|---|----|----|----|
| <u>1</u> | 1 | -1 | -3 | -6 |
|          | 1 | 2  | 1  | -2 |
| 1        | 2 | 1  | -2 | -8 |
|          | 1 | 3  | 4  |    |
| 1        | 3 | 4  | 2  |    |
|          | 1 | 4  |    |    |
| 1        | 4 | 8  |    |    |
|          | 1 |    |    |    |
| 1        | 5 |    |    |    |

The new equation  $f_1(x) = x^4 + 5x^3 + 8x^2 + 2x - 8 = 0$  has a root between 0 and 1 which is exactly 1 less than the root  $r$  of the original equation. Hence in the new equation  $f_1(x) = 0$  we may try successive tenths. We find that  $f(0.7) = -0.7249$  and  $f(0.8) = 1.6896$ . Therefore  $f_1(x)$  has a root  $0.7 \dots$ , and the original equation has the root  $r = 1.7 \dots$ .

Instead of substituting 0.71, 0.72, 0.73, . . . in  $f_1(x)$ , we

\* Horner's method, or something closely resembling it, was anticipated by Ruffini in 1804, and by the Chinese Ch'in Chiu-shao more than five centuries earlier



transform  $f_1(x) = 0$  into a new equation whose roots are equal to those of  $f_1(x) = 0$ , each diminished by 0.7, thus:

|            |   |      |       |        |         |
|------------|---|------|-------|--------|---------|
| <u>0.7</u> | 1 | 5    | 8     | 2      | −8      |
|            |   | 0.7  | 3.99  | 8.393  | 7.2751  |
|            | 1 | 5.7  | 11.99 | 10.393 | −0.7249 |
|            |   | 0.7  | 4.48  | 11.529 |         |
|            | 1 | 6.4  | 16.47 | 21.922 |         |
|            |   | 0.7  | 4.97  |        |         |
|            | 1 | 7.1  | 21.44 |        |         |
|            |   | 0.7  |       |        |         |
|            | 1 | 7.8. |       |        |         |

The new equation  $f_2(x) = x^4 + 7.8x^3 + 21.44x^2 + 21.922x - 0.7249 = 0$  has a root between 0 and 0.1, exactly 0.7 less than the root of  $f_1(x) = 0$ , and exactly 1.7 less than the root  $r$  of  $f(x) = 0$ . Hence in  $f_2(x)$  we substitute successive hundredths 0.01, 0.02, 0.03, and so on. We find that  $f_2(0.03) = -0.04773259$  and  $f_2(0.04) = 0.18678576$ . Hence the positive root of  $f_2(x) = 0$  is  $0.03 \dots$ , the positive root of  $f_1(x) = 0$  is  $0.73 \dots$ , and the root  $r$  of  $f(x) = 0$  is  $r = 1.73 \dots$ .

Instead of substituting 0.031, 0.032, 0.033, . . . , we may diminish the roots of  $f_2(x) = 0$  by 0.03, thus:

|             |   |      |         |           |             |
|-------------|---|------|---------|-----------|-------------|
| <u>0.03</u> | 1 | 7.8  | 21.44   | 21.922    | −0.72490000 |
|             |   | 0.03 | 0.2349  | 0.650247  | 0.67716741  |
|             | 1 | 7.83 | 21.6749 | 22.572247 | −0.04773259 |
|             |   | 0.03 | 0.2358  | 0.657321  |             |
|             | 1 | 7.86 | 21.9107 | 23.229568 |             |
|             |   | 0.03 | 0.2367  |           |             |
|             | 1 | 7.89 | 22.1474 |           |             |
|             |   | 0.03 |         |           |             |
|             | 1 | 7.92 |         |           |             |

The equation  $f_3(x) = x^4 + 7.92x^3 + 22.1474x^2 + 23.229568x - 0.04773259 = 0$  has a positive root between 0 and 0.01 exactly 0.03 less than that of  $f_2(x)$  and 1.73 less than the root  $r$  of  $f(x)$ . If we wished to find the thousandths place of  $r$ , we would substitute 0.001, 0.002, and so on in  $f_3(x)$ . To find the value of  $r$  to the nearest hundredth, however, it is sufficient to substitute 0.005 in  $f_3(x)$ . By considering the graph (compare Fig. 55) it is obvious that if  $f_3(0.005)$  has the same sign as at the left-hand end of the interval, we must correct the hundredths place upward; while if  $f_3(0.005)$  has the same sign as at the right-hand end of the interval,



we leave the hundredths place unaltered. In our example, we find that  $f_3(0.005)$  is positive; hence  $r = 1.73$  correct to the nearest hundredth.

Clearly this process will yield as many places as we wish in the decimal expression of a positive real root. *To find a negative root of  $f(x) = 0$ , we find the corresponding positive root of  $f(-x) = 0$  and reverse the sign.*

*Note 1.* After the tenths place of the root has been determined, an estimate of the next decimal place can be made by neglecting the powers of  $x$  higher than the first, as follows. In the equation,  $f_2(x) = 0$  above, we obtain the equation  $21.922x - 0.7249 = 0$  by neglecting powers higher than the first. This yields

$$x = \frac{0.7249}{21.922} = 0.033 = 0.03 \text{ approximately,}$$

which is accurate for the next place. The student may verify that this method will give a very inaccurate prediction if applied to  $f_1(x) = 0$ , or  $f(x) = 0$ . The reason for this becomes apparent when we understand the justification of the method, which is as follows. If the root is very close to zero, then its square, cube, and so on are very much smaller. For example, if the root is 0.03, its square is 0.0009 and its cube is 0.000027, and so on. Hence higher powers of  $x$  will contribute a negligible amount to the value of the function unless their coefficients happen to be correspondingly large, which is not usually the case. While the estimate obtained in this way is not reliable, it may be used to save unnecessary trials. Thus if we get an estimate of 0.07 with this method, we would not bother to try 0.01, 0.02, . . . , 0.05 at first, but would start with 0.07. The accuracy of this estimate increases as we work with later decimal places.

*Note 2.* If it is desired to avoid the use of decimals, we may transform each equation after the original one into a new equation whose roots are equal to the roots of the preceding one, each multiplied by 10, by transformation (B) of section 114. Then instead of substituting 0.1, 0.2, and so on, we could substitute 1, 2, and so on.

*Note 3.* Horner's method incidentally provides a technique for calculating real  $n$ th roots. For  $\sqrt[3]{2}$ , for example, is nothing else than the positive root of  $x^3 - 2 = 0$ .

*Note 4.* Horner's process applies only to polynomial or integral rational equations, while the process of section 112 may be applied to a wider class of equations. A still more efficient method,



devised by Isaac Newton (English, 1642–1727), applies to the wider class of equations, but requires some preliminary study of differential calculus and cannot be taken up here. For a discussion of Newton's method see textbooks on the theory of equations.

*Note 5.* In solving a polynomial equation with rational or integral coefficients, it is advisable to obtain information from Descartes' rule first, then search for rational roots, then approximate irrational roots.

## EXERCISES

1–10. Using Horner's method, work exercises 1–10 of section 112.

11–16. Using Horner's method, work exercises 11–16 of section 113.

*Using Horner's method, approximate to the nearest thousandth:*

17. The larger real root of  $x^4 - 4x^3 + 2x^2 - 4x + 1 = 0$ .

18. The smaller positive root of  $x^4 + 4x^3 - 4x^2 - 12x + 3 = 0$ .

19. The two roots of  $x^4 + 2x^3 - 6x^2 - 4x + 8 = 0$  lying between 1 and 2.

20. The negative root of  $x^4 + x^3 - 5x^2 - 6x - 6 = 0$ .

21.  $\sqrt[3]{50}$ .      22.  $\sqrt[3]{-47}$ .      23.  $\sqrt[3]{-23}$ .      24.  $\sqrt[5]{16}$ .

25. All the real roots of  $x^4 - 6x^3 + 8x^2 + 2x - 1 = 0$ .

26. All the real roots of  $x^4 - 8x^3 + 15x^2 + 10x - 28 = 0$ .

27. In studying the performance of a certain airplane, it is necessary to solve the equation\*  $x - x^4 = Z$ . If  $Z = 0.2$ , show that there are two real roots between 0 and 1 and find the larger one to the nearest hundredth.

28. An open box is to be made from a rectangular sheet of tin 8 inches by 10 inches by cutting equal squares from the corners and folding up the sides. Find the side of the cutout square to the nearest hundredth if the volume is to be 60 cubic inches.

29. In studying the lateral stability of a certain airplane, it is necessary to solve the equation\*  $x^4 + 45x^3 + 214x^2 + 1492x + 595 = 0$ . Find the real roots to the nearest tenth.

30. Find the thickness of the wall of a hollow spherical shell of uniform thickness if its outer diameter is 10 inches and its capacity or volume is 300 cubic inches.

**116. Relations between coefficients and roots.** Since the values of the coefficients of an equation determine the values of its roots, it is natural to expect the existence of some definite relations between the coefficients and the roots.

Consider the equation

$$(1) \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad (a_0 \neq 0)$$

\* The equations of exercises 27 and 29 are reprinted by permission from *Theory of Flight* by R. von Mises, copyrighted, 1945, by the McGraw-Hill Book Co., Inc.



of degree  $n$ , or, in factored form,

$$(2) \quad a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0.$$

Dividing both members of (1) and of (2) by  $a_0$ , we get the equivalent equations

$$(3) \quad x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0,$$

and

$$(4) \quad (x - r_1)(x - r_2) \cdots (x - r_n) = 0.$$

Since the left members of (3) and (4) are identically equal, it follows from corollary 2, section 104, that we may equate coefficients of like powers.

For simplicity, let us do this first for  $n = 3$ . Multiplying  $(x - r_1)(x - r_2)(x - r_3)$ , we obtain

$$x^3 + (-r_1 - r_2 - r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x + (-r_1r_2r_3) = 0.$$

Comparing these coefficients with those of

$$x^3 + \frac{a_1}{a_0}x^2 + \frac{a_2}{a_0}x + \frac{a_3}{a_0} = 0,$$

we find the relations

$$(5) \quad \left\{ \begin{array}{l} -\frac{a_1}{a_0} = r_1 + r_2 + r_3 \\ \frac{a_2}{a_0} = r_1r_2 + r_1r_3 + r_2r_3 \\ -\frac{a_3}{a_0} = r_1r_2r_3. \end{array} \right.$$

Similarly, it can be shown in general that

$$(6) \quad \left\{ \begin{array}{l} -\frac{a_1}{a_0} = r_1 + r_2 + \cdots + r_n \text{ (= the sum of all the roots taken one at a time)} \\ \frac{a_2}{a_0} = r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n \text{ (= the sum of all possible products of the roots taken two at a time)} \\ -\frac{a_3}{a_0} = r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n \text{ (= the sum of all possible products of the roots taken three at a time)} \\ \vdots \\ (-1)^n \frac{a_n}{a_0} = r_1r_2 \cdots r_n \text{ (= the product of the roots taken all at a time).} \end{array} \right.$$



*Note 1.* These relations (6) are a generalization of the relations of section 78 for quadratics.

*Note 2.* The first relation of (6) constitutes a convenient check on the correctness of the solution of an equation. If the roots have been found exactly, they should add up to  $-a_1/a_0$ . If they have been found approximately as decimals, they should add up to something very close to  $-a_1/a_0$ .

*Example.* Solve the equation  $x^3 - 3x^2 + kx + 12 = 0$ , given that one root is the negative of another.

*First solution.* Here  $a_0 = 1$ ,  $a_1 = -3$ ,  $a_2 = k$ ,  $a_3 = 12$ , and  $r_2 = -r_1$ . Then by the first relation of (5) or (6), we have  $-\frac{(-3)}{1} = r_1 + r_2 + r_3$ , or  $3 = r_1 + (-r_1) + r_3$ , or  $r_3 = 3$ .

Hence 3 is a root. Therefore 3 must satisfy the equation, or  $3^3 - 3 \cdot 3^2 + 3k + 12 = 0$ . Hence  $k = -4$ . The remaining roots can be found by depressing the equation:

$$\begin{array}{r} 3 \overline{) 1 \quad -3 \quad -4 \quad +12} \\ \underline{\phantom{3} 3 \quad 0 \quad -12} \\ 1 \quad 0 \quad -4 \quad 0. \end{array}$$

Hence the remaining roots are found to be  $\pm 2$  from the depressed equation  $x^2 - 4 = 0$ .

*Second solution.* After finding  $r_3 = 3$  as before, we may use the third relation of (5) thus:

$$-r_1(-r_1) \cdot 3 = 12, \text{ or } r_1^2 = 4.$$

Hence  $r_1 = \pm 2$  and  $r_2 = \mp 2$ . As before, the roots are  $\pm 2, 3$ .

## EXERCISES

1. Without solving, find the sum and product of the roots of  $2x^3 - 3x^2 + 3x + 1 = 0$ .

2. By the method used in deriving equations (5), derive similar relations for the equation  $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ .

3. Solve  $x^3 + 2x^2 + kx + 6 = 0$  and find the value of  $k$ , given that the sum of two of the roots is  $-4$ .

4. Solve  $2x^3 + 3x^2 + kx - 3 = 0$  and find the value of  $k$ , given that one root is the negative of another.

5. Solve  $x^3 + kx^2 - 3x - 4 = 0$  and find the value of  $k$ , given that one root is 4.

6. Solve  $x^3 + kx + k = 0$  and find the values of  $h$  and  $k$ , given that one root is 2 and the difference between the other roots is 4.

7. Solve  $x^4 + 2x^3 - 3x^2 + kx + 4 = 0$  and find the value of  $k$ , given that there are two distinct double roots which are real.



8. Show that if one root of  $x^3 + bx^2 + cx + d = 0$  is the negative of another, then  $d = bc$ .

9. Solve  $2x^3 - 3x^2 + hx + k = 0$  and find the values of  $h$  and  $k$ , given that 3 and  $-2$  are roots.

10. Solve  $2x^3 - 7x^2 + hx + k = 0$  and find the values of  $h$  and  $k$ , given that  $-1$  is the first root and the third root is twice the second.

**117. Algebraic solution of equations.** While the relations (6) of the preceding section between the coefficients and the roots are interesting and often useful, they do not suffice to determine the roots in terms of the coefficients. However, the formulas

$$r_1 = -\frac{b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = -\frac{b - \sqrt{b^2 - 4ac}}{2a} \quad \text{give the}$$

roots of the general quadratic  $ax^2 + bx + c = 0$  as algebraic expressions\* in the coefficients. An equation with literal coefficients is said to be **solved algebraically** when the roots are written as algebraic expressions in the coefficients. The algebraic solution of the quadratic is attributed to the Hindus. It is natural to ask whether we can find formulas for the roots of the general equations of degree higher than 2 which similarly give the roots as algebraic expressions in the coefficients. Such formulas were obtained for the general cubic by Tartaglia, and possibly del Ferro, in the 16th century. They are sometimes called Cardano's formulas because they were first published in 1545 in the *Ars Magna* by Cardano, who obtained them from Tartaglia under a pledge of secrecy.† For the general equation of degree 4, similar formulas were obtained by Ferrari at about the same time. The ingenious solutions of the cubic and quartic will be sketched in the next sections. The next natural task is to find such formulas for equations of degree 5 or more. This question was attacked from the 16th century until 1824 when a brilliant young Norwegian mathematician, N. H. Abel, proved, at the age of 22, that such formulas cannot exist for the general equation of degree higher than 4. Abel's proof is too advanced to be presented here. While the general equation of degree 5 or more cannot be solved by algebraic expressions, some equations of degree 5 or more can be solved by such formulas. For example, the pure equation of degree 5,  $ax^5 + f = 0$ , has a solution of the

\* The definition of *algebraic expression* is given in section 24.

† In those days it was customary to withhold a new discovery and challenge other mathematicians to solve the problem within a set time limit, say a month. This often gave the discoverer an opportunity to gloat, and at other times led to disputes over priority of discovery.



form  $x = \sqrt[5]{-f/a}$ , which is an algebraic expression in the coefficients. Hence a natural question is to find criteria for deciding which equations have such solutions and which have not. This task was completed by E. Galois, a young French mathematician, early in the 19th century. His work is likewise too advanced to be presented here. It is interesting to note that Abel died before he was 27, leaving behind him much original work which stimulated research for years afterward. Galois was killed in 1832 in a duel before he was 21, and before he received recognition for his remarkable work. It may comfort the student to know that Galois' ability was not appreciated by his teachers.

**\*118. Algebraic solution of the general cubic equation.** The general cubic equation  $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$  ( $a_0 \neq 0$ ) can be written in the form

$$(1) \quad x^3 + bx^2 + cx + d = 0$$

by dividing through by  $a_0$ . Setting

$$(2) \quad x = y - \frac{b}{3},$$

equation (1) becomes

$$(3) \quad y^3 + py + q = 0$$

where  $p = c - \frac{b^2}{3}$  and  $q = d - \frac{bc}{3} + \frac{2b^3}{27}$ . Substituting

$$(4) \quad y = z - \frac{p}{3z}$$

in (3) and simplifying, we get  $z^3 - \frac{p^3}{27z^3} + q = 0$ . Multiplying by  $z^3$ , this becomes

$$(5) \quad z^6 + qz^3 - \frac{p^3}{27} = 0.$$

This is a quadratic in  $z^3$ . Solving it for  $z^3$  by the quadratic formula, we get  $\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right)/2$ , or

$$(6) \quad z^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

and

$$(7) \quad z^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

\* This section may be omitted on first reading.



By exercise 23, section 98, any number  $N$  has three cube roots, such that if  $\sqrt[3]{N}$  is one of them then the others are  $\omega \sqrt[3]{N}$  and  $\omega^2 \sqrt[3]{N}$ ,

where  $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$  is one of the imaginary cube roots of 1.

Hence denoting one of the cube roots of the right member of (6) by  $\alpha$  and one of the cube roots of the right member of (7) by  $\beta$ , we have, from (6) and (7), six values of  $x$ , namely  $\alpha, \omega\alpha, \omega^2\alpha, \beta, \omega\beta, \omega^2\beta$ . It is easy to verify that  $\alpha$  and  $\beta$  can be chosen so that

$$\alpha = \frac{-p}{3\beta}, \omega\alpha = \frac{-p}{3\omega^2\beta}, \omega^2\alpha = \frac{-p}{3\omega\beta},$$

because the product of the right members of (6) and (7) is  $(-p/3)^3$ . By (4), the roots of (3) are

$$y_1 = \alpha + \beta, y_2 = \omega\alpha + \omega^2\beta, y_3 = \omega^2\alpha + \omega\beta.$$

Hence, by (2), the roots of (1) are

$$(8) \quad r_1 = y_1 - \frac{b}{3}, r_2 = y_2 - \frac{b}{3}, r_3 = y_3 - \frac{b}{3}.$$

If one traces back the values of  $y_1, y_2, y_3, \alpha, \beta, p, q$ , we get algebraic expressions for  $r_1, r_2, r_3$  in terms of  $b, c, d$ .

*Note 1.* In practice, with given numerical coefficients, it is usually less tedious to solve a cubic by the methods explained in earlier sections of this chapter than by the method of this section. The principal interest of this section is that it proves that algebraic expressions for the roots of the general cubic in terms of the coefficients exist.\*

## EXERCISES

Following the steps in the text above, solve each of the following cubics:

$$1. x^3 + 2x^2 + 2x + 1 = 0. \quad 2. x^3 - 2x^2 - x + 2 = 0.$$

$$3. y^3 - 3\omega y - 2 = 0 \text{ where } \omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3} \text{ is an imaginary cube root of unity.}$$

$$4. y^3 - 6y - 2 = 0.$$

**\*119.** Algebraic solution of the general quartic equation. The general quartic† equation  $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$  ( $a_0 \neq 0$ ) can be written in the form

\* For a more detailed discussion of cubics, see L. E. Dickson, *Elementary Theory of Equations*, Wiley, 1914. W. V. Lovitt, *Elementary Theory of Equations*, Prentice-Hall, 1939, and L. Weisner, *Introduction to the Theory of Equations*, Macmillan, 1938.

† This section may be omitted on first reading.

‡ Or *biquadratic*, as it is often called.



$$(1) \quad x^4 + bx^3 + cx^2 + dx + e = 0$$

by dividing by  $a_0$ . Setting

$$(2) \quad x = y - \frac{b}{4}$$

equation (1) becomes

$$(3) \quad y^4 + gy^2 + hy + k = 0,$$

where  $g$ ,  $h$ , and  $k$  are certain polynomials in  $b$ ,  $c$ ,  $d$ ,  $e$  which should be calculated by the student. Adding and subtracting  $y^2z + \frac{z^2}{4}$ , where  $z$  is a constant whose value is to be determined presently, we get

$$y^4 + y^2z + \frac{z^2}{4} - y^2z - \frac{z^2}{4} + gy^2 + hy + k = 0$$

or

$$(4) \quad \left(y^2 + \frac{z}{2}\right)^2 - \left([z - g]y^2 - hy + \left[\frac{z^2}{4} - k\right]\right) = 0.$$

The quantity in the second parenthesis is a quadratic function of  $y$  which will be a perfect square if and only if its discriminant is zero, or

$$(5) \quad h^2 - 4[z - g]\left[\frac{z^2}{4} - k\right] = 0.$$

Simplifying this, we get the so-called **resolvent cubic** equation

$$(6) \quad z^3 - gz^2 - 4kz - (h^2 - 4gk) = 0.$$

Let  $z_1$  be any root of (6), which may be found by the method of the preceding section or some other method. Then (4) becomes

$$(7) \quad \left(y^2 + \frac{z_1}{2}\right)^2 - \left(y\sqrt{z_1 - g} - \frac{h}{2\sqrt{z_1 - g}}\right)^2 = 0$$

by virtue of (5). But an equation of the form  $A^2 - B^2 = 0$  is equivalent to two equations  $A + B = 0$  and  $A - B = 0$ . Hence (7) is equivalent to the two quadratics in  $y$ :

$$(8) \quad y^2 + \frac{z_1}{2} + y\sqrt{z_1 - g} - \frac{h}{2\sqrt{z_1 - g}} = 0$$

and

$$(9) \quad y^2 + \frac{z_1}{2} - y\sqrt{z_1 - g} + \frac{h}{2\sqrt{z_1 - g}} = 0.$$



Hence the four roots of (3) are the roots of (8) and (9). The four roots of (1) may be obtained by substituting these in (2).

*Note.* In practice, with given numerical coefficients, it is usually easier to solve quartic equations by the methods of earlier sections of this chapter than by the method of this section. The principal interest of this section is that it proves that there exist algebraic expressions for the roots of the general quartic in terms of the coefficients.\*

## EXERCISES

*Following the steps in the text above, solve each of the following quartics:*

1.  $x^4 + 2x + \frac{1}{2} = 0$ . (*Hint:* The resolvent cubic has a rational root.)

2.  $x^4 + 4x^2 + 32x - 48 = 0$ .

3. The discussion in the text breaks down if  $z_1 = g$ . (Why?) Show that then  $g$  satisfies (6) so that  $h = 0$ , and (3) becomes a quadratic in  $y^2$ , which is thus easily solved.

4. Solve  $x^4 + 4x^3 + 7x^2 + 10x + 3 = 0$ .

\* For a more detailed discussion of quartics, see L. E. Dickson, *Elementary Theory of Equations*, Wiley, 1914, and L. Weisner, *Introduction to the Theory of Equations*, Macmillan, 1938.



## CHAPTER XVI

### Determinants and Elimination Theory

After studying the solution of polynomial equations in one unknown in the preceding chapter, it is natural to investigate systems of equations in more than one unknown. The technique of solving such systems involves eliminating all unknowns but one, and therefore the theory of such systems may be called **elimination theory**.

**120. Solution of linear systems in two unknowns by determinants.** Let us solve the general system

$$\begin{aligned} (1) \quad & a_1x + b_1y = k_1 \\ (2) \quad & a_2x + b_2y = k_2 \end{aligned}$$

by multiplication and addition (section 59). Multiplying (1) by  $b_2$  and (2) by  $-b_1$  and adding, we get

$$(3) \quad (a_1b_2 - a_2b_1)x = k_1b_2 - k_2b_1.$$

Multiplying (1) by  $-a_2$  and (2) by  $a_1$  and adding, we get

$$(4) \quad (a_1b_2 - a_2b_1)y = a_1k_2 - a_2k_1.$$

Hence if  $a_1b_2 - a_2b_1 \neq 0$ , we have, from (3) and (4),

$$(5) \quad x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1}$$

$$(6) \quad y = \frac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1}.$$

The student should check by substitution.

These formulas for the solutions can be written in a convenient pattern if we introduce a new symbol.

**DEFINITION.** A symbol of the form  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ , consisting of four numbers called **elements** arranged in two (horizontal) rows and two (vertical) columns flanked by vertical bars, will be called a **determinant of order two** and will denote the number  $ad - bc$ . That is,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



For example,

$$\begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix} = 2(-2) - 4(-3) = 8.$$

Using determinants of order 2, the solutions of the system of equations (1) and (2) may be written as

$$(7) \quad x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

if  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ .

The determinant in the denominators of (7) is denoted by  $\Delta$ .

Formula (7) may be called **Cramer's rule** for the special case of a system of two equations in two unknowns. Note that the denominator is the same in the expressions for  $x$  and  $y$ . If the given equations are written in the form of (1) and (2) with like terms under one another in the left member and the constant terms in the right member, then the denominator for both unknowns consists of the determinant  $\Delta$  of the coefficients of the unknowns as they stand in the given equations. The numerator for either unknown is the same determinant except that the column of coefficients of the unknown being sought is replaced by the column of constant terms. When determinants of orders higher than 2 have been defined, it will turn out that this rule applies to systems of  $n$  linear equations in  $n$  unknowns.

**Example 1.** Solve the system  $\begin{cases} x = y + 1 \\ 2x + 3y - 12 = 0. \end{cases}$

**Solution.** Rewriting the equations as  $\begin{cases} x - y = 1 \\ 2x + 3y = 12, \end{cases}$  we find

that

$$\Delta = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 - (-2) = 5$$

$$\text{and } x = \frac{\begin{vmatrix} 1 & -1 \\ 12 & 3 \end{vmatrix}}{5} = \frac{15}{5} = 3, \quad y = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 12 \end{vmatrix}}{5} = \frac{10}{5} = 2.$$

The student should check by substitution.



## EXERCISES

*Evaluate each determinant:*

$$1. \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} \quad 2. \begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} \quad 3. \begin{vmatrix} 4 & -2 \\ 5 & -3 \end{vmatrix} \quad 4. \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}$$

*Solve, by means of determinants, and check:*

5-34. Exercises 5-34, section 59.

*Solve each of the following equations and check:*

$$35. \begin{vmatrix} 2 & 4 \\ 3 & x \end{vmatrix} = 0.$$

$$36. \begin{vmatrix} x & 7 \\ 2 & 3 \end{vmatrix} = 1.$$

$$37. \begin{vmatrix} 4-x & 2 \\ -1 & 1-x \end{vmatrix} = 0.$$

$$38. \begin{vmatrix} x-10 & 2 \\ 2+x & 2+x \end{vmatrix} = 0.$$

$$39. \text{ Show that } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

$$40. \text{ Show that } \begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

$$41. \text{ Show that } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}.$$

$$42. \text{ Show that } \begin{vmatrix} a_1 + a_1' & b_1 \\ a_2 + a_2' & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 \\ a_2' & b_2 \end{vmatrix}.$$

$$43. \text{ Show that } \begin{vmatrix} a_1 & ka_1 \\ a_2 & ka_2 \end{vmatrix} = 0.$$

121. Determinants of third order. A determinant of third order is a symbol of the form

$$(1) \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

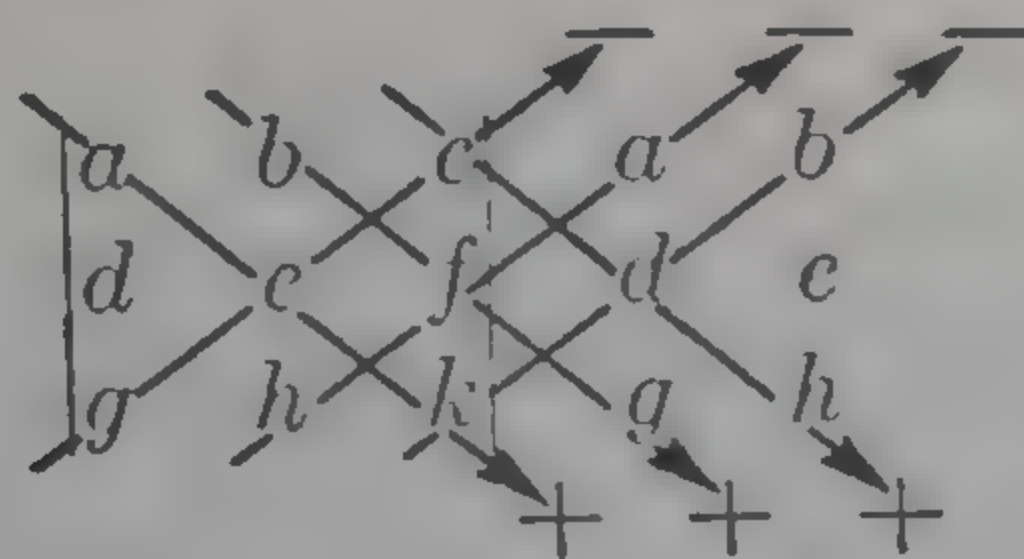
consisting of nine numbers called elements of the determinant, arranged in three (horizontal) rows and three (vertical) columns. The symbol (1) stands for the number

$$(2) \quad aek + bfg + cdh - gec - hfa - dbk$$

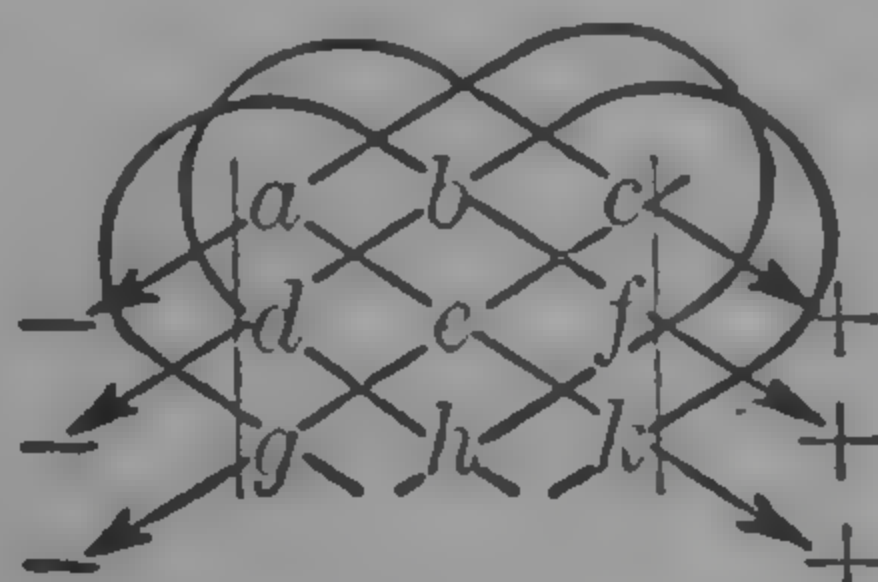
which is called the **value of the determinant (1)**. The expression (2) is called the **expansion of the determinant (1)**.

One easy way to remember (2) is to copy over the first two columns at the right of the determinant and take the "diagonal products" which go downward from left to right with a plus sign, and the "diagonal products" which go upward from left to right with a minus sign. The algebraic sum of these six terms is the expression (2), thus:





Another scheme for remembering (2) is to follow the arrows in the following diagram:



*Example.* Evaluate the determinant  $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$ .

*Solution.* By either scheme above, we get

$$2 \cdot 2 \cdot 1 + (-1) \cdot 3 \cdot 3 + 3 \cdot 1 \cdot (-2) - 3 \cdot 2 \cdot 3 - (-2) \cdot 3 \cdot 2 - 1 \cdot 1 \cdot (-1) = 4 - 9 - 6 - 18 + 12 + 1 = 17 - 33 = -16.$$

## EXERCISES

Evaluate each of the following determinants:

1.  $\begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{vmatrix}$ .

2.  $\begin{vmatrix} -1 & 3 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix}$ .

3.  $\begin{vmatrix} 4 & 2 & -1 \\ 4 & 4 & -1 \\ 9 & 7 & -1 \end{vmatrix}$ .

4.  $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 2 & 3 \end{vmatrix}$ .

5.  $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ .

6.  $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 4 & -2 \end{vmatrix}$ .

Solve each of the following equations and check:

7.  $\begin{vmatrix} 1 & x & -1 \\ 2 & 4 & 2 \\ 3 & 6 & -3 \end{vmatrix} = 0$ .

8.  $\begin{vmatrix} 4 & 2x & -1 \\ 2 & x & 3 \\ -2 & -1 & 2 \end{vmatrix} = 0$ .

9.  $\begin{vmatrix} 3x & 3 & 2 \\ 1 & x & 2 \\ -1 & -1 & 1 \end{vmatrix} = 0$ .

10.  $\begin{vmatrix} -2 & -1 & -2 \\ 4 & x & 1 \\ 6 & 3 & x \end{vmatrix} = 0$ .

Evaluate each of the following determinants:

11.  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ .

12.  $\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$ .



13. Show that 
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}.$$

14. Show that 
$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

15. Show that 
$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0.$$

122. Solution of linear equations in three unknowns by determinants. Let us solve the general system of three linear equations in three unknowns

$$\begin{aligned} (1) \quad & a_1x + b_1y + c_1z = k_1 \\ (2) \quad & a_2x + b_2y + c_2z = k_2 \\ (3) \quad & a_3x + b_3y + c_3z = k_3 \end{aligned}$$

by multiplication and addition. Multiplying (1) by  $c_2$  and (2) by  $-c_1$  and adding, we eliminate  $z$ , obtaining

$$(4) \quad (a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y = k_1c_2 - k_2c_1.$$

Multiplying (2) by  $c_3$  and (3) by  $-c_2$  and adding, we eliminate  $z$ , obtaining

$$(5) \quad (a_2c_3 - a_3c_2)x + (b_2c_3 - b_3c_2)y = k_2c_3 - k_3c_2.$$

Solving the system of equations (4) and (5) for  $x$  by any previous method, we get

$$\begin{aligned} & (a_1c_2b_2c_3 - a_2c_1b_2c_3 - a_1c_2b_3c_2 + a_2c_1b_3c_2 - a_2c_3b_1c_2 + a_2c_3b_2c_1 \\ & + a_3c_2b_1c_2 - a_3c_2b_2c_1)x = b_2c_3k_1c_2 - b_2c_3k_2c_1 - b_3c_2k_1c_2 \\ & + b_3c_2k_2c_1 - b_1c_2k_2c_3 + b_2c_1k_2c_3 + b_1c_2k_3c_2 - b_2c_1k_3c_2 \end{aligned}$$

or, simplifying,

$$\begin{aligned} (6) \quad & c_2(a_1b_2c_3 - a_1b_3c_2 + a_2c_1b_3 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1)x \\ & = c_2(k_1b_2c_3 - k_1b_3c_2 + k_2b_3c_1 - k_2b_1c_3 + k_3b_1c_2 - k_3b_2c_1) \end{aligned}$$

or\*

$$(7) \quad x = \frac{k_1b_2c_3 + k_2b_3c_1 + k_3b_1c_2 - k_3b_2c_1 - k_2b_1c_3 - k_1b_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2},$$

provided the denominator is not zero. Solving similarly for  $y$  and  $z$ , we get

\* This step is valid only if  $c_2 \neq 0$ . But the check by substitution reveals that the result is correct even if  $c_2 = 0$ . Another proof of these formulas will be given in section 127.



$$(8) \quad y = \frac{a_1 k_2 c_3 + a_2 k_3 c_1 + a_3 k_1 c_2 - a_3 k_2 c_1 - a_2 k_1 c_3 - a_1 k_3 c_2}{a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2},$$

$$(9) \quad z = \frac{a_1 b_2 k_3 + a_2 b_3 k_1 + a_3 b_1 k_2 - a_3 b_2 k_1 - a_2 b_1 k_3 - a_1 b_3 k_2}{a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2}.$$

The student may check the solution (7), (8), and (9) by substitution in the original equations.

Comparing (7), (8), and (9) with the definition of determinant of order 3, we have the following result.

If  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ , then the solution of the system of equations (1), (2), and (3) is given by

$$(10) \quad x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\Delta}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\Delta}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\Delta}.$$

Formula (10) is known as **Cramer's rule** for the special case of three linear equations in three unknowns. Note that the denominator for each unknown is the determinant  $\Delta$  of the coefficients of the unknowns as they appear in the given equations when they are written with like terms under each other on the left and constant terms on the right. The numerator for any unknown is the same determinant  $\Delta$  except that the column of coefficients of that unknown is replaced by the column of constant terms. The same rule will apply verbatim to systems of  $n$  equations in  $n$  unknowns after we have defined determinants of order  $n$ .

*Example.* Solve the system

$$\begin{aligned} x + y + z &= 4 \\ 2x - y - 2z &= -1 \\ x - 2y - z &= 1. \end{aligned}$$

*Solution.* We have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = 1 - 4 - 2 + 1 + 2 - 4 = -6,$$

$$x = \frac{\begin{vmatrix} 4 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}}{-6} = \frac{4 + 2 - 2 + 1 - 1 - 16}{-6} = \frac{-12}{-6} = 2,$$



$$y = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix}}{-6} = \frac{1 + 2 - 8 + 1 + 8 + 2}{-6} = \frac{6}{-6} = -1,$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 4 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix}}{-6} = \frac{-1 - 16 - 1 + 4 - 2 - 2}{-6} = \frac{-18}{-6} = 3.$$

The student may check by substitution in the original equations.

## EXERCISES

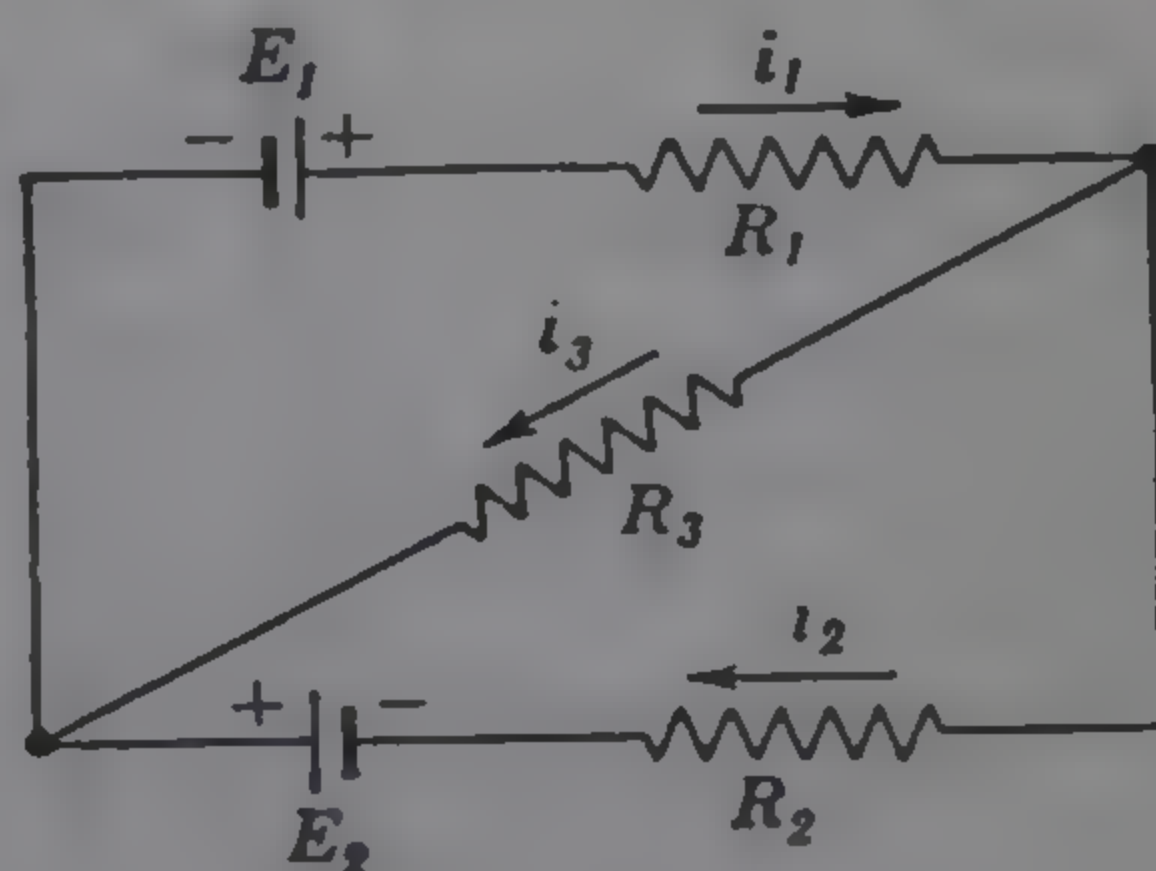
*Solve by means of determinants, and check:*

1-10. Exercises 1-10 of section 61.

11. In the electrical circuit pictured in the accompanying diagram, suppose  $E_1 = 10$  volts,  $E_2 = 20$  volts,  $R_1 = 5$  ohms,  $R_2 = 10$  ohms,  $R_3 = 5$  ohms. We find by Kirchoff's laws, the system of equations

$$\begin{cases} i_1 - i_2 - i_3 = 0 \\ 5i_1 + 5i_3 = 10 \\ 10i_2 - 5i_3 = 20. \end{cases}$$

Find the currents  $i_1$ ,  $i_2$ ,  $i_3$  in amperes.



\*123. Area of a triangle by determinants. We digress to point out an interesting geometric application of determinants.

THEOREM. If the vertices of a triangle have the rectangular coordinates  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ , the area of the triangle is given by the absolute value of

$$(1) \quad \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$

*Proof.* Suppose that all three vertices are in the first quadrant, as in Fig. 56; it can be shown that the result is unaffected if they are anywhere else. Then the area of triangle  $ABC$  equals the area of trapezoid  $ABED$  plus the area of trapezoid  $BCFE$  minus the area of trapezoid  $ACFD$ . Now trapezoid  $ABED$  has the altitude  $DE = x_2 - x_1$  and the bases  $y_1$  and  $y_2$ . Hence the area of  $ABED$  is

$$\frac{1}{2}h(b + b') = \frac{1}{2}(x_2 - x_1)(y_1 + y_2) = \frac{1}{2}(x_2y_1 + x_2y_2 - x_1y_1 - x_1y_2).$$

\*This section may be omitted on first reading.



Similarly the area of  $BCFE$  is

$$\frac{1}{2}(x_3 - x_2)(y_2 + y_3) = \frac{1}{2}(x_3y_2 + x_3y_3 - x_2y_2 - x_2y_3),$$

and the area of  $ACFD$  is

$$\frac{1}{2}(x_3 - x_1)(y_1 + y_3) = \frac{1}{2}(x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3).$$

Hence the area of  $ABC$  is

$$\frac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_3y_1 - x_1y_3).$$

But this is the expansion of (1), except for sign, which proves the theorem.

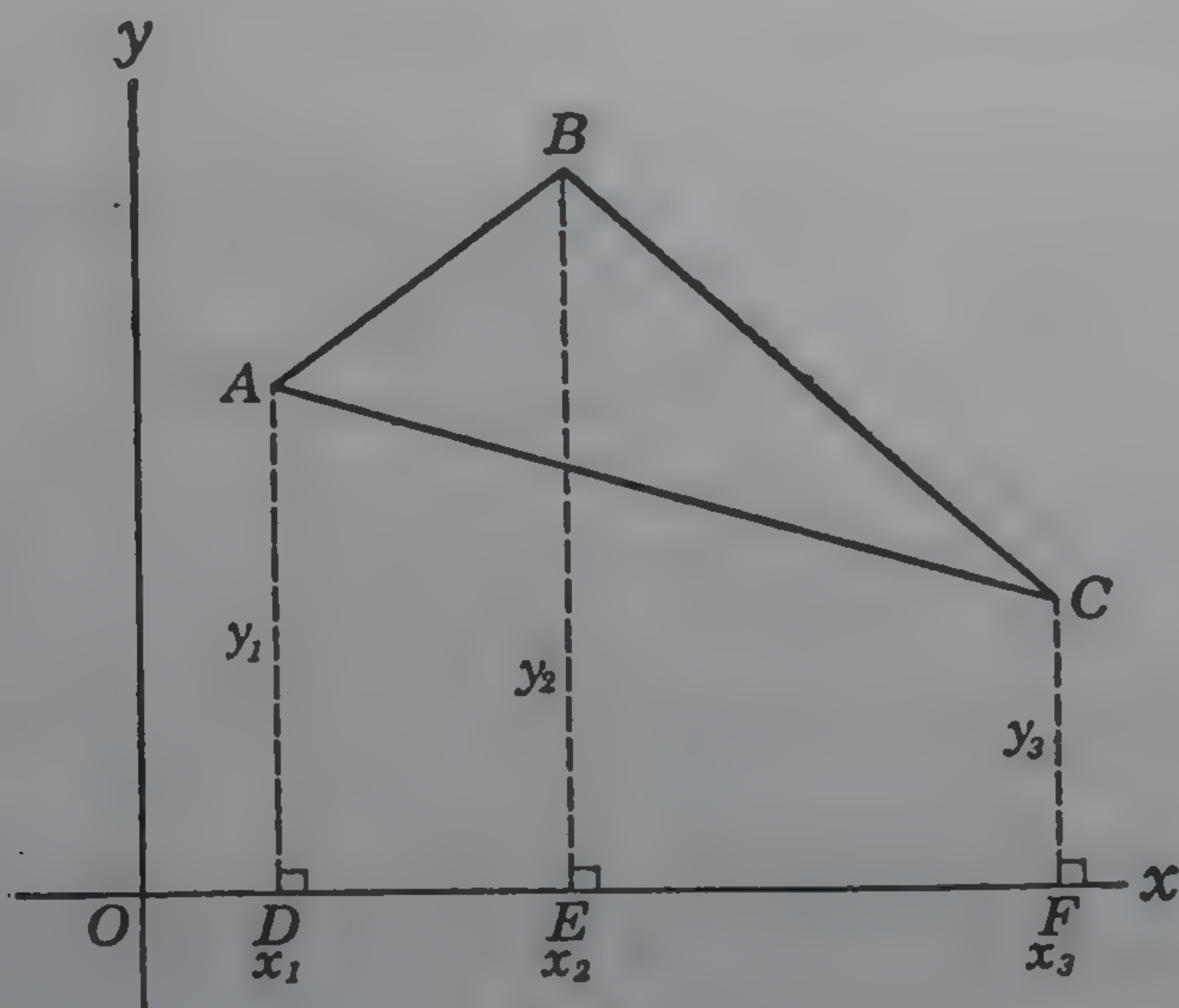


Fig. 56.

**COROLLARY 1.** *Three points  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$  are in the same straight line if and only if*

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0.$$

*Proof.* Left to the student.

**COROLLARY 2.** *An equation of the straight line passing through two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is*

$$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 0.$$

*Proof.* Left to the student.



## EXERCISES

Find, by determinants, the area of the triangle having the following vertices:

1.  $(-1, -2), (3, -2), (1, 5)$ .
2.  $(0, 0), (0, -5), (-12, 0)$ .
3.  $(-2, 3), (-6, -3), (1, 1)$ .
4.  $(-4, -3), (6, -5), (-3, 2)$ .

Show by determinants that the following sets of points are collinear:

5.  $(-3, -3), (3, 5), (6, 9)$ .
6.  $(1, 2), (2, 6), (4, 14)$ .

Write in determinant form an equation of the straight line joining the following pairs of points, expand and simplify:

7.  $(2, 3), (5, 6)$ .
8.  $(2, -3), (-1, -2)$ .
9.  $(2, -3), (2, 5)$ .
10.  $(3, -2), (-1, -2)$ .

**124. Determinants of order  $n$ .** Consider any arrangement of the whole numbers  $1, 2, 3, \dots, n$  in some order in a line. An **inversion** is said to occur whenever a larger number precedes a smaller one.

*Example 1.* The arrangement 14325 has three inversions because 4 precedes 3, 4 precedes 2, and 3 precedes 2.

In an arrangement of letters in a line, an **inversion** is said to occur whenever a letter precedes one which occurs earlier than it in alphabetical order.

*Example 2.* The arrangement  $b\ e\ c\ d\ a$  has six inversions because  $b$  precedes  $a$ ,  $e$  precedes  $c$ ,  $e$  precedes  $d$ ,  $e$  precedes  $a$ ,  $c$  precedes  $a$ , and  $d$  precedes  $a$ .

**DEFINITION.** A **determinant of order  $n$**  is a symbol consisting of  $n^2$  numbers, called **elements of the determinant**, arranged in  $n$  (horizontal) rows and  $n$  (vertical) columns, written in the form

$$(1) \quad \begin{vmatrix} a_1 & b_1 & c_1 & \dots & h_1 \\ a_2 & b_2 & c_2 & \dots & h_2 \\ a_3 & b_3 & c_3 & \dots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & b_n & c_n & \dots & h_n \end{vmatrix}.$$

The symbol (1) stands for the algebraic sum of all possible products of  $n$  factors each such that



(a) *each product has as its factors one and only one element from each row and each column;*

(b) *each product is preceded by a plus or minus sign according as the number of inversions in the subscripts is even or odd, after the letters have been written in the order in which they appear in the first row of the determinant.*

The algebraic sum of these products is called the **expansion of the determinant**, and each product is called a **term** of the expansion. The number resulting from calculating the expansion is called the **value of the determinant**. The elements lying in a straight line from the upper left-hand corner to the lower right-hand corner of the determinant constitute the **principal diagonal** of the determinant.

*Example 3.* Consider the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

of order 3. Every term in the expansion of  $D$  must be a product of an  $a$ , a  $b$ , and a  $c$ , and the subscripts must be the numbers 1, 2, 3 in some order. No matter which of the three numbers is chosen as the subscript for  $a$ , there are two choices for the subscript for  $b$ . Hence there are  $3 \cdot 2 \cdot 1$  choices in all, since the subscript for  $c$  is determined by the choice of the other two. Therefore there are 6 terms in the expansion, namely

$$a_1b_2c_3, a_2b_3c_1, a_3b_1c_2, a_3b_2c_1, a_2b_1c_3, \text{ and } a_1b_3c_2.$$

The numbers of inversions in these terms are 0, 2, 2, 3, 1, 1, respectively. Hence

$$D = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$$

which is in agreement with the definition of section 121.

For  $n > 3$ , it is too tedious to expand a determinant directly from this definition. Better ways of evaluating determinants of order higher than 3 will emerge from the following sections.

A more systematic notation is obtained by using double subscripts, such as  $a_{32}$ , where the first subscript indicates the row and the second indicates the column. Thus  $a_{32}$  is the element in the third row and second column, while  $a_{23}$  is the element in the second row and third column. Then part (b) of the definition above should read: each term is preceded by a plus or minus



sign according as the number of inversions in the row-subscripts is even or odd, after the factors have been written so that the column subscripts appear in order of magnitude. The determinant (1) is written in this notation as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3n} \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix}.$$

Determinants were invented by G. W. Leibniz (German, 1646–1716) and also independently by Seki-Kowa (Japanese, 1642–1708). Because of the isolation of Japan, the latter's work had no influence outside of Japan.

## EXERCISES

1. Write out the expansion of

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

according to the definition of this section.

- ★ 2. Write out the expansion of

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

according to the definition of this section.

**125. Some properties of determinants.** Some useful properties of determinants are given in the following theorems. While examples will be determinants of order 3, for convenience in writing, the theorems apply to determinants of any order.

Let the symbol  $n!$ , read “factorial  $n$ ,” denote the number  $n(n-1)(n-2) \cdots 1$ , where  $n$  is any positive integer. For example,  $3! = 3 \cdot 2 \cdot 1 = 6$ ,  $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ , and so on.

**THEOREM 1.** *The expansion of a determinant of order  $n$  contains  $n!$  terms.*

*Proof.* By (a) of the definition of section 124, in forming a term of the expansion, there are  $n$  choices for the subscript attached to the  $a$ -factor,  $(n-1)$  choices for the subscript attached to the

★ May be omitted without disturbing continuity.



$b$ -factor, and so on. But there are  $n$  factors in each term. Hence there are  $n(n-1)(n-2) \cdots 1$  terms.

**THEOREM 2.** *The value of a determinant is unchanged if corresponding rows and columns are interchanged.*

*Proof for order 3.* The student should expand the determinants

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

by the scheme of section 121, and verify directly that the resulting expansions are equal.

*Proof for order  $n$ .* Suppose as above that  $D$  has the same letter for elements in the same column, while  $D'$  has the same letter for elements in the same row. By definition of determinant,  $D$  and  $D'$  have the same terms in their expansions except possibly for signs. (In reading the definition for  $D'$  we must of course reverse the roles of letters and subscripts.) We shall show that corresponding terms of  $D$  and  $D'$  have the same sign by showing that they have the same number of inversions. Suppose a term of  $D$  is written with the letters in alphabetical order. The inversions in the subscripts can be removed by interchanging adjacent factors, repeatedly. But each such interchange introduces one inversion in the letters. Hence, when the subscripts have been restored to order of magnitude, we have introduced into the letters the same number of inversions that the subscripts had originally.

From theorem 2 it follows that *for every theorem concerning the columns of a determinant, there is a corresponding theorem concerning the rows, and conversely.* The proof of one of these theorems automatically suffices to prove the other.

**THEOREM 3.** *If all the elements of some column (or row) of a determinant are equal to zero, the value of the determinant is zero.*

*Proof.* By definition, each term of the expansion contains an element from the column of zeros as a factor. Hence each term of the expansion is equal to zero.

**THEOREM 4.** *If two columns (or rows) of a determinant are interchanged, the sign of the determinant is reversed.*

*Proof.* If we interchange two adjacent rows, we interchange two adjacent subscripts in each term of the expansion. This either increases or decreases the number of inversions in the



subscripts of each term by one. Therefore the sign of each term and hence of the entire determinant is reversed.

Now suppose we interchange two rows separated by  $m$  intermediate rows. It will take  $m$  interchanges of adjacent rows to bring the lower row to just below the upper one, one to interchange them, and  $m$  more to bring the upper row to where the lower one was originally. Hence there have been  $2m + 1$  interchanges of adjacent rows. Since  $2m + 1$  is odd, this is equivalent to reversing the sign of the determinant an odd number of times, which leaves the sign reversed.

*Example 1.* If

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

then  $D' = -D$ . The student should verify this by expanding  $D$  and  $D'$ .

**THEOREM 5.** *If two columns (or rows) of a determinant are identical, the value of the determinant is zero.*

*Proof.* If we interchange the two identical columns of the determinant  $D$ , it is unaltered in value. But by theorem 4, its sign must be reversed. Hence  $D = -D$ , or  $2D = 0$ , or  $D = 0$ .

*Example 2.* By inspection,

$$\begin{vmatrix} 2 & 2 & 7 \\ 3 & 3 & -2 \\ 4 & 4 & 5 \end{vmatrix} = 0,$$

since the first two columns are identical. The student should verify this by expanding the determinant.

**THEOREM 6.** *If each of the elements of a column (or row) of a determinant is multiplied by the same quantity  $k$ , the value of the determinant is multiplied by  $k$ .*

*Proof.* Since each term in the expansion has as one factor an element from the column in question, each term is multiplied by  $k$ . The factor  $k$  is thus common to all the terms of the expansion and hence the entire determinant is multiplied by  $k$ .

*Example 3.*

$$\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The student should verify this by expanding both determinants.



**THEOREM 7.** *If corresponding elements of two columns (or rows) of a determinant are proportional, the value of the determinant is zero.*

*Proof.* The hypothesis that corresponding elements of two columns are proportional means that there is a constant factor of proportionality  $k$  such that every element of one column is  $k$  times the corresponding element of the other column. Then, if this factor is taken out, by theorem 6, the two columns become identical, and the determinant is zero, by theorem 5. Illustrating this for a third order determinant, we have

$$\begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = k \cdot 0 = 0.$$

**Example 4.** By inspection,

$$\begin{vmatrix} 2 & 6 & 5 \\ 3 & 9 & 4 \\ -4 & -12 & 3 \end{vmatrix} = 0,$$

since the first two columns are proportional.

**THEOREM 8.** *If each element of a certain column (or row) of a determinant is expressed as a sum of two terms, the whole determinant may be expressed as a sum of two determinants, each of which has one of these terms in the corresponding position and both of which are identical with the given determinant in all other elements.*

Illustrating this for a determinant of third order, we may write

$$(1) \quad \begin{vmatrix} (a_1 + a_1') & b_1 & c_1 \\ (a_2 + a_2') & b_2 & c_2 \\ (a_3 + a_3') & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix}.$$

*Proof.* Consider a typical term of the left member of (1), such as  $(a_1 + a_1')b_2c_3$ . It is equal to  $a_1b_2c_3 + a_1'b_2c_3$ ; but each of these two terms is a term of one of the determinants in the right member of (1). Hence the expansion of the left member of (1) is the sum of the expansions of the determinants in the right member of (1). While the proof has been given for the illustration, which is a third order determinant, it is clear that the argument applies to determinants of any order.

**THEOREM 9.** *If to every element of any column (or row) of a determinant is added the corresponding element of any other column*



(or row), each multiplied by the same quantity  $k$ , then the value of the determinant is unaltered.

Illustrating this for a determinant of order 3, we may write

$$(2) \quad \begin{vmatrix} a_1 + kc_1 & b_1 & c_1 \\ a_2 + kc_2 & b_2 & c_2 \\ a_3 + kc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

*Proof.* In the illustration,

$$\begin{vmatrix} a_1 + kc_1 & b_1 & c_1 \\ a_2 + kc_2 & b_2 & c_2 \\ a_3 + kc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & b_1 & c_1 \\ kc_2 & b_2 & c_2 \\ kc_3 & b_3 & c_3 \end{vmatrix},$$

by theorem 8. By theorem 7 the last determinant is zero. This proves the theorem for the illustration. Clearly the argument applies in general.

*Example 5.* Write a determinant equal to

$$\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{vmatrix}$$

but having zeros everywhere in the first row except the upper right-hand corner.

*Solution.* Multiplying the (elements of the) last column by  $k = 2$  and adding to the (corresponding elements of the) first column, we get

$$\begin{vmatrix} 0 & 3 & 1 \\ 7 & 2 & 3 \\ 8 & 3 & 3 \end{vmatrix}.$$

Multiplying the last column by  $-3$  and adding to the second column, we get

$$\begin{vmatrix} 0 & 0 & 1 \\ 7 & -7 & 3 \\ 8 & -6 & 3 \end{vmatrix}.$$

## EXERCISES

1. Without expanding show that  $\begin{vmatrix} 2 & 1 & 4 & 0 \\ 3 & 2 & 6 & 1 \\ 4 & 3 & 8 & -1 \\ -1 & 4 & -2 & 2 \end{vmatrix} = 0.$

2. Without expanding show that  $\begin{vmatrix} 1 & 2 & -3 \\ 4 & 1 & 2 \\ 3 & 6 & -9 \end{vmatrix} = 0.$



3. Without expanding find the root of the equation  $\begin{vmatrix} 1 & 2 & 4 \\ 3 & x & 12 \\ -1 & 2 & 1 \end{vmatrix} = 0$ .

4. Write a determinant equal to  $\begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & 1 \\ 4 & -2 & 2 \end{vmatrix}$  but having zeros

everywhere in the first row except the upper left-hand corner. (*Hint:* See example 5 above.)

5. Write a determinant equal to  $\begin{vmatrix} 2 & 1 & -3 \\ -5 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix}$  but having zeros

everywhere in the first column except the lower left-hand corner.

6. Write a determinant equal to  $\begin{vmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & -1 & -3 \\ 3 & 0 & 2 & 0 \\ 4 & 3 & 1 & 2 \end{vmatrix}$  but having

zeros everywhere in the third column except for the element in the last row.

7. Write a determinant equal to that given in exercise 6 but having zeros everywhere in the second row except for the element in the first column.

\*8. Without expanding show that  $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$ .

**126. Expansion by minors.** By the **minor** of an element in a determinant of order  $n$  is meant the determinant of order  $n - 1$  obtained by removing both the row and the column to which the given element belongs. The minor of an element is denoted by the corresponding capital letter; thus the minor of  $b_1$  is denoted by  $B_1$ , and so on. For example, in the determinant

$$(1) \quad D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

the minor of  $a_1$  is

$$A_1 = \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix},$$

\* May be omitted without disturbing continuity.



the minor of  $b_3$  is

$$B_3 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_4 & c_4 & d_4 \end{vmatrix},$$

and so on.

**THEOREM 1.** *The value of a determinant  $D$  may be obtained as follows:*

- (a) *Select any column (or row);*
- (b) *Multiply each element in this column (or row) by its minor;*
- (c) *Insert a plus or minus sign before each of these products obtained in (b) according to the following rule: if the element is in the  $i$ th row and  $j$ th column, we use a plus sign if  $i + j$  is even and a minus sign if  $i + j$  is odd;*
- (d) *Then  $D$  is equal to the algebraic sum of the signed terms obtained in (c).*

For example, the value of  $D$  in (1) may be obtained in any of the following ways:

$$\begin{aligned} D &= a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4, \\ \text{or, } D &= -b_1B_1 + b_2B_2 - b_3B_3 + b_4B_4, \\ \text{or, } D &= a_3A_3 - b_3B_3 + c_3C_3 - d_3D_3, \\ \text{or, } D &= -a_4A_4 + b_4B_4 - c_4C_4 + d_4D_4, \end{aligned}$$

and so on.

The signs determined in (c) of theorem 1 may be found also by means of the so-called "checkerboard rule";

$$\begin{vmatrix} + & - & + & - & \cdots & \cdot \\ - & + & - & + & \cdots & \cdot \\ + & - & + & - & \cdots & \cdot \\ - & + & - & + & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \end{vmatrix};$$

that is, the signs alternate, beginning with plus in the upper left-hand corner of the determinant.

*Example 1.* Evaluate by expanding by minors:

$$D = \begin{vmatrix} 2 & -1 & 4 \\ 5 & 1 & -2 \\ 3 & -3 & -4 \end{vmatrix}.$$



*Solution.* Selecting the third column to work with, we have

$$\begin{aligned} D &= 4 \begin{vmatrix} 5 & 1 \\ 3 & -3 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} + (-4) \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} \\ &= 4(-18) + 2(-3) - 4(7) = -72 - 6 - 28 = -106. \end{aligned}$$

*Example 2.* Evaluate by expanding by minors:

$$(2) \quad D = \begin{vmatrix} 1 & -1 & 3 & 2 \\ 2 & -2 & 2 & -1 \\ 3 & 0 & 0 & -2 \\ 4 & 3 & 1 & 1 \end{vmatrix}.$$

*Solution.* In order to simplify the calculation, we use the third row, since it contains two zeros. Thus

$$\begin{aligned} D &= 3 \begin{vmatrix} -1 & 3 & 2 \\ -2 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 2 & -1 \\ 4 & 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 & 2 \\ 2 & -2 & -1 \\ 4 & 3 & 1 \end{vmatrix} \\ &\quad - (-2) \begin{vmatrix} 1 & -1 & 3 \\ 2 & -2 & 2 \\ 4 & 3 & 1 \end{vmatrix}. \end{aligned}$$

The second and third minors need not be evaluated nor even written down, since they are to be multiplied by zero. Hence  $D = 3(-22) + 2(28) = -66 + 56 = -10$ .

Expansion by minors reduces the task of evaluating a determinant of order  $n$  to that of evaluating several determinants of order  $n - 1$ . The technique of example 5, section 125, may be used to produce zeros by a judicious choice of the multiplier  $k$ , thus diminishing the number of minors to be worked out in an expansion by minors.

*Example 3.* In (2) we may add  $-3$  times the last row to the first row, and add  $-2$  times the last row to the second row, obtaining

$$D = \begin{vmatrix} -11 & -10 & 0 & -1 \\ -6 & -8 & 0 & -3 \\ 3 & 0 & 0 & -2 \\ 4 & 3 & 1 & 1 \end{vmatrix}.$$

Using the third column, we have only one minor to evaluate, obtaining

$$D = -1 \begin{vmatrix} -11 & -10 & -1 \\ -6 & -8 & -3 \\ 3 & 0 & -2 \end{vmatrix} = -10.$$



*Proof of theorem 1.* (A) The sum of all terms in the expansion of  $D$  which have  $a_1$  as a factor is equal to  $a_1A_1$ . This is so because each such term is the product of  $a_1$  by one element from each of the remaining rows and columns of  $D$ , that is, the product of  $a_1$  by a term of the expansion of  $A_1$ ; furthermore, the sign before each such term of  $D$  is the same as the corresponding term of  $A_1$ , for writing  $a_1$  to the left of any term of  $A_1$  cannot alter the number of inversions in either subscripts or letters.

(B) Let  $x$  be the element in the  $i$ th row and  $j$ th column of  $D$ , and let  $D'$  be the determinant formed from  $D$  by successive interchanges of rows with rows and columns with columns so that  $x$  is in the upper left-hand corner of  $D'$ . Since it takes  $i - 1$  interchanges of adjacent rows to bring  $x$  to the top row, and  $j - 1$  interchanges of adjacent columns to bring it to the first column, we have

$$D' = (-1)^{i+j-2}D = (-1)^{i+j}D,$$

since each such interchange multiplies the value of the determinant by  $-1$ . By (A), the sum of all terms in the expansion of  $D'$  which have  $x$  as a factor is equal to  $xX$  where  $X$  is the minor of  $x$  in  $D'$ . But the minor of  $x$  in  $D'$  is the same as its minor in  $D$ . Hence the sum of all terms having  $x$  as a factor in the expansion of  $D$  is equal to  $(-1)^{i+j}xX$ .

(C) Each term in the expansion of  $D$  contains as a factor one and only one element of a given column (or row). By (B), the sum of all the terms containing each element of the given column (or row) is equal to this element times its minor times  $(-1)^{i+j}$  where the element is in the  $i$ th row and  $j$ th column. This completes the proof of theorem 1.

**THEOREM 2.** *If, in the expansion of a determinant by minors, using the elements of a certain column (or row), we replace the elements of this column (or row) by the corresponding elements of any other column (or row), the resulting expression has the value zero.*

*Proof.* Consider the expansion of (1):

$$D = -b_1B_1 + b_2B_2 - b_3B_3 + b_4B_4.$$

Replacing  $b_1, b_2, b_3, b_4$  by  $a_1, a_2, a_3, a_4$  respectively in the right member, we get

$$(3) \quad -a_1B_1 + a_2B_2 - a_3B_3 + a_4B_4.$$

But this is the expansion of the determinant



$$\begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_2 & a_2 & c_2 & d_2 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

according to minors of the second column, and is therefore equal to zero by theorem 5, section 125. Clearly, any expression of the type (3) may be treated similarly, and the argument obviously applies to determinants of any order.

*Note.* If  $x$  is the element in the  $i$ th row and  $j$ th column and  $X$  is its minor, then the quantity  $(-1)^{i+j}X$  is often called the **cofactor** of  $x$ .

### EXERCISES

1. Expand  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  by minors of the first column and verify that the same result is obtained as when the scheme of section 121 is used.

2. Expand  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$  by minors of the first column and verify that the same result is obtained as by the scheme of section 120. (We here regard an element as a determinant of order 1.)

*Expand each of the following by minors:*

3–6. The determinants of exercises 3–6 of section 121.

7.  $\begin{vmatrix} 3 & -1 & 3 \\ 2 & 5 & -3 \\ 5 & 4 & -1 \end{vmatrix}$

8.  $\begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & 1 \\ -3 & 2 & 1 \end{vmatrix}$

9.  $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 0 & -1 & -3 \end{vmatrix}$

10.  $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 1 & 1 & 0 & 2 \\ 2 & -1 & 0 & 3 \\ -2 & -1 & 1 & 2 \end{vmatrix}$

11.  $\begin{vmatrix} 2 & -2 & 1 & 3 \\ 0 & 2 & -1 & -1 \\ 2 & -3 & 2 & 4 \\ 0 & -1 & 1 & 1 \end{vmatrix}$

12.  $\begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & -2 \\ 1 & 3 & 2 & -1 \\ -3 & -2 & 1 & 2 \end{vmatrix}$

13.  $\begin{vmatrix} 3 & -2 & 6 & 4 \\ 1 & 0 & 2 & -1 \\ 5 & 4 & 3 & 0 \\ 2 & 2 & -5 & 6 \end{vmatrix}$

14.  $\begin{vmatrix} 1 & 2 & 1 & 1 \\ -1 & -3 & 1 & 2 \\ 2 & 2 & 3 & -2 \\ 3 & 2 & 1 & 1 \end{vmatrix}$

15.  $D = \begin{vmatrix} -1 & 2 & -1 & 1 & 1 \\ -4 & 4 & -1 & 1 & 1 \\ -4 & 6 & -3 & 2 & 3 \\ 2 & -4 & 1 & 1 & 2 \\ -1 & 1 & -1 & 1 & 0 \end{vmatrix}$

(*Hint:* First express  $D$  in terms of minors [of fourth order] of some row or column. Then evaluate each of these minors separately and insert their values.)



127. Solution of linear systems in  $n$  unknowns by determinants. Consider a system of  $n$  linear equations in  $n$  unknowns. For example, for  $n = 4$ ,

$$\begin{array}{lcl} (1) & a_1x + b_1y + c_1z + d_1t = k_1 & \\ (2) & a_2x + b_2y + c_2z + d_2t = k_2 & \\ (3) & a_3x + b_3y + c_3z + d_3t = k_3 & \\ (4) & a_4x + b_4y + c_4z + d_4t = k_4. & \end{array} \quad \left. \vphantom{\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}} \right\}$$

Let  $\Delta$  be the determinant of the coefficients as they appear in the equations when like terms are written below one another; that is,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Then we have the following theorem, known as Cramer's rule, after G. Cramer (Swiss, 1704–1752).

**THEOREM 1.** *If  $\Delta \neq 0$ , the solutions of the above system are given by*

$$(5) \quad x = \frac{N_1}{\Delta}, y = \frac{N_2}{\Delta}, z = \frac{N_3}{\Delta}, t = \frac{N_4}{\Delta}$$

where

$$N_1 = \begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}, N_2 = \begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix},$$

$$N_3 = \begin{vmatrix} a_1 & b_1 & k_1 & d_1 \\ a_2 & b_2 & k_2 & d_2 \\ a_3 & b_3 & k_3 & d_3 \\ a_4 & b_4 & k_4 & d_4 \end{vmatrix}, \text{ and } N_4 = \begin{vmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}.$$

*Proof.* (A) Multiplying equation (1) by  $A_1$ , (2) by  $-A_2$ , (3) by  $A_3$ , (4) by  $-A_4$ , and adding, we get

$$\begin{aligned} (a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4)x + (b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4)y \\ + (c_1A_1 - c_2A_2 + c_3A_3 - c_4A_4)z + (d_1A_1 - d_2A_2 + d_3A_3 - d_4A_4)t = (k_1A_1 - k_2A_2 + k_3A_3 - k_4A_4). \end{aligned}$$

By theorems 1 and 2 of section 126, this reduces to  $\Delta \cdot x = N_1$ . Since  $\Delta \neq 0$ , this yields  $x = N_1/\Delta$ , which is the first equality in (5). Similarly, multiplying (1), (2), (3), (4) by  $-B_1$ ,  $B_2$ ,  $-B_3$ ,  $B_4$  respectively, and adding, we get  $\Delta \cdot y = N_2$ , and so on.



This proves that if there exist values of  $x, y, z, t$  which satisfy the given equations, they can only be the values given by (5).

(B) To show, conversely, that these values really satisfy these equations, we may check by substitution, or proceed as follows.

The determinant of order 5

$$\begin{vmatrix} k_1 & a_1 & b_1 & c_1 & d_1 \\ k_1 & a_1 & b_1 & c_1 & d_1 \\ k_2 & a_2 & b_2 & c_2 & d_2 \\ k_3 & a_3 & b_3 & c_3 & d_3 \\ k_4 & a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0,$$

since the first two rows are identical. Expanding it by minors of the first row, we get

$$k_1\Delta - a_1N_1 - b_1N_2 - c_1N_3 - d_1N_4 = 0.$$

Transposing  $k_1\Delta$  and dividing by  $-\Delta (\neq 0)$ , we get

$$a_1 \frac{N_1}{\Delta} + b_1 \frac{N_2}{\Delta} + c_1 \frac{N_3}{\Delta} + d_1 \frac{N_4}{\Delta} = k_1,$$

which proves that the values (5) satisfy equation (1). A similar device shows that they satisfy the other equations. This completes the proof for the case  $n = 4$ . It is clear that the argument applies as well for any value of  $n$ .

Cramer's rule can be stated in words as follows: *In a system of  $n$  linear equations in  $n$  unknowns, let  $\Delta$  be the determinant of the coefficients as they appear in the equations after like terms have been written below each other in the left member, and constant terms in the right member. Then if  $\Delta \neq 0$ , the solution can be written down by expressing each unknown as a quotient of two determinants, the denominator being  $\Delta$  and the numerator for a given unknown being the determinant obtained from  $\Delta$  by replacing the column of coefficients of that unknown by the column of constant terms.*

*Example.* Solve the system 
$$\begin{cases} x + 2y + z = 3 \\ 2x - 2z + t = 6 \\ 4y + 3z + 2t = -1 \\ -x + 6y - z - t = 2. \end{cases}$$

*Solution.* The denominator  $\Delta = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & -2 & 1 \\ 0 & 4 & 3 & 2 \\ -1 & 6 & -1 & -1 \end{vmatrix} = -92.$



The numerator for  $x$  is  $N_1 = \begin{vmatrix} 2 & 2 & 1 & 0 \\ 6 & 0 & -2 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & 6 & -1 & -1 \end{vmatrix} = -184.$

Hence  $x = \frac{-184}{-92} = 2.$  Similarly  $y = \frac{-46}{-92} = \frac{1}{2}, z = \frac{92}{-92} = -1,$   
and  $t = \frac{0}{-92} = 0.$

The student may check by substituting in the given equations.

*Note.* While determinants are valuable for the theory of systems of linear equations, the methods explained here are too tedious for the numerical solution of a system of  $n$  equations in  $n$  unknowns if  $n$  is large. Some progress has been made recently in methods of solution suitable for use with calculating machines.

## EXERCISES

*Solve by determinants and check:*

1.  $\begin{cases} x + y + z + 3t = 3 \\ 3x + y - z = 0 \\ 2x - 2y - z + 6t = 4 \\ 4x - y - 2z - 3t = 0. \end{cases}$

2.  $\begin{cases} -2x + y + z + 2w = 5 \\ 3x + 2z - w = -5 \\ x + 3y + 3z + 3w = 0 \\ 4x + 6z - 3w = 0. \end{cases}$

3.  $\begin{cases} x + y + z + t = 3 \\ 2x - 2y - z + 2t = 0 \\ 3x - y + 2z + 2t = 2 \\ x - y - 2z + t = 0. \end{cases}$

4.  $\begin{cases} x + y + z + w = 0 \\ 2x + z - w = 6 \\ x + y + 2w = -4 \\ 2y + z + w = -2. \end{cases}$

5.  $\begin{cases} 2x + 3y + z + t = 1 \\ x - y - z + t = 1 \\ 3x + y + z + 2t = 0 \\ -x + z - t = -2. \end{cases}$

6.  $\begin{cases} x + y + z + w = 4 \\ x - z = 0 \\ y + z - w = 3 \\ 2x - z - 3w = 1. \end{cases}$

7.  $\begin{cases} x - 2y + z + t = 3 \\ x + z = t \\ 2y - z = t \\ x + 4y + 2z - t = 1. \end{cases}$

8.  $\begin{cases} 3x - 2y + 6z - 4w = 2 \\ x + 2z = w \\ 5x + 3z + 4y = 0 \\ 6w + 2y + 2x = 5z. \end{cases}$

**128. Systems of homogeneous linear equations.** If in a linear equation in two or more unknowns the constant term is zero, the equation is called **homogeneous**. Clearly any homogeneous equation or system of several homogeneous equations is satisfied if we let every unknown equal zero; this solution is called the **trivial solution**. Consider a system of  $n$  linear homogeneous



equations in  $n$  unknowns. The proof of the theorem of section 127 shows that if  $\Delta \neq 0$ , the trivial solution is the only solution. Hence we have the following theorem.

**THEOREM 1.** *A system of  $n$  homogeneous linear equations in  $n$  unknowns can have a non-trivial solution only if  $\Delta = 0$ .*

**THEOREM 2.** *Conversely, if, in a system of  $n$  homogeneous linear equations in  $n$  unknowns, we have  $\Delta = 0$ , then there exist infinitely many non-trivial solutions.*

The proof of theorem 2 is beyond the scope of this book.\* We shall take up one particular case.

**THEOREM 3.** *If, in a system of  $n$  linear homogeneous equations with  $\Delta = 0$ , the minor of at least one element is not zero, then the system has infinitely many solutions.*

*Proof.* For simplicity, take  $n = 3$ . Suppose in the system

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

we have  $\Delta = 0$  but  $A_1 \neq 0$ , say. Then we may solve the last two equations for  $y$  and  $z$  in terms of  $x$ , obtaining

$$y = \frac{\begin{vmatrix} -a_2x & c_2 \\ -a_3x & c_3 \end{vmatrix}}{A_1} = \frac{-xB_1}{A_1}, \quad z = \frac{\begin{vmatrix} b_2 & -a_2x \\ b_3 & -a_3x \end{vmatrix}}{A_1} = \frac{xC_1}{A_1}.$$

Substituting these values in the first equation, we have

$$a_1x - \frac{b_1B_1}{A_1}x + \frac{c_1C_1}{A_1}x = 0$$

or

$$x(a_1A_1 - b_1B_1 + c_1C_1) = 0,$$

which is true since the parenthesis has the value  $\Delta = 0$ . Hence for any value of  $x$  whatever, say  $x = m$ , the values  $x = m$ ,  $y = -mB_1/A_1$ , and  $z = mC_1/A_1$  satisfy the system.

The same proof may be applied for any value of  $n$ .

*Example.* Find infinitely many solutions of the system

$$\begin{cases} x + y + z = 0 \\ 4x + y + 2z = 0 \\ 3x - 3y - z = 0. \end{cases}$$

*Solution.* The student should verify that  $\Delta = 0$  while the minor  $A_1 \neq 0$ . Hence we may solve the latter two equations for  $y$  and  $z$  in terms of  $x$ , obtaining  $y = 2x$ ,  $z = -3x$ . Hence for any value  $x = m$  whatever, the numbers  $x = m$ ,  $y = 2m$ ,  $z = -3m$

\* See, for a more complete treatment of linear systems, Bôcher, *Introduction to Higher Algebra*, Macmillan, 1907, or Birkhoff-MacLane, *Survey of Modern Algebra*, Macmillan, 1941, or Dickson, *Elementary Theory of Equations*, Wiley, 1914, etc.



will satisfy the given system. For example,  $x = 1$ ,  $y = 2$ ,  $z = -3$ , or  $x = 4$ ,  $y = 8$ ,  $z = -12$ , etc.

## EXERCISES

Find an expression for infinitely many non-trivial solutions, and list two particular non-trivial solutions, or else show that no non-trivial solutions exist:

$$1. \begin{cases} x + y + z = 0 \\ x + 2y + z = 0 \\ 2x + 3y + 2z = 0. \end{cases}$$

$$2. \begin{cases} x + y - z = 0 \\ x + 3y + z = 0 \\ 2x - y + 5z = 0. \end{cases}$$

$$3. \begin{cases} 2x + 2y + z = 0 \\ 2x - y - 2z = 0 \\ 4x + y - z = 0. \end{cases}$$

$$4. \begin{cases} 2x - y - z = 0 \\ x + y - z = 0 \\ x - 5y + z = 0. \end{cases}$$

$$5. \begin{cases} 2x + y + z = 0 \\ 3x + 2y + 4z = 0 \\ x - 2y - 3z = 0. \end{cases}$$

$$6. \begin{cases} x + 4y - 5z = 0 \\ 2x + 3y - z = 0 \\ 3x + 7y + 2z = 0. \end{cases}$$

Find the value which  $k$  must have in order that non-trivial solutions may exist:

$$7. \begin{cases} 3kx + 4y + 3z = 0 \\ kx + 2y + z = 0 \\ 2x + 3y + 2z = 0. \end{cases}$$

$$8. \begin{cases} 5x + ky + z = 0 \\ x - 4ky + 3z = 0 \\ 3x + 3ky - z = 0. \end{cases}$$

**129. Systems of  $m$  linear equations in  $n$  unknowns.** A system of  $m$  linear equations in  $n$  unknowns is called **consistent** if all the equations of the system possess at least one solution in common; otherwise **inconsistent**. By section 127, if  $m = n$  and  $\Delta \neq 0$ , the system is consistent.

**THEOREM 1.** If  $m = n$ , and  $\Delta = 0$ , and at least one of the numerator determinants  $N_1, N_2, N_3, \dots$  is not zero, then the system is inconsistent.

*Proof.* By the proof of theorem 1, section 127, if there were a common solution,  $x, y, z, \dots$ , we would have

$$(1) \quad \Delta \cdot x = N_1, \Delta \cdot y = N_2, \Delta \cdot z = N_3, \dots$$

But the left members are all zero, while at least one of the right members is not, by hypothesis. Hence this is impossible.

If  $m > n$ , the system is usually inconsistent, although under special circumstances it may be consistent. If it is possible to solve  $n$  of the equations for the  $n$  unknowns, then the values thus obtained can be substituted into the remaining  $m - n$  equations; if these are satisfied, the system is consistent. A special case is given by the following theorem.



**THEOREM 2.** *If a system of  $m$  linear equations in  $m - 1$  unknowns is written with the constant terms in the left member, it is inconsistent unless the resulting determinant is zero.*

*Proof.* For simplicity, we take  $m = 3$ . The given system

$$(2) \quad \begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \\ a_3x + b_3y + c_3 = 0 \end{cases}$$

can have a solution only if the system

$$(3) \quad \begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

of homogeneous equations has a non-trivial solution with  $z = 1$ . By section 128, this implies that

$$(4) \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

This completes the proof.

*Note 1.* This does not imply that the vanishing of the determinant (4) guarantees the consistency of the system (2).

*Note 2.* If the system (2) is consistent, the equations represent straight lines having a point in common.

*Example 1.* If possible, solve the system

$$\begin{cases} x + y - 7 = 0 \\ x - y - 1 = 0 \\ 2x - 3y + 1 = 0. \end{cases}$$

*Solution.* Since the determinant  $\begin{vmatrix} 1 & 1 & -7 \\ 1 & -1 & -1 \\ 2 & -3 & +1 \end{vmatrix} = 0$ , it is pos-

sible that the system may be consistent. Solving the first two equations, we find  $x = 4$ ,  $y = 3$ . Substituting these values in all three equations, we find that they are satisfied.

If  $m < n$ , then usually the system has infinitely many solutions, although, under special circumstances, it may be inconsistent. To find the solutions, when they exist, we may solve for  $m$  of the unknowns in terms of the remaining  $n - m$  unknowns, which may then have arbitrary values,\* as in the following example.

\* For a more complete discussion of the material in this section, see the works of Bôcher, Birkhoff-MacLane, and Dickson, cited in section 128. Also, Dresden, *Solid Analytical Geometry and Determinants*, Wiley, 1930.



*Example 2.* Solve the system

$$\begin{cases} x + y - 2z = 3 \\ x - 2y + z = 1. \end{cases}$$

*Solution.* Solving for  $x$  and  $y$  in terms of  $z$  we obtain  $x = z + \frac{7}{3}$ ,  $y = z + \frac{2}{3}$ . Hence,  $z$  may be given an arbitrary value  $a$ , and  $x = a + \frac{7}{3}$ ,  $y = a + \frac{2}{3}$ ,  $z = a$  will satisfy the system.

## EXERCISES

Show, by means of theorem 1, that the following systems are inconsistent:

$$1. \begin{cases} x + 2y - z = 3 \\ 2x + 4y + 3z = 1 \\ 3x + 6y - 2z = 2. \end{cases}$$

$$2. \begin{cases} 2x - y + 3z = 1 \\ 4x - 2y + 6z = 3 \\ x + y + z = 1. \end{cases}$$

Solve each system, or else show that it is inconsistent:

$$3. \begin{cases} x + y - 7 = 0 \\ x - y - 3 = 0 \\ x - 2y - 1 = 0. \end{cases}$$

$$4. \begin{cases} x + y = 2 \\ x - 2y = 5 \\ 2x + 5y = 1. \end{cases}$$

$$5. \begin{cases} x + y = 1 \\ 2x - y = 11 \\ x - y = 3. \end{cases}$$

$$6. \begin{cases} 2x + y = 0 \\ x - 2y + 10 = 0 \\ 3x + y + 2 = 0. \end{cases}$$

$$7. \begin{cases} 2x - y + 3 = 0 \\ 4x - 2y + 5 = 0 \\ 6x - 3y + 1 = 0. \end{cases}$$

$$8. \begin{cases} x + y + 4 = 0 \\ x - 4y - 1 = 0 \\ 2x - 5y + 1 = 0. \end{cases}$$

$$9. \begin{cases} x + y - 5z = 0 \\ x - y + 3z = 1. \end{cases}$$

$$10. \begin{cases} 2x + y - z = 1 \\ x - 2y - 2z = -3. \end{cases}$$

$$11. \begin{cases} x + 2y + 3z = 1 \\ 2x + 4y + 6z = 5. \end{cases}$$

$$12. \begin{cases} 2x - y + 3z = 1 \\ 4x - 2y + 6z = 3. \end{cases}$$

13. Prove that if  $x$ ,  $y$ , and  $z$  satisfy the system

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}$$

and if at least one of the three determinants below is different from zero, then

$$x:y:z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

14. Prove that if  $x_1, y_1, z_1$  and  $x_2, y_2, z_2$  are two solutions of the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

and if none of the determinants below are zero, then



$$\frac{x_1 - x_2}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y_1 - y_2}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z_1 - z_2}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

**\*130. The resultant of two polynomials.** Having studied the simultaneous solution of linear equations, it is natural to investigate when equations of degree higher than the first have a solution in common.

**THEOREM 1.** If  $f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$  and  $g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n$ , and if not both  $a_0$  and  $b_0$  are equal to zero, then  $f(x)$  and  $g(x)$  have a common factor involving  $x$  if and only if there exist polynomials  $h(x)$  and  $k(x)$  of degrees not greater than  $n-1$  and  $m-1$  respectively, not both identically zero, such that

$$(1) \quad f(x) \cdot h(x) - g(x) \cdot k(x) = 0$$

identically.

*Proof.* (A) If (1) holds, and say  $a_0 \neq 0$ , then  $f(x)$  has  $m$  linear factors of which at most  $m-1$  can be factors of  $k(x)$ . Therefore at least one of these factors is a factor of  $g(x)$ . A similar argument can be given if  $b_0 \neq 0$ .

(B) Conversely, if  $f(x)$  and  $g(x)$  have a common factor  $F(x)$ , then writing  $h(x) = g(x)/F(x)$  and  $k(x) = f(x)/F(x)$ , we see by substitution that condition (1) is satisfied. This completes the proof.

Now let the polynomials  $h(x)$  and  $k(x)$  be given by  $h(x) = \alpha_0x^{n-1} + \alpha_1x^{n-2} + \cdots + \alpha_{n-1}$  and  $k(x) = \beta_0x^{m-1} + \beta_1x^{m-2} + \cdots + \beta_{m-1}$ , respectively. Then condition (1) says that

$$(2) \quad (a_0x^m + a_1x^{m-1} + \cdots + a_m)(\alpha_0x^{n-1} + \alpha_1x^{n-2} + \cdots + \alpha_{n-1}) - (b_0x^n + b_1x^{n-1} + \cdots + b_n)(\beta_0x^{m-1} + \beta_1x^{m-2} + \cdots + \beta_{m-1}) = 0$$

identically. But a polynomial can be identically zero only if all its coefficients are zero. Hence multiplying (2) out and equating the total coefficient of each power of  $x$  to zero, we get

$$(3) \quad \begin{cases} a_0\alpha_0 - b_0\beta_0 = 0 \\ a_1\alpha_0 + a_0\alpha_1 - b_1\beta_0 - b_0\beta_1 = 0 \\ \cdot \\ \cdot \\ \cdot \\ a_m\alpha_{n-2} + a_{m-1}\alpha_{n-1} - b_n\beta_{m-2} - b_{n-1}\beta_{m-1} = 0 \\ a_m\alpha_{n-1} - b_n\beta_{m-1} = 0. \end{cases}$$

\* May be omitted without disturbing continuity.



This is a system of  $m + n$  linear homogeneous equations in the  $m + n$  unknowns  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, -\beta_0, -\beta_1, \dots, -\beta_{m-1}$  with the determinant

$$\begin{vmatrix} a_0 & & 0 & b_0 & & 0 \\ & a_1 & & b_1 & b_0 & \\ & & a_1 & & b_1 & \\ & & & & & \\ & & & & & \\ & & & & & \\ a_m & & & a_0 & & b_0 \\ & a_m & & a_1 & b_n & b_1 \\ & & & & b_n & \\ & & & & & \\ & & & & & \\ 0 & & & & 0 & \\ & & & a_m & & b_n \end{vmatrix},$$

where the large zeros indicate that all other elements are zero. It is customary to interchange the rows and columns, and to denote the determinant thus arrived at by  $R$ . Thus,

$$(4) \quad R = \begin{vmatrix} a_0 & a_1 & \dots & a_m & & 0 \\ & a_0 & a_1 & \dots & a_m & \\ & & & & & \\ & & & & & \\ 0 & & & a_0 & a_1 & \dots & a_m \\ b_0 & b_1 & \dots & & b_n & & 0 \\ & b_0 & b_1 & \dots & & b_n & \\ & & & & & & \\ & & & & & & \\ 0 & & & b_0 & b_1 & \dots & b_n \end{vmatrix} \begin{matrix} \left. \begin{matrix} \\ \\ \\ \end{matrix} \right\} n \text{ rows} \\ \left. \begin{matrix} \\ \\ \end{matrix} \right\} m \text{ rows} \end{matrix}$$

By theorem 1 above, and theorems 1 and 2 of section 128, we have the following theorem.

**THEOREM 2.** *Either  $f(x)$  and  $g(x)$  have a common factor involving  $x$ , or  $a_0 = b_0 = 0$ , if and only if  $R = 0$ .*

The determinant  $R$  given by (4) is called the **resultant** of the polynomials  $f(x)$  and  $g(x)$ .



*Example.* Show, by means of the resultant, that the equations  $f(x) = x^2 - 5x + 6 = 0$  and  $g(x) = 2x - 6 = 0$  have a common solution.

*Solution.* The resultant is

$$R = \begin{vmatrix} 1 & -5 & 6 \\ 2 & -6 & 0 \\ 0 & 2 & -6 \end{vmatrix} = 0.$$

Since  $a_0 = 1 \neq 0$  and  $b_0 = 2 \neq 0$ ,  $f(x)$  and  $g(x)$  have a common factor involving  $x$ . Therefore  $f(x) = 0$  and  $g(x) = 0$  have a common solution.

## EXERCISES

1. Find the resultant of  $a_0x + a_1$  and  $b_0x^2 + b_1x + b_2$ .

Show by means of the resultant that the following pairs of equations have a common root:

2.  $\begin{cases} 2x - 4 = 0 \\ x^2 - 5x + 6 = 0. \end{cases}$

3.  $\begin{cases} 2x - 1 = 0 \\ 2x^2 - 5x + 2 = 0. \end{cases}$

4.  $\begin{cases} 3x^2 - 5x + 2 = 0 \\ 6x^2 - x - 2 = 0. \end{cases}$

5.  $\begin{cases} 2x^2 - 3x - 2 = 0 \\ 6x^2 + x - 1 = 0. \end{cases}$

6.  $\begin{cases} x^3 - 1 = 0 \\ x^2 + x + 1 = 0. \end{cases}$

7.  $\begin{cases} x^3 - 8 = 0 \\ x^2 - 5x + 6 = 0. \end{cases}$

**\*131. Systems of higher degree in two unknowns.** Nothing in the preceding section requires that the coefficients  $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n$  be constants. Hence we may use theorem 2 of section 130 to solve systems of two equations of degree higher than the first in two unknowns  $x$  and  $y$ , by writing the equations as polynomial equations in  $x$  with coefficients which are polynomials in  $y$ , as in the following example.

*Example.* Solve the system

$$\begin{cases} x^2 - 2y^2 + x - 8y - 8 = 0 \\ 2x^2 - 5y^2 + 4x - 17y - 12 = 0. \end{cases}$$

*Solution.* Writing the given equations as polynomial equations in  $x$  with coefficients which are polynomials in  $y$ , we have

$$\begin{cases} x^2 + x + (-2y^2 - 8y - 8) = 0 \\ 2x^2 + 4x + (-5y^2 - 17y - 12) = 0. \end{cases}$$

\* May be omitted without disturbing continuity.



Hence

$$R = \begin{vmatrix} 1 & 1 & (-2y^2 - 8y - 8) & 0 \\ 0 & 1 & 1 & (-2y^2 - 8y - 8) \\ 2 & 4 & (-5y^2 - 17y - 12) & 0 \\ 0 & 2 & 4 & (-5y^2 - 17y - 12) \end{vmatrix}.$$

Evaluating  $R$  and setting it equal to zero, we have

$$R = y^4 + 2y^3 - 13y^2 - 38y - 24 = 0,$$

which has the roots  $y = -1, -2, -3, +4$ .

For  $y = -1$ , the given equations reduce to  $\begin{cases} x^2 + x - 2 = 0 \\ 2x^2 + 4x = 0 \end{cases}$

which have the root  $x = -2$  in common.

For  $y = -2$ , the given equations reduce to  $\begin{cases} x^2 + x = 0 \\ 2x^2 + 4x + 2 = 0 \end{cases}$

which have the root  $x = -1$  in common.

For  $y = -3$ , the given equations reduce to  $\begin{cases} x^2 + x - 2 = 0 \\ 2x^2 + 4x - 6 = 0 \end{cases}$

which have the root  $x = 1$  in common.

For  $y = 4$ , the given equations reduce to  $\begin{cases} x^2 + x - 72 = 0 \\ 2x^2 + 4x - 160 = 0 \end{cases}$

which have the root  $x = 8$  in common.

Hence the common solutions of the given equations are

$$\begin{array}{c|c|c|c|c} x & -2 & -1 & 1 & 8 \\ \hline y & -1 & -2 & -3 & 4 \end{array}.$$

The real solutions are the coordinates of the points of intersection of the graphs of the given equations.

The form of the resultant used in the method of elimination of one unknown from a pair of equations explained here is attributed to J. J. Sylvester (English, 1814–1897) who was one of the first great mathematicians to teach in an American university. Sylvester made many important contributions to mathematics.

For a further study of resultants and systems of equations see Böcher, *Introduction to Higher Algebra*, Macmillan, 1907.

## EXERCISES

Find the common solutions of each of the following systems of equations:

1.  $\begin{cases} x^2 - 2y^2 - x = 0 \\ 2x^2 - 5y^2 + 3y = 0 \end{cases}$
2.  $\begin{cases} x^2 - 3x - 2y^2 - 4y = 0 \\ 2x^2 - 4x - 5y^2 - 7y = 0 \end{cases}$
3.  $\begin{cases} x^2 + 3xy + 2y^2 + 16x - 28y = 0 \\ x^2 - xy - 2y^2 - 5x - 5y = 0 \end{cases}$



4. 
$$\begin{cases} x^2 - 4x + y^2 = 1 \\ x^2 + y^2 - 2y = 9. \end{cases}$$

5. Find the (real) coordinates of the points of intersection of the graphs of the equations

$$\begin{cases} y^2 - x^2 = 0 \\ (y - 1)^2 + x^2 - 1 = 0. \end{cases}$$

*Solve the system of equations given in:*

- 6. Exercise 1, section 86.
- 8. Exercise 7, section 86.
- 10. Exercise 15, section 86.
- 12. Exercise 21, section 86.
- 14. Exercise 25, section 86.

- 7. Exercise 3, section 86.
- 9. Exercise 9, section 86.
- 11. Exercise 17, section 86.
- 13. Exercise 23, section 86.
- 15. Exercise 27, section 86.



## CHAPTER XVII

### Permutations and Combinations

**132. Fundamental principles.** The following principles will provide the basis for the work of this chapter.

(A) *If one event can occur in  $h$  different ways, and if, after it has happened in one of these ways, another event can occur in  $k$  different ways, then both events can occur, in the stated order, in  $h \cdot k$  different ways.*

Since any one of the  $h$  ways in which the first event can occur can be coupled with any one of the  $k$  ways in which the second event can occur, there are obviously  $h$  groups of  $k$  pairs or  $h \cdot k$  pairs of possible occurrences.

*Example 1.* A game consists of drawing 2 cards in succession from a pack of 4 different cards marked  $a, b, c$ , and  $d$ , respectively, the first card not being replaced before the second is drawn. How many different results can there be?

*Solution.* The first drawing can produce any one of 4 different results. The second drawing can produce any one of 3 different results. Both drawings in the stated order can produce  $4 \cdot 3 = 12$  different results, namely  $ab, ac, ad; ba, bc, bd; ca, cb, cd; da, db, dc$ .

*Example 2.* The same as example 1 except that the first card is to be replaced before the second card is drawn.

*Solution.* The first drawing can produce 4 different results. The second can then produce 4 different results again. Both therefore can occur in the stated order in  $4 \cdot 4 = 16$  different ways, namely  $aa, ab, ac, ad; ba, bb, bc, bd; ca, cb, cc, cd; da, db, dc, dd$ .

If the two events are **independent** of each other, that is, if neither is influenced by the outcome of the other, then we may state the following principle:

(B) *If one of two independent events can occur in  $h$  different ways, and the other in  $k$  different ways, then both can occur, disregarding their order, in  $h \cdot k$  ways.*

*Example 3.* A luncheon tray is to contain a sandwich and a beverage. There are 4 different kinds of sandwich and 3 different kinds of beverage. How many different lunches can be formed?



*Solution.* The order is immaterial; we do not care whether the sandwich or the beverage is put on the tray first. Hence, according to principle *B*, there are  $4 \cdot 3 = 12$  different lunches.

Two events are called **mutually exclusive** if they cannot both occur together.

(*C*) If two events are mutually exclusive, and one can occur in  $h$  different ways while the other can occur in  $k$  different ways, then either one or the other (but not both) can occur in  $h + k$  different ways.

*Example 4.* The same as example 3, except that, because of financial embarrassment, a lunch is now to consist of either a sandwich or a beverage but not both.

*Solution.* According to principle *C*,  $4 + 3 = 7$  different lunches are possible.

In connection with principle *C*, care must be taken that the events are mutually exclusive.

*Example 5.* We are to choose either a heart or an ace from a deck of 52 playing cards. How many choices are there?

*Solution.* There are 13 hearts and 4 aces. But it is wrong to say that there are 17 choices, because these events are not mutually exclusive; there is one heart which is also an ace. Actually there are only 16 different choices.

*Example 6.* How many different symbols, each consisting of 6 different letters in succession, can be formed from the letters of the word "algorithm," if each symbol is to begin and end with a vowel?

*Solution.* There are 3 vowels and 6 consonants, and 6 places to be filled: — — — — —. For the first place we may choose any one of the 3 vowels. For the last place we may choose any one of the 2 remaining vowels. The first of the 4 remaining places may be filled by any one of the 7 remaining letters, the next by any one of the 6 remaining letters, the next by any one of the 5 remaining letters, and the last by any one of the 4 remaining letters. We write the numbers of choices over the appropriate blank spaces, thus:  $3 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 2$  and multiply these numbers, by virtue of principle (*A*). Hence there are 5040 different symbols.

*Example 7.* (a) How many different arrangements of 6 distinct books each can be made on a shelf with space for 6 books? (b) If 3 of the books are English and 3 are French, how many arrangements of the 6 books can be made on the same shelf if books in the same language are to be kept together?



*Solution.* (a) In the first space we may put any one of the 6 books, in the next space any one of the 5 remaining books, and so on. Hence there are  $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$  arrangements. (b) If the English books are at the left, we may put any one of the 3 English books in the first position, any one of the 2 remaining English books in the next place, and so on; hence there are  $3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 36$  such arrangements. But there are 36 others having the French books at the left. Hence there are  $36 + 36 = 72$  arrangements in all.

## EXERCISES

1. In how many ways can 5 students be seated in a row of 5 seats?
2. How many different signals can be made from 5 different flags if each signal is to consist of 3 flags hung in a horizontal row?
3. How many different numbers of 2 different digits each can be formed using the digits 1, 3, 5, 7, 9?
4. How many different numbers of 3 different digits each can be formed using the digits 0, 2, 4, 6, 8?
5. How many different symbols, each symbol consisting of 3 letters in succession, can be formed using the letters  $a, b, x, u, r$  if (a) no letter may be repeated in the same symbol, (b) repetitions are allowed?
6. How many different numbers of 3 different digits each can be formed from the digits 2, 3, 5, 6, 7, 9, if each number is to be (a) odd, (b) even?
7. In how many ways can the positions of president, vice-president, and secretary be filled in a club of 20 members if no person is to hold more than one position?
8. Two cubical dice, one red and one white, are thrown. In how many ways can they fall?
9. Three different cubical dice are thrown. In how many different ways can they fall?
10. If a penny, nickel, dime, quarter, and half dollar are tossed together, in how many ways may they fall?
11. How many numbers of 4 different digits, each greater than 5000, can be formed from the digits 2, 4, 5, 7, 8, 0?
12. In how many ways can a baseball team of 9 men be arranged in batting orders, if a certain 4 men must occupy the first 4 positions in some order?
13. In how many ways can a party of 5 people be seated in a row of 5 seats (a) if a certain 2 insist on sitting next to each other, (b) if the same 2 refuse to sit next to each other?
14. In how many ways can a set of 4 different mathematics books and 3 different physics books be placed on a shelf with space for 7 books if all books on the same subject are to be kept together?



15. A telephone dial has 10 holes. How many different signals, each consisting of 7 impulses in succession, can be formed (a) if no impulse is to be repeated in any given signal, (b) if repetitions are permitted?

16. In how many different ways can 4 people be seated in a row of 7 seats?

17. (a) How many different symbols, each consisting of 4 different letters in succession, can be formed from the letters of the word *certain*? (b) How many of these begin and end with a vowel? (c) How many begin with a consonant and end with a vowel?

18. In how many ways can 4 boys and 3 girls be seated in a row of 7 seats if boys and girls are to occupy alternate seats?

19. In how many different ways can the letters of the name *Naomi* be arranged if all vowels are to be kept together?

20. In how many different ways can the letters of the name *Leonard* be arranged if consonants and vowels are to alternate?

21. (a) How many numbers can be formed using some or all of the digits 1, 3, 4, 7, 8 if no number is to contain repeated digits? (b) How many of these numbers will be even? (c) How many will be greater than 350?

22. How many arrangements of 3 different English books and 5 different French books can be made on a shelf with space for 8 books if all the English books are to be kept together?

23. From 3 English, 4 French, and 5 German books, all different, how many different sets can be chosen, each set consisting of 2 books in different languages?

24. How many 3-digit numbers begin with an odd digit and end with an even digit? (*Hint*: Zero is to be counted among the even digits.)

**133. Permutations of  $n$  distinct objects taken  $r$  at a time.** An arrangement of a set of objects in some order in a straight line is called a **permutation** of these objects. More precisely, if we have  $n$  distinct objects, any arrangement of  $r$  of them in some order in a straight line is called a **permutation of the  $n$  distinct objects taken  $r$  at a time**. The number of permutations of  $n$  distinct objects taken  $r$  at a time is denoted by  $P(n, r)$ ; other symbols in common use are  ${}^nP_r$ ,  ${}_nP_r$ ,  $P_r^n$ ,  $P_{n,r}$ .

*Example 1.* Find the number of permutations of the four letters  $a, b, c, d$  taken two at a time.

*Solution.* Any one of the four letters may be put in the first position, and any one of the three remaining letters in the second position. Hence there are  $4 \cdot 3 = 12$  permutations, which are listed in example 1, section 132. Thus  $P(4, 2) = 12$ .

*Example 2.* The number of permutations of the four letters



$a, b, c, d$  taken all at a time is similarly  $\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 24$ . Thus  $P(4,4) = 24$ .

These permutations are:

|        |        |        |          |
|--------|--------|--------|----------|
| $abcd$ | $bacd$ | $bcad$ | $bcda$   |
| $abdc$ | $badc$ | $bdac$ | $bdca$   |
| $acbd$ | $cabd$ | $cbad$ | $cbda$   |
| $acdb$ | $cadb$ | $cdab$ | $cdba$   |
| $adbc$ | $dabc$ | $dbac$ | $dbca$   |
| $adcb$ | $dacb$ | $dcab$ | $dcba$ . |

*Example 3.* The number of permutations of the four letters  $a, b, c, d$  taken three at a time is  $\underline{4} \cdot \underline{3} \cdot \underline{2} = 24$ . Thus,  $P(4,3) = 24$ . These permutations are:

|       |       |       |         |
|-------|-------|-------|---------|
| $abc$ | $abd$ | $acd$ | $bcd$   |
| $acb$ | $adb$ | $adc$ | $bdc$   |
| $bac$ | $bad$ | $cad$ | $cbd$   |
| $bca$ | $bda$ | $cda$ | $cdb$   |
| $cab$ | $dab$ | $dac$ | $dbc$   |
| $cba$ | $dba$ | $dca$ | $dcb$ . |

We have

$$(1) \quad P(n,n) = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1,$$

since the first position can be filled by any one of the  $n$  distinct objects, the next by any one of the remaining  $(n-1)$ , and so on.

Introducing the symbol

$$(2) \quad n! = n(n-1)(n-2) \cdot \cdots \cdot 2 \cdot 1,$$

read "factorial  $n$ ," we have

$$(3) \quad P(n,n) = n!$$

For example,  $1! = 1$ ,  $2! = 2 \cdot 1 = 2$ ,  $3! = 3 \cdot 2 \cdot 1 = 6$ ,  $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ,  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ , and so on.

If  $r$  is less than  $n$ , then

$$(4) \quad P(n,r) = n(n-1)(n-2) \cdot \cdots (n-r+1)$$

where there are  $r$  factors in the right member, since there are  $r$  places to be filled by  $n$  distinct objects. Multiplying numerator and denominator of the right-hand member of (4) by

$$(n-r)(n-r-1)(n-r-2) \cdot \cdots \cdot 2 \cdot 1,$$



we get

$$P(n, r) = \frac{n(n-1)(n-2) \cdots (n-r+1) \cdot (n-r)(n-r-1) \cdots 2 \cdot 1}{(n-r)(n-r-1) \cdots 2 \cdot 1}$$

or

$$(5) \quad P(n, r) = \frac{n!}{(n-r)!} \quad \text{if } r < n.$$

In practice, it will sometimes be desirable to work problems directly by use of the fundamental principles, and sometimes by use of these formulas.

## EXERCISES

*Evaluate each of the following symbols:*

1.  $6!$       2.  $\frac{5!}{3!}$       3.  $\frac{9!}{7!}$       4.  $P(5, 2)$       5.  $P(9, 2)$ .

6.  $P(6, 6)$       7.  $P(7, 3)$       8.  $P(7, 4)$       9.  $\frac{P(6, 2)}{2!}$       10.  $\frac{P(8, 3)}{3!}$ .

11. How many different arrangements can be formed on a shelf with space for 3 books if there are 5 different books available?

12. A bookshelf has space for 4 books. If there are 6 different books available, how many different arrangements can be made on the shelf?

13. A bookshelf has space for 6 books. There are 5 different English books and 6 different French books available. If we wish to put 3 English books and 3 French books on the shelf, how many arrangements can be made keeping all books in the same language together?

14. In how many ways can a coach assign positions on a baseball team of 9 men if only 2 men are qualified to be pitcher and only 2 others are qualified to be catcher but all are able to play any other position?

15. In how many ways can 7 people sit in a row of 7 seats if a certain 4 people must sit side by side?

16. How many numbers, each of 5 different digits, can be formed from the digits 3, 4, 5, 6, 7 if (a) the numbers 4 and 5 are to be next to each other, (b) the numbers 4 and 5 are not to be next to each other?

17. How many different signals, each consisting of one flag or more hung in a horizontal line, can be formed from a set of 4 different flags?

18. How many positive whole numbers can be formed from some or all of the digits 2, 3, 4, 5, 6 if no number is to have repeated digits?

19. How many positive even numbers can be formed from some or all of the digits 2, 3, 4, 5, 6 if no number is to have repeated digits?

20. How many positive whole numbers can be formed from some or all of the digits 0, 1, 2, 3, 4 if no number is to have repeated digits?

\*21. How many positive even numbers can be formed from some or all of the digits 0, 1, 2, 3, 4 if no number is to have repeated digits?

\* May be omitted without disturbing continuity.



**134. Permutations where some things are alike.** The number of permutations of the distinct letters  $a, b, c, d$  taken all at a time is 24. The permutations are listed in example 2, section 133. If three of the letters were indistinguishable, for example, if instead of  $a, b, c, d$  we had  $a, x, x, x$ , then there would be only four permutations, namely

$$axxx, \quad xaxx, \quad xxax, \quad xxxa.$$

Clearly each column of  $3! = 6$  permutations of  $a, b, c, d$  listed in example 2, section 133, gives rise to only one permutation of  $axxx$ , since the letters  $bcd$  may be permuted among themselves in  $3!$  ways which are indistinguishable for  $xxx$ . Hence we may arrive at the answer 4 by dividing the answer of example 2, section 133, by  $3!$ ; thus  $4!/3! = 4 \cdot 3 \cdot 2 \cdot 1/3 \cdot 2 \cdot 1 = 4$ .

In general, if a set of  $n$  objects is such that a set of  $p$  of them are indistinguishable, another set of  $q$  of them are indistinguishable, and so on, then the number of permutations of the  $n$  objects taken all at a time is

$$(1) \quad \frac{n!}{p!q! \cdots}$$

*Example 1.* How many permutations can be made of the letters in the word *Mississippi*, taken all at a time?

*Solution.* There are 11 letters of which 4  $i$ 's, 4  $s$ 's, and 2  $p$ 's are indistinguishable. Hence the answer is

$$\frac{11!}{4!4!2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 2} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5 = 34650.$$

## EXERCISES

1. How many permutations can be formed from the letters of the word *reiterate* taken all at a time?

2. How many permutations can be formed from the letters of the word *institution* taken all at a time?

3. How many permutations can be formed from the letters  $a, a, a, b, b, c, c, c, d$  taken all at a time?

4. How many different eight-digit numbers can be formed from the digits 2, 2, 2, 5, 5, 5, 5, 3?

5. How many different signals, each consisting of 7 flags hung in a horizontal line, can be formed from a set of 4 indistinguishable blue flags and 3 indistinguishable yellow flags?

6. How many different eight-digit numbers can be formed from the digits 5, 5, 5, 5, 3, 3, 7, 7?



7. A modern town is laid out in square blocks. By how many different routes can we go from the corner of East First Street and North First Avenue to the corner of East Seventh Street and North Fifth Avenue, each route being of the smallest possible length, that is, 10 blocks?

**135. Circular permutations.** We agree that when objects are arranged in a circle, only their relative order as we traverse the circle in a definite sense is to be considered. Hence the arrangements

$$(1) \quad \begin{array}{c} a \\ \circlearrowleft \\ c \end{array} \begin{array}{c} b \\ \circlearrowleft \\ a \end{array} \quad \begin{array}{c} c \\ \circlearrowleft \\ b \end{array} \begin{array}{c} a \\ \circlearrowleft \\ c \end{array} \quad \begin{array}{c} b \\ \circlearrowleft \\ a \end{array} \begin{array}{c} c \\ \circlearrowleft \\ b \end{array}$$

are regarded as the same. Similarly the arrangements

$$(2) \quad \begin{array}{c} a \\ \circlearrowright \\ b \end{array} \begin{array}{c} c \\ \circlearrowright \\ a \end{array} \quad \begin{array}{c} b \\ \circlearrowright \\ c \end{array} \begin{array}{c} a \\ \circlearrowright \\ b \end{array} \quad \begin{array}{c} c \\ \circlearrowright \\ b \end{array} \begin{array}{c} a \\ \circlearrowright \\ c \end{array}$$

are regarded as the same. Thus, there are only two possible arrangements of three distinct objects in a circle. Therefore, to calculate the number of arrangements in circular order, we must first fix the position of one object arbitrarily, and then we may calculate the number of permutations of the remaining objects as if they were in a straight line. Hence  $n$  distinct objects can be arranged in a circle in  $(n - 1)!$  ways.

*Example.* A group of 6 people can be arranged at a circular table in  $5! = 120$  ways.

*Note.* In arranging keys on a ring, it is usually agreed that two arrangements are the same if one can be gotten from the other by turning the ring over like a flapjack. Thus the arrangements (1) and (2) above are the same with this agreement. In this case we must divide the answer by 2. Thus, there is only one possible arrangement of three different keys on a ring. In general, there are  $\frac{(n - 1)!}{2}$  different arrangements of  $n$  keys on a ring.

## EXERCISES

The following questions are to be answered in accordance with the conventions set forth above:

1. In how many different orders can 6 people take seats at a round table?
2. In how many different orders can 7 children join hands in a ring?
3. In how many different orders can 6 keys be placed on a key ring?
4. How many different necklaces can be formed from 7 differently colored beads?



5. In how many orders can a party of 3 girls and 3 boys be placed at a round table so that boys and girls alternate?

6. In how many orders can a party of 4 girls and 4 boys be placed at a round table so that boys and girls alternate?

7. In how many orders can 6 people be seated at a round table (a) if a certain 2 insist on sitting next to each other, (b) if they refuse to sit next to each other?

**136. Combinations of  $n$  distinct objects taken  $r$  at a time.** A set of  $r$  objects chosen from a given set of  $n$  distinct objects, without regard to the order in which they are chosen or arranged, is called a **combination of  $n$  distinct objects taken  $r$  at a time**. The number of combinations of  $n$  distinct objects taken  $r$  at a time is denoted by  $C(n,r)$ . Other symbols in common use are  ${}^nC_r$ ,  ${}_nC_r$ ,  $C_r^n$ ,  $C_{n,r}$  and  $\binom{n}{r}$ .

*Example 1.* The number of combinations of the four letters  $a, b, c, d$  taken two at a time is 6. The combinations are  $ab, ac, ad, bc, bd, cd$ . Clearly each of these combinations gives rise to  $2! = 2$  permutations which are listed in example 1, section 132.

Hence  $C(4,2) = \frac{P(4,2)}{2!} = 6$ .

*Example 2.* The number of combinations of the four letters  $a, b, c, d$  taken three at a time is 4. The combinations are  $abc, abd, acd, bcd$ . Each of these gives rise to  $3! = 6$  permutations which are listed in columns in example 3, section 133. Hence

$$C(4,3) \cdot 3! = P(4,3), \text{ or } C(4,3) = \frac{P(4,3)}{3!} = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2} = 4.$$

Clearly, any combination of  $n$  distinct things taken  $r$  at a time gives rise to  $r!$  permutations since each set of  $r$  objects can be permuted among themselves in  $r!$  ways. Thus  $C(n,r) \cdot r! = P(n,r)$ , or

$$(1) \quad C(n,r) = \frac{P(n,r)}{r!} \quad (0 < r < n).$$

Using (5), section 133, we have

$$(2) \quad C(n,r) = \frac{n!}{r!(n-r)!} \quad (0 < r < n).$$

Using (4), section 133, we have

$$(3) \quad C(n,r) = \frac{n(n-1)(n-2) \cdots (n-r+1)}{r!} \quad (0 < r < n).$$

We write  $C(n,0) = C(n,n) = 1$ , since there is only one way to select no things from the given  $n$  things, and only one way to select  $n$  things from the given  $n$  things, namely to take them all.



*Note 1.* Whenever we select a set of  $r$  things from a set of  $n$  distinct things, we also automatically select a set of  $n - r$  things to be left behind. Hence

$$(4) \quad C(n, r) = C(n, n - r).$$

This can also be proved by using (2).

*Note 2.* Notice that  $9! = 9 \cdot 8! = 9 \cdot 8 \cdot 7!$ , and so on. Hence, by (2), we may write  $C(9, 7) = C(9, 2) = \frac{9!}{2!7!} = \frac{9 \cdot 8 \cdot 7!}{2 \cdot 7!} = 36$ .

*Example 3.* How many committees of 5 can be formed from a group of 6 Democrats and 3 Republicans if each committee is to have at least 3 Democrats?

*Solution.* The number of committees with:

exactly 3 Democrats is  $C(6, 3) \cdot C(3, 2) = 20 \cdot 3 = 60$ ;

exactly 4 Democrats is  $C(6, 4) \cdot C(3, 1) = 15 \cdot 3 = 45$ ;

exactly 5 Democrats is  $C(6, 5) = 6$ ;

either 3, 4, or 5 Democrats is the total  $111$ .

Hence the total number of committees with at least 3 Democrats is 111.

## EXERCISES

*Evaluate each symbol:*

1.  $C(5, 2)$ .

2.  $C(7, 3)$ .

3.  $C(7, 4)$ .

4.  $C(10, 4)$ .

5.  $C(20, 18)$ .

6.  $C(6, 2) \cdot C(5, 3)$ .

*In how many ways can we select:*

7. A committee of 3 from a group of 10 people?

8. A committee of 5 from a group of 8 people?

9. A set of 3 books from a set of 7 different books?

10. A set of 5 books from a set of 9 different books?

11. How many different sums of money, each composed of 3 coins, can be formed from a cent, a nickel, a dime, and a quarter?

12. (a) How many straight lines are determined by the vertices of a regular hexagon? (b) How many of these lines go through a given vertex?

13. How many triangles are determined by the vertices of a regular hexagon?

14. In how many ways can we select a committee of 3 Democrats and 2 Republicans from a group of 8 Democrats and 6 Republicans?

15. In how many ways can we select a set of 3 mathematics books and 2 physics books from a set of 7 mathematics books and 5 physics books, all different?



16. From a group of 12 Democrats and 10 Republicans, how many different committees of 7 can be chosen which contain (a) exactly 4 Democrats? (b) at least 4 Democrats? (c) at most 4 Democrats?

17. (a) How many committees of 5 can be chosen from a group of 15 men? (b) How many of these will include a specified man  $A$ ? (c) From how many will  $A$  be excluded?

18. How many committees of 5 can be chosen from a group of 10 men if each committee must include 2 certain men  $A$  and  $B$ ?

19. In how many ways can we select a committee of 5 from a group of 10 men (a) if a certain 2 men insist on serving together or not at all? (b) if a certain 2 men refuse to serve together?

20. How many different committees of 7 can be chosen from a group of 8 Democrats and 5 Republicans if each committee is to contain (a) exactly 4 Democrats? (b) at least 4 Democrats? (c) at most 4 Democrats?

21. In how many different orders can we shelve sets of 5 books, each set consisting of 3 mathematics books and 2 physics books, if the books are to be chosen from a set of 10 mathematics books and 8 physics books, all different?

22. If  $P(n,4) = 11,880$ , find  $C(n,4)$ .

23. If  $C(n,4) = 210$ , find  $P(n,4)$ .

24. Prove that  $C(n-1, r) + C(n-1, r-1) = C(n, r)$ .

25. (a) How many straight lines are determined by 10 points, no 3 of which are in the same straight line? (b) How many of these lines pass through any given point?

26. From an urn containing 5 black and 7 white balls, in how many ways can we draw a set of 6 balls of which 4 are white and 2 black?

27. In how many ways can 10 different books be divided among  $A$ ,  $B$ , and  $C$  so that 5 are given to  $A$ , 3 to  $B$ , and 2 to  $C$ ?

28. In a league of 8 baseball teams, how many games will be played in a season if each team plays 10 games with every other team?

29. How many numbers of 5 different digits, each number to contain 3 odd and 2 even digits, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9?

30. How many different sums of money can be formed from a cent, a nickel, a dime, a quarter, and a half dollar, using any number of coins at a time?

31. A cent, a nickel, a dime, a quarter, and a half dollar are tossed simultaneously. In how many ways can it happen that 2 coins fall heads and the remaining 3 tails?

32. (a) How many different hands of 13 cards each can be dealt from a pack of 52 different cards? (b) How many different sets of 4 hands can be dealt, ignoring the order in which the 4 hands are distributed around the bridge table?



## CHAPTER XVIII

### Probability

**137. Definition of probability.** In everyday speech, the word “probable” is used in a vague sense, as in the statements “It will probably rain tomorrow,” or “I shall probably pass mathematics.” In fact, some have asserted that probability is merely a measure of belief. But different people believe opposite statements with equal firmness, and the same person often believes opposite statements at different times. To be scientifically useful, a definition of probability must be more objective.

**DEFINITION.** *If an event can occur in a total number  $t$  of alternative ways, all of which are considered equally likely, and if a certain number  $s$  of these ways are considered successful or favorable, then the number  $s/t$  is called the **probability** of success.*

*Example 1.* What is the probability of tossing “head” in a single toss of a coin?

*Solution.* Here  $t = 2$ ,  $s = 1$ . Hence the probability is  $1/2$ .

*Example 2.* An urn contains 10 balls, of which 3 are red, 4 are blue, 2 are green, and 1 is yellow. Then the probability of obtaining a red ball in a single draw is  $3/10$ , the probability of drawing a blue ball is  $4/10$ , green  $2/10$ , yellow  $1/10$ . The probability of drawing a white ball is 0, while the probability of drawing a colored ball is  $10/10 = 1$ .

As seen in example 2, a probability of 1 indicates certainty, while a probability of 0 indicates impossibility. The probability of any event is a number between 0 and 1.

The student should now review carefully the fundamental principles of section 132, chapter XVII, which will be useful in working problems on probability.

*Example 3.* Three different English books and 3 different French books are to be arranged on a shelf with space for 6 books. If they are stacked at random, what is the probability that they will be arranged with all books in the same language together?

*Solution.* According to the fundamental principles of section 132, chapter XVII, there are  $\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$  ways in which the books can be stacked. Of these only twice  $\underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 36$



or 72 will satisfy the condition that all books in the same language are together. (See example 7, section 132, chapter XVII.) Hence the probability is  $72/720 = 1/10$ .

In practice, the statement that the probability of tossing “head” is  $1/2$  is understood to mean that “head” should turn up close to half the times if enough tosses are made. That is, probability is interpreted as relative frequency; if  $f(n)$  is the number of favorable events in  $n$  trials, the **relative frequency** is  $f(n)/n$ . Thus by experiment, we might find that, in tossing a coin, we get

|                               |      |      |       |        |         |     |
|-------------------------------|------|------|-------|--------|---------|-----|
| $n =$ number of trials        | 10   | 100  | 1000  | 10000  | 100000  | ... |
| $f(n) =$ number of heads      | 2    | 45   | 512   | 4982   | 50137   | ... |
| $f(n)/n =$ relative frequency | 0.20 | 0.45 | 0.512 | 0.4982 | 0.50137 | ... |

Here it appears that the relative frequency is approaching the probability  $1/2$  as  $n$  increases indefinitely. Needless to say, this is an assumption, though well verified by experience.

Also of great importance is the assumption, in the definition of probability, that the alternative possibilities are equally likely. For instance, in example 2, it would be wrong to say that the probability of drawing a red ball is  $1/2$ , because there are only two possibilities, drawing a red ball and not drawing a red ball, since these two possibilities are not equally likely.

Even if the computation is correct, the relative frequency cannot be expected to correspond to the probability in a practical application unless the alternative possibilities are in fact equally likely (or very nearly so). For instance, in example 1, we would not expect a relative frequency of nearly  $1/2$  if the coin were weighted on one side, or if it had two heads. The ultimate test of the applicability of a theory to reality is whether or not it works. The theory of probability, properly used, has been found to be of great value in such diverse subjects as physical chemistry, statistical mechanics, heredity, biometrics, econometrics, the theory of gases, quality control in mass production industries, insurance, social sciences, games of chance, experimental method in general, etc.

*Note.* The theory of probability was founded by two great French mathematicians Blaise Pascal and Pierre de Fermat in the 17th century. Pascal was a child prodigy who did valuable research in mathematics at the age of 16.



## EXERCISES

1. Find the probability of throwing a six with a single die.
2. Throw a single die 100 times and record the number of sixes obtained in 10, 20, 30, . . . , 100 throws.
3. Find the probability of throwing a total of two with a pair of dice.
4. Find the probability of throwing a total of three with a pair of dice; a total of four; a total of seven. Which is more probable, a total of three or a total of seven?
5. (a) Find the probability of drawing a spade from a pack of 52 playing cards; (b) an ace; (c) the ace of spades.
6. Toss a coin 100 times and record the number of heads obtained in 10, 20, 30, . . . , 100 tosses.
7. Find the probability of obtaining (a) no heads; (b) 1 head; (c) 2 heads in tossing a pair of coins.
8. Toss a pair of coins 100 times and record the relative frequencies of 0, 1, and 2 heads.
9. Four different books are to be placed at random on a bookshelf with space for 4 books. What is the probability that a certain 2 books will be next to each other?
10. An urn contains 5 white, 3 black, and 2 green balls. (a) Find the probability of drawing a black ball in a single draw; (b) a white ball; (c) a white or green ball.
11. In a single throw of a single die, what is the probability of obtaining a three or anything larger?
12. In a single throw of a pair of dice, what is the probability of obtaining a total of ten or more?
13. If 3 balls are drawn simultaneously from the urn of exercise 10, what is the probability that 2 will be white and 1 black?
14. If 3 balls are drawn simultaneously from an urn containing 5 white and 3 black balls, what is the probability that all 3 will be white?
15. If 7 people are seated at random in a row of 7 seats, what is the probability that a certain 4 people will occupy adjacent seats?
16. If 5 cards are drawn from a deck of 52 playing cards, what is the probability that they will be, regardless of order, the ace, king, queen, jack, and ten of spades?
17. A number of 3 different digits is formed from the digits 1, 2, 3, 4, 5. (a) What is the probability that it will be even? (b) What is the probability that it will be greater than 325?
18. If 6 people are seated at random at a round table, what is the probability that a certain 2 will be neighbors?
19. Find the probability of drawing (a) either an ace or a king in a single draw from a deck of 52 playing cards; (b) either an ace or a spade.
20. If 5 cards are drawn from a deck of 52 playing cards, what is the probability that they will be all of the same suit?



**138. Statistical probability.** Probability as defined in section 137 is called a *a priori probability*, because we determine its value without conducting experiments. A *a priori probability* is of great importance in both theoretical and practical work, and we shall develop it further in the next section. But it can be used only where one has sufficient data to determine the probability without experiment or statistical observation. It would hardly do for such things as life expectancy in insurance, or other statistical questions. For such considerations, we use relative frequency as the definition of probability. That is, if  $f(n)$  is the number of favorable events in  $n$  trials, the relative frequency  $f(n)/n$  is called the *statistical probability*, or a *posteriori probability*, or *empirical probability of success*. It may be expected to change if  $n$  is changed, that is, if the basic data or number of cases examined is changed.

For life insurance, the Commissioner's Standard Ordinary Mortality Table (table VIII) is based on many observations of holders of life insurance policies.

*Example.* What is the probability that a man aged 32 will live to 50?

*Solution.* From the table, of 917880 men alive at age 32, 810900 are alive at age 50. Hence the probability is  $810900/917880 = 0.883$  or 88.3%.

Needless to say, this means nothing to any particular man since it takes no account of his health, occupation, etc. But it has statistical significance.

## EXERCISES

*Using table VIII, estimate the statistical probability of:*

1. A 10-year-old reaching the age of 35.
2. A 10-year-old reaching the age of 60.
3. A 10-year-old reaching the age of 70.
4. A 30-year-old reaching the age of 60.
5. A 35-year-old reaching the age of 70.
6. A 31-year-old reaching the age of 90.

**139. Some theorems concerning probability.** The following theorems are useful in more complicated examples of probability.

**THEOREM 1.** *If the probability of success is  $p$ , then the probability of failure is  $1 - p$ .*

*Proof.* If the event can happen in  $s$  successful ways out of a total number  $t$  of possible equally likely ways, then it fails in



$t - s$  ways. Hence the probability of failure is  $\frac{t - s}{t} = 1 - p$  where  $p = s/t$  is the probability of success.

*Example 1.* The probability of throwing a two with a single die is  $1/6$  while the probability of failing to throw a two is  $1 - 1/6 = 5/6$ .

Two or more events are called **independent** if the occurrence or non-occurrence of any one of them is not affected by the occurrence or non-occurrence of the rest.

**THEOREM 2.** *If  $p_1$  is the probability that a first event will occur, and  $p_2$  is the probability that a second independent event will occur, then the probability that both will occur is  $p_1p_2$ . A similar theorem can be stated for more than two events.*

*Proof.* Let  $p_1 = s_1/t_1$ , and  $p_2 = s_2/t_2$ . By the fundamental principles of section 132, chapter XVII, any of the  $t_1$  possibilities for the first event may be combined with any of the  $t_2$  possibilities for the second event, yielding a total number  $t_1t_2$  of possibilities for both events. Similarly, the total number of successful possibilities for both events together is  $s_1s_2$ . Hence the probability of both events occurring successfully is  $\frac{s_1s_2}{t_1t_2} = p_1p_2$ .

*Example 2.* The probability that an ace be drawn from a pack of 52 playing cards is  $4/52 = 1/13$ . Suppose that, after a card is drawn, it is replaced in the pack. Then the probability of drawing two aces in succession is  $\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$ , since the two drawings are independent of each other.

**THEOREM 3.** *If the probability of a first event is  $p_1$  and if the probability of a second event, after the first event has happened, is  $p_2$ , then the probability that both events will occur in succession is  $p_1p_2$ . A similar theorem can be stated for more than two events.*

*Proof.* Similar to theorem 2.

*Example 3.* Suppose that when a card is drawn from a pack, it is not to be replaced before the second drawing. Then the probability of drawing two aces in succession from a pack of 52 playing cards is  $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$ , since, after the first ace has been drawn and not replaced, the probability of drawing the second ace is  $3/51$ .

Two or more events are said to be **mutually exclusive** if not more than one of them can occur.



**THEOREM 4.** *If the probability that one event will occur is  $p_1$  and the probability that a second event will occur is  $p_2$  and if the two events are mutually exclusive, then the probability that either the first or the second event will occur is  $p_1 + p_2$ . A similar theorem can be stated for more than two events.*

*Proof.* Let the total number of possibilities be  $t$ , of which  $s_1$  cases are favorable for the first event and  $s_2$  cases are favorable for the second event. The  $s_1$  cases and the  $s_2$  cases are all different since the two events are mutually exclusive. The probability of the first event is  $p_1 = s_1/t$  and the probability of the second event is  $p_2 = s_2/t$ . One or the other of the two events can occur in  $s_1 + s_2$  ways. Hence the probability that either one or the other will occur is  $\frac{s_1 + s_2}{t} = \frac{s_1}{t} + \frac{s_2}{t} = p_1 + p_2$ .

*Example 4.* The probability that either an ace or a king will be drawn in a single drawing from a pack of 52 playing cards is  $\frac{4}{52} + \frac{4}{52} = \frac{2}{13}$ .

One must be careful, in applying theorem 4, to be sure that the events are mutually exclusive.

*Example 5.* The probability of drawing a heart in a single drawing from a pack of 52 playing cards is  $13/52$ . The probability of drawing an ace is  $4/52$ . But the probability of drawing either an ace or a heart is not  $17/52$  but  $16/52 = 4/13$ .

**THEOREM 5.** *If  $p$  is the probability that an event will occur in a single trial, then the probability that it will occur exactly  $r$  times out of  $n$  trials is  $C(n,r) \cdot p^r q^{n-r}$ , where  $q = 1 - p$  is the probability of failure in a single trial.*

*Proof.* The probability that the event will occur in any particular set of  $r$  trials and fail in the remaining  $n - r$  trials is  $p^r q^{n-r}$ , by theorem 2. But the particular set of  $r$  trials may be selected from the set of  $n$  trials in  $C(n,r)$  ways. Hence, by theorem 4, we must add  $C(n,r)$  terms each of which has the value  $p^r q^{n-r}$ . Therefore the desired probability is  $C(n,r)p^r q^{n-r}$ .

*Example 5.* Find the probability of throwing a two with a single die exactly 3 times out of 5 trials.

*Solution.* The probability of throwing a two in a single trial is  $1/6$ . Hence the answer is  $C(5,3) \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 = 10 \cdot \frac{1}{216} \cdot \frac{25}{36} = \frac{125}{3888}$ .



*Example 6.* A mass production factory makes rods which are to be 1 inch in diameter, with a tolerance of 0.1 inch. That is, they are acceptable if their diameters lie between 0.9 inch and 1.1 inches. In normal production it is found that 1 out of 10 will fail to meet this requirement. Suppose, on a given occasion, a sample of 20 shows 5 unsatisfactory rods. Can this be attributed to chance? That is, what is the likelihood that 5 unsatisfactory rods should occur out of 20? By theorem 5, the probability is  $C(20,5) \left(\frac{1}{10}\right)^5 \left(\frac{9}{10}\right)^{15} = 0.03$  approximately. Since this is a very small probability, careful investigation of the production process is indicated to discover the cause of the excessive failures.

If  $p$  is the probability of a person winning an amount  $A$  of money, then  $pA$  is called his **mathematical expectation**.

*Example 7.* A man will win \$12 if he throws a two with a single throw of a single die. His mathematical expectation is  $\frac{1}{6} \cdot \$12 = \$2$ .

If the probability of an event is  $s/t$ , the **odds in favor** of the event are  $s$  to  $t - s$ , and the **odds against** the event are  $t - s$  to  $s$ .

*Example 8.* The odds against throwing a two with a single throw of a single die are 5 to 1.

## EXERCISES

1. What is the probability of throwing a six twice in succession with a single die?

2. What is the probability of drawing an ace twice in succession from a pack of 52 playing cards (a) if each drawn card is replaced in the pack before drawing again; (b) if drawn cards are not replaced?

3. Find the probability of drawing an ace 4 times in succession from a pack of 52 playing cards (a) if each drawn card is replaced in the pack before drawing again; (b) if drawn cards are not replaced.

4. An urn contains 6 white and 5 black balls. If drawn balls are to be replaced in the urn before the next drawing, what is the probability of drawing (a) 2 white balls in succession; (b) 3 black balls in succession?

5. The same as exercise 4 except that drawn balls are not to be replaced.

6. What is the probability that a six will turn up exactly once in 3 successive throws of a single die?

7. What is the probability that a six will turn up exactly twice in 5 successive throws of a single die?

8. Find the probability that a six will turn up at least once in 4 successive throws of a single die. (*Hint: Consider the probability of failure.*)



9. If 10 coins are tossed in succession, what is the probability that (a) exactly 3 will be heads; (b) at least 3 will be heads?

10. If 10 coins are tossed, find the probability that (a) exactly 8 will be heads; (b) at least 8 will be heads; (c) at most 8 will be heads.

11. If 5 dice are thrown, find the probability that (a) exactly 3 of them will turn up a six; (b) at least 3 of them will turn up a six; (c) at most 3 of them will turn up a six.

12. If a man will win \$60 if he throws a total of seven in a single throw of a pair of dice, what is his mathematical expectation?

13. What is the probability of throwing a total of seven with a pair of dice (a) 5 times in succession? (b) 10 times in succession? (c) Is the probability in (b) half as great as that in (a)?

14. If 13 cards are drawn from a pack of 52 playing cards, what is the probability that they will all be spades?

15. The probability that  $A$  will be alive next Christmas is  $3/4$ . The probability that  $B$  will be alive next Christmas is  $2/3$ . What is the probability that at least one of them will be alive next Christmas? (*Hint: Consider the probability of failure.*)

16. Three different red and 3 different blue books are shelved at random on a shelf with space for 6 books. Find the probability that they will be shelved with colors alternating.

17. In a tennis tournament, the odds that  $A$  will win the tournament are 4 to 3 and the odds that  $B$  will win are 1 to 4. Find the odds that either  $A$  or  $B$  will win.

18. The probability that  $A$  will solve a problem is  $2/3$ . The probability that  $B$  will solve the problem is  $3/5$ . If both work on it independently, what is the probability that the problem will be solved?

19. Two people  $A$  and  $B$  are to draw alternately one ball at a time from an urn containing 3 white and 2 black balls, drawn balls not being replaced. If  $A$  takes the first turn, what is the probability that  $A$  will be the first to draw white?

20. A machine has two component parts,  $A$  and  $B$ . If either one breaks down, the machine breaks down. The probability that  $A$  will break down in 1 year is  $1/4$ . The probability that  $B$  will break down in 1 year is  $1/5$ . Find the probability that the machine will break down in 1 year.

21. A machine has three parts  $A$ ,  $B$ ,  $C$ . If any one of them breaks down, the machine breaks down. Suppose that the probability that  $A$  breaks down in 1 year is  $1/5$ ; similarly for  $B$  and for  $C$ . Find the probability that the machine will break down in 1 year.

22. (a) Two men are to be chosen at random from a group of 8. What is the probability that a certain 2,  $A$  and  $B$ , will be those chosen? (b) The 8 men are to be seated in a row of 8 seats. What is the probability that a certain 2,  $A$  and  $B$ , will be seated next to each other?

23. (a) Find the probability of throwing a total of seven at least once in 3 successive throws of a pair of dice. (b) What are the odds against this happening?



24. Five students are assigned seats in a row of 5 seats. (a) If they sit down at random, find the probability that every one will sit in his own seat. (b) Find the probability that at least 3 will sit in their own seats.

25. Under normal conditions 10% of a certain mass production item will be defective. In a certain sample of 10 items, 3 of them are found to be defective. What is the probability that this happened by chance?

26. One urn contains 5 black balls and 2 white balls. A second urn contains 3 black balls and 4 white balls. If we are to draw just 1 ball, what is the probability that it will be white?

27. In a "multiple choice" test, each question is to be answered by selecting 1 of 5 different proposed answers, of which only 1 is right. If there are 10 questions on the test, what is the probability of (a) getting exactly 6 right by pure guesswork; (b) getting at least 6 right by pure guesswork?

28. A gambler bets \$72 that he will obtain a total of either eight or five in a single throw of a pair of dice. What are the odds in his favor and what is his mathematical expectation?



## CHAPTER XIX

### Mathematical Induction

**140. The  $n$ th term of a sequence.** By a sequence of numbers is meant a succession  $a_1, a_2, a_3, \dots, a_n, \dots$ . The subscript  $n$  indicates the position of the term  $a_n$  in the sequence. Thus  $a_1$  is the first term,  $a_2$  the second term, and in general,  $a_n$  is the  $n$ th term, or **general term**, of the sequence. A sequence may be specified by stating its  $n$ th term as a function of  $n$ , as follows.

*Example 1.* The sequence whose  $n$ th term is  $2n$  has the numbers  $2 \cdot 1 = 2, 2 \cdot 2 = 4, 2 \cdot 3 = 6, 2 \cdot 4 = 8, 2 \cdot 5 = 10$  as the first five terms.

*Example 2.* The sequence whose  $n$ th term is  $\frac{1}{n^2}$  has the numbers  $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$  as the first five terms.

However, there is no possibility of conversely determining the  $n$ th term as a function of  $n$  from a knowledge of the first five (or first thousand) terms alone. For example, the  $n$ th term of the sequence of example 1, namely

$$(1) \qquad 2, 4, 6, 8, 10, \dots$$

is  $2n$ . But the first five terms given in (1) are also the first five terms of the sequence whose  $n$ th term is

$$(2) \qquad 2n + (n-1)(n-2)(n-3)(n-4)(n-5),$$

and also the first five terms of the sequence whose  $n$ th term is

$$(3) \qquad 2n + (n-1)(n-2)(n-3)(n-4)(n-5)f(n),$$

where  $f(n)$  may be almost any function of  $n$  at all.

*Note.* It follows that the question, "What is the next term of the sequence  $2, 4, 6, 8, 10, \dots$ ?" which is often asked on intelligence tests, cannot be answered logically. For if the  $n$ th term were  $2n$ , the next term would be  $2 \cdot 6 = 12$ ; but if the  $n$ th term were (2), then the next term would be  $2 \cdot 6 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 132$ ; indeed, the next term may be any number at all if the function  $f(n)$  in (3) is properly chosen and (3) then taken as the  $n$ th term.



## EXERCISES

Write (a) the first 5 terms, (b) the 10th term, (c) the  $k$ th term, (d) the  $(k + 1)$ th term, of the sequence whose  $n$ th term is:

1.  $2n$ .
2.  $3n$ .
3.  $n^2$ .
4.  $2n - 1$ .
5.  $n(n + 1)$ .
6.  $\frac{(-1)^n}{n^2}$ .
7.  $\frac{(-1)^{n+1}}{n(n + 1)}$ .
8.  $2n + 3$ .

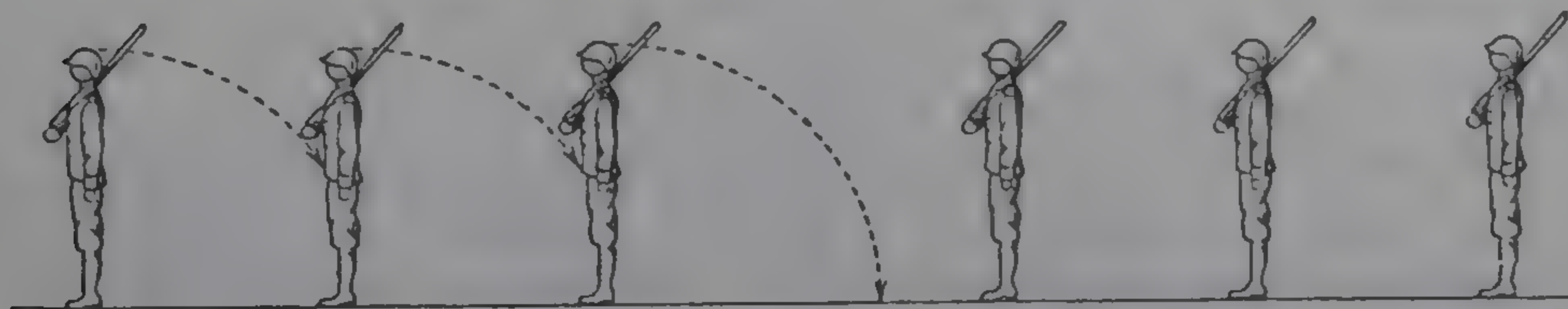
**141. The axiom of mathematical induction.** The following so-called axiom of mathematical induction is one of the basic axioms of algebra.

*If a proposition involving the positive integer  $n$  can be proved to have the following properties:*

(A) *The proposition is correct for  $n = 1$ ;*

(B) *If  $k$  is any value of  $n$  for which the proposition is true, then the proposition is also true for the next value  $n = k + 1$ ,*

*then the proposition is true for all positive integral values of  $n$ .*



Condition (b) not satisfied

Fig. 57.

This axiom is extremely plausible. For if (A) is established, the proposition is true for  $n = 1$ . But if (B) is established, then it is true for  $1 + 1$  or 2. But then (B) also implies that it is true for  $2 + 1$  or 3. And so on. But we cannot go on so for more than a limited number of cases in a lifetime. Hence we assume the axiom above.

The axiom of mathematical induction may be made intuitively plausible by considering an endless single file of toy soldiers (Fig. 57). Suppose we wish to knock over all the soldiers. We cannot knock over more than a limited number of them if we knock over one at a time. But we can be sure they will all fall if we can make sure of two things, namely that:

(a) The first soldier is knocked over;

(b) The soldiers are spaced so that if any one is knocked over, it automatically knocks over the next. Clearly, (a) corresponds to (A) and (b) to (B) in the axiom of mathematical induction.



**142. Proofs by mathematical induction.** Let us prove the following theorem:

**THEOREM 1.** *If  $n$  is any positive integer, then*

$$(1) \quad 3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2}.$$

Note that (1) asserts that

$$\text{for } n = 1, \quad 3 = \frac{3 \cdot 1(1+1)}{2},$$

$$\text{for } n = 2, \quad 3 + 6 = \frac{3 \cdot 2(2+1)}{2},$$

$$\text{for } n = 3, \quad 3 + 6 + 9 = \frac{3 \cdot 3(3+1)}{2},$$

$$\text{for } n = 4, \quad 3 + 6 + 9 + 12 = \frac{3 \cdot 4(4+1)}{2},$$

and so on. But we cannot prove the infinite number of propositions contained in theorem 1 by checking one case at a time in this way. Hence we prove it by means of the axiom of mathematical induction, as follows:

*Proof.* (A) For  $n = 1$ , we verify that (1) is true by direct substitution of 1 for  $n$ ; that is, we verify that

$$3 = \frac{3 \cdot 1(1+1)}{2}.$$

(B) We prove the so-called auxiliary proposition, that *if the proposition is true for  $n = k$ , then it must be true for the next value  $n = k + 1$ .*

The hypothesis of this proposition is obtained by substituting  $n = k$  in the proposed formula (1). The conclusion is obtained by substituting  $n = k + 1$  in (1). Thus,

$$\text{Hypothesis.} \quad 3 + 6 + 9 + \cdots + 3k = \frac{3k(k+1)}{2} = \frac{3k^2 + 3k}{2}.$$

$$\begin{aligned} \text{Conclusion.} \quad 3 + 6 + 9 + \cdots + 3(k+1) \\ = \frac{3(k+1)(k+1+1)}{2} = \frac{3k^2 + 9k + 6}{2}. \end{aligned}$$

*Proof.* By hypothesis,

$$3 + 6 + 9 + \cdots + 3k = \frac{3k^2 + 3k}{2}.$$

Adding the  $(k+1)$ th term,  $3(k+1)$ , to both sides, we obtain

$$(2) \quad 3 + 6 + 9 + \cdots + 3k + 3(k+1) = \frac{3k^2 + 3k}{2} + 3(k+1).$$



which is then true, because if equals are added to equals the results are equal. The left member of (2) is the same as the left member of the conclusion. It remains only to verify that the right member of (2) is the same as the right member of the conclusion. Since

$$\frac{3k^2 + 3k}{2} + 3(k + 1) = \frac{3k^2 + 9k + 6}{2},$$

the auxiliary proposition is proved.

By virtue of (A) and (B), the axiom of mathematical induction assures us that (1) is true for all positive integral values of  $n$ . That is, theorem 1 is proved.

The work may be arranged compactly as in the following example:

THEOREM 2. *If  $n$  is any positive integer, then*

$$(3) \quad 2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}.$$

*Proof.* (A) We verify (3) for the case  $n = 1$ :

$$2 = \frac{1(3 \cdot 1 + 1)}{2}$$

$$2 = 2.$$

$$(B) \text{ Hypothesis. } 2 + 5 + 8 + \cdots + (3k - 1) = \frac{k(3k + 1)}{2} \\ = \frac{3k^2 + k}{2}.$$

*Conclusion.*

$$2 + 5 + 8 + \cdots + (3[k + 1] - 1) = \frac{(k + 1)(3[k + 1] + 1)}{2}, \\ \text{or } 2 + 5 + 8 + \cdots + (3k + 2) = \frac{3k^2 + 7k + 4}{2}.$$

*Proof.* By hypothesis,

$$2 + 5 + 8 + \cdots + (3k - 1) = \frac{3k^2 + k}{2}.$$

Adding the  $(k + 1)$ th term,  $(3k + 2)$ , to both sides, we obtain

$$2 + 5 + 8 + \cdots + (3k + 2) = \frac{3k^2 + k}{2} + 3k + 2 \\ = \frac{3k^2 + 7k + 4}{2},$$

which agrees with the right member of the conclusion.



This completes the proof of (B), and hence, by the axiom of mathematical induction, completes the proof of theorem 2.

*Note 1.* The analogy with the endless file of tin soldiers in section 141 makes clear that both part (A) and part (B) must be established. We give two algebraic examples to show that neither (A) nor (B) alone is sufficient. The student should fill in all details.

(a) Consider the false formula

$$3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2} + (n-1).$$

Part (A) is correct, since  $3 = \frac{3 \cdot 1(1+1)}{2} + 0$ . But part (B) will not work. For, adding  $3(k+1)$  to  $\frac{3k(k+1)}{2} + (k-1)$ , we do not get  $\frac{3(k+1)(k+2)}{2} + k$ .

(b) Consider the false formula

$$3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2} + 1.$$

Here part (B) is correct, since, if we add  $3(k+1)$  to  $\frac{3k(k+1)}{2} + 1$ , we do get  $\frac{3(k+1)(k+2)}{2} + 1$ . But part (A) is false, since  $3 \neq \frac{3 \cdot 1(1+1)}{2} + 1$ .

*Note 2.* A proof by mathematical induction is deductive logic and must be distinguished from the induction or inductive logic used in experimental science. In experimental science we might test a formula like (1) for  $n = 1, 2, 3, \cdots, 1000$ . If it worked for 1000 trials we might conclude that it was (very) probably true. But we could not conclude more without further information. For the false formula

$$\begin{aligned} 3 + 6 + 9 + \cdots + 3n \\ = \frac{3n(n+1)}{2} + (n-1)(n-2)(n-3) \cdots (n-1000) \end{aligned}$$

would also work correctly for  $n = 1, 2, 3, \cdots, 1000$ , but would be false thereafter. Why?

*Note 3.* The axiom of mathematical induction is one of five axioms, stated by Peano in the 19th century, upon which all of



elementary algebra can be based, but the idea of proving theorems by mathematical induction can be traced, in some form, to the 16th century.

*Note 4.* In the examples above and many of the exercises below, theorems giving the sum of the first  $n$  terms of a given sequence are verified by mathematical induction. But no hint is given as to how the right-hand member of the given formula is discovered. Many of these can be obtained by methods discussed in chapter XXI.

## EXERCISES

(a) In each of the following exercises, verify the correctness of the given formula for  $n = 1, 2, 3$ . (b) Does the work done in part (a) suffice to establish the correctness of the given formula for all positive integers  $n$ ? Explain. (c) Prove by mathematical induction that the given formula is correct for all positive integers  $n$ .

$$1. 1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

$$2. 2 + 4 + 6 + \cdots + 2n = n(n + 1).$$

$$3. 4 + 8 + 12 + \cdots + 4n = 2n(n + 1).$$

$$4. 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.$$

$$5. 5 + 10 + 15 + \cdots + 5n = \frac{5n(n + 1)}{2}.$$

$$6. 5 + 7 + 9 + \cdots + (2n + 3) = n(n + 4).$$

$$7. \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}.$$

$$8. 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}.$$

$$9. 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.$$

$$10. 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}.$$

$$11. 3 + 3^2 + 3^3 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}.$$

$$12. 1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1).$$

$$13. 5 + 8 + 11 + \cdots + (3n + 2) = \frac{n(3n + 7)}{2}.$$

$$14. 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n}{6} (n + 1)(2n + 7).$$

$$15. a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}.$$



$$10 + 20 + 30 + \dots + 100 + 110 + 120 + \dots + 190 + 200 = 1000$$

$$*17. \text{ Prove } 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

\*18. Prove  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$  is true for all  $n \in \mathbb{N}$ . (Use Mathematical Induction.)

\*19. Prove by mathematical induction that  $x - y$  is a factor of  $x^n - y^n$  for all positive integral values of  $n$ . [Hint:  $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \dots + y^{n-1})$ ]

\*20. Prove by mathematical induction that

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Let all positive integral values of  $n$ .

\*21. Prove by mathematical induction that



## CHAPTER XX

### The Binomial Theorem

143. The binomial theorem for positive integral exponents. By direct multiplication, the student may verify that

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5,$$

and so on. The right member is called the **expansion** of the left member. From these examples, we might guess that the expansion of  $(a + b)^n$ , where  $n$  is any positive integer, has the following properties:

(A) The expansion has  $n + 1$  terms;

(B) The first term is  $a^n$  and the last is  $b^n$ ;

(C) In each successive term after the first, the exponent of  $a$  decreases by 1 and the exponent of  $b$  increases by 1 so that the sum of the exponents in each term is  $n$ ;

(D) The coefficients of terms equidistant from the ends of the expansion are equal;

(E) The coefficients may be determined from the triangular array

$$\begin{array}{ccccccc}
 & & & & 1 & & 1 \\
 & & & 1 & & 2 & & 1 \\
 & & 1 & & 3 & & 3 & & 1 \\
 & 1 & & 4 & & 6 & & 4 & & 1 \\
 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 . & . & . & . & . & . & . & . & . & . & .
 \end{array}$$

in which each number, except those at the ends, is the sum of the two nearest numbers in the line above. This scheme is known as Pascal's triangle, although it was known, both in Europe and in China, considerably before Pascal's time (17th century).



Pascal studied it in connection with permutations, combinations, and probability.

*Example 1.* From properties (A) to (E), we can write

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

To find the coefficients from Pascal's triangle is tedious if the exponent  $n$  is high. For example, to find the coefficients of  $(a + b)^{150}$  by this scheme requires writing out the first 149 lines of Pascal's triangle before the 150th can be determined. A better way of writing the expansion is given by the following theorem, called the **binomial theorem**.

**THEOREM.** *If  $n$  is any positive integer,*

$$\begin{aligned} (1) \quad (a + b)^n &= a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 \\ &+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4}b^4 \\ &+ \cdots + \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} a^{n-r}b^r + \cdots + b^n. \end{aligned}$$

*Proof.\** Recall that  $(a + b)^n$ , where  $n$  is a positive integer, stands for the product of  $n$  factors each equal to  $(a + b)$ . Hence each term in the result is a sum of terms each of which is a product of one letter from each factor  $(a + b)$ . Thus, the term involving  $b^r$  will be a sum of terms of the form  $a^{n-r}b^r$ , each of which is a product of  $r$   $b$ 's [one from each of  $r$  factors  $(a + b)$ ] and  $n - r$   $a$ 's [one from each of the remaining  $n - r$  factors  $(a + b)$ ]. There are as many such terms as there are choices of  $r$  factors from the  $n$  factors, from which to take a  $b$ . Hence there are  $C(n, r)$  such terms and the total coefficient of  $a^{n-r}b^r$  will be  $C(n, r)$ , or, by (3) of section 136, chapter XVII,

$$C(n, r) = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}.$$

*Note 1.* The coefficient of the term involving  $b^r$  in the expansion of  $(a + b)^n$  has the product of the first  $r$  positive integers (or  $r!$ ) in the denominator, and  $r$  factors in the numerator beginning with  $n$  and successively decreasing by 1. That is, the term

\* Chapter XVII on Permutations and Combinations is needed for this proof. Another proof by mathematical induction will be given in the next section.



involving  $b^r$  is

$$(2) \quad C(n, r) a^{n-r} b^r = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} a^{n-r} b^r.$$

This is the  $(r+1)$ th term of the expansion. The  $r$ th term would be, accordingly,

$$(3) \quad C(n, r-1) a^{n-r+1} b^{r-1} \\ = \frac{n(n-1)(n-2) \cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} a^{n-r+1} b^{r-1}.$$

The number of factors in either numerator or denominator of the coefficient is equal to the exponent of  $b$ .

*Note 2.* The binomial formula (1) may be written, for positive integral exponents, as

$$(4) \quad (a+b)^n = a^n + C(n, 1) a^{n-1} b + C(n, 2) a^{n-2} b^2 + \cdots \\ + C(n, r) a^{n-r} b^r + \cdots + b^n.$$

In particular, setting  $a = b = 1$  in (4), it follows that

$$C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n, n) = 2^n$$

where we have written  $C(n, 0) = C(n, n) = 1$ . That is, the total number of sets of any size that can be selected from a given set of  $n$  things is  $2^n$  provided we include just 1 "empty" set [ $C(n, 0) = 1$ ]. If we do not wish to count this "empty" set, the number is  $2^n - 1$ .

*Note 3.* If the product of the coefficient of any term and the exponent of  $a$  in that term is divided by the number of that term, we obtain the coefficient of the next term. For example, the coefficient of the 4th term in the expansion of  $(a+b)^6$  (see example 1) may be obtained from the third term by calculating  $15 \cdot 4/3 = 20$ .

*Note 4.* If the entire expansion is desired, for a low value of  $n$ , the coefficients may be obtained by means of note 3, or (4), or (1), or Pascal's triangle. If a particular term only is desired, (2) or (3) will be useful.

*Note 5.* Since  $a - b = a + [-b]$ , the expansion of  $(a - b)^n$  will have alternating signs beginning with plus. The minus signs will appear whenever the exponent of  $[-b]$  is odd, namely in the even-numbered terms.

*Example 2.* Expand  $\left(2x - \frac{b^2}{2}\right)^6$ , and simplify each term.



*Solution.* By (1), or various other considerations above,

$$\begin{aligned}
 \left(2x - \frac{b^2}{2}\right)^6 &= \left(2x + \left[-\frac{b^2}{2}\right]\right)^6 \\
 &= (2x)^6 + \frac{6}{1} (2x)^5 \left[-\frac{b^2}{2}\right] + \frac{6 \cdot 5}{1 \cdot 2} (2x)^4 \left[-\frac{b^2}{2}\right]^2 \\
 &\quad + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} (2x)^3 \left[-\frac{b^2}{2}\right]^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} (2x)^2 \left[-\frac{b^2}{2}\right]^4 \\
 &\quad + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (2x) \left[-\frac{b^2}{2}\right]^5 + \left[-\frac{b^2}{2}\right]^6 \\
 &= 64x^6 - 96x^5b^2 + 60x^4b^4 - 20x^3b^6 + \frac{15}{4}x^2b^8 \\
 &\quad - \frac{3}{8}xb^{10} + \frac{b^{12}}{64}
 \end{aligned}$$

*Example 3.* Write and simplify the 8th term in the expansion of  $\left(\frac{u}{2} - v^2\right)^{12}$ .

*Solution.* The 8th term will involve  $b^7$ . Hence it is

$$\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \left(\frac{u}{2}\right)^5 (-v^2)^7 = -99 \cdot 8 \frac{u^5}{2^5} v^{14} = -\frac{99}{4} u^5 v^{14}.$$

*Example 4.* Write and simplify the term involving  $k^6$  in the expansion of  $(h^3 - k^2)^8$ .

*Solution.* This is the fourth term since we must have  $b$  or  $k^2$  raised to the third power. This term is

$$\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} (h^3)^5 (-k^2)^3 = -56h^{15}k^6.$$

*Example 5.* By means of the binomial theorem, calculate  $(1.02)^8$  to the nearest hundredth.

*Solution.*  $(1.02)^8 = (1 + 0.02)^8 = 1^8 + \frac{8}{1} \cdot 1^7(0.02) + \frac{8 \cdot 7}{1 \cdot 2} \cdot 1^6(0.02)^2 + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \cdot 1^5(0.02)^3 + \dots = 1 + 0.16 + 0.0112 + 0.000448 + \dots = 1.171648 = 1.17$  to the nearest hundredth. It is intuitively clear that further terms will contribute negligible increases since higher powers of 0.02 are very small.

*Note 6.* Some elementary practical applications of the binomial theorem will be found in chapter XXIV. It will prove to be a useful theorem in more advanced courses in mathematics.



## EXERCISES

Write the expansion of each of the following and simplify each term:

1.  $(x + h)^7$ .    2.  $(a + b)^8$ .    3.  $(p + q)^5$ .    4.  $(2a + 3b)^4$ .
5.  $(2a^3 + 3b^2)^5$ .    6.  $\left(\frac{a}{2} + \frac{4b^2}{a}\right)^6$ .    7.  $(a - b)^3$ .
8.  $(a - b)^4$ .    9.  $(a - b)^5$ .    10.  $\left(2a^3 - \frac{b^2}{2a}\right)^6$ .
11.  $\left(\frac{x}{y} - \frac{1}{x}\right)^4$ .    12.  $\left(\frac{a^2}{2} - \frac{1}{a}\right)^5$ .    13.  $\left(\frac{x^2}{2} - \frac{2}{x}\right)^4$ .
14.  $\left(\frac{ab^2}{2x} - \frac{4x}{a^2b}\right)^5$ .

Write, in simplified form, only the specified term:

15. The term involving  $h^3$  in the expansion of  $(x + h)^{17}$ .
16. The term involving  $b^5$  in the expansion of  $(a + b)^{24}$ .
17. The sixth term of  $(x + h)^{15}$ .
18. The eighteenth term of  $(x + h)^{19}$ .
19. The term involving  $b^6$  in the expansion of  $\left(\frac{a^2}{2} + 2b^2\right)^{10}$ .
20. The term involving  $q^{10}$  in the expansion of  $\left(\frac{p^2}{3} + 6q^2\right)^{12}$ .
21. The middle term of  $(a - b)^8$ .
22. The middle terms of  $(2a - 3b)^{11}$ .
23. The sixth term of  $\left(\frac{2}{\sqrt{x}} - \frac{x^2}{2}\right)^9$ .
24. The term involving  $x^6$  in the expansion of  $(2x^2 + \frac{1}{2})^{10}$ .
25. The eighth term of  $\left(2x^2 - \frac{1}{2x^2}\right)^{12}$ .
26. The term involving  $y^2$  in the expansion of  $(\frac{1}{2}x^2 + 2\sqrt[3]{y})^8$ .
27. The term involving  $x^6$  in the expansion of  $\left(x^3 - \frac{1}{2x}\right)^{10}$ .
28. The fourth term in the expansion of  $(3x^2 + 2x^{1/2})^7$ .
29. The term involving  $x^7$  in the expansion of  $\left(\frac{x^2}{2} - \frac{2}{x}\right)^8$ .
30. The term involving  $x^2$  in the expansion of  $(2x^4 - x^{-2})^8$ .
31. The term involving  $x^3$  in the expansion of  $\left(x + \frac{1}{x^2}\right)^6$ .
32. The fourth term of  $\left(\frac{x^2}{y} - 2x\right)^{15}$ .



33. The term involving  $x^4$  in the expansion of  $\left(3x^2 - \frac{2}{x}\right)^8$ .

34. The fifth term of  $(\sqrt{x} - 2x^{-2})^8$ .

Compute by means of the binomial theorem, using only enough terms to get the result accurate to three decimal places:

35.  $(1.01)^9 = (1 + 0.01)^9$ .

36.  $(1.02)^6$ .

37.  $(0.99)^6 = (1 - 0.01)^6$ .

38.  $(0.98)^6$ .

39.  $(1.02)^{11}$ .

40.  $(0.99)^8$ .

41.  $(0.98)^7$ .

42.  $(1.03)^{10}$ .

43. Explain the connection between Pascal's triangle and exercise 24 of section 136, chapter XVII.

44. Expand  $(a + b + c)^4$  by the binomial theorem. (*Hint:*  $a + b + c = [a + b] + c$ .)

\*144. Proof of the binomial theorem by mathematical induction. We give a second proof of theorem 1 or formula (4), section 143, for positive integral exponents, by mathematical induction.

*Proof.* (A) For  $n = 1$  the formula (4), section 143, is true, since  $(a + b)^1 = a + b$ .

(B) If the formula (4) is true for  $n = k$ , then it must be true for  $n = k + 1$ .

*Hypothesis.*  $(a + b)^k = a^k + C(k, 1)a^{k-1}b + \cdots + C(k, r-1)a^{k-r+1}b^{r-1} + C(k, r)a^{k-r}b^r + \cdots + b^k$ .

*Conclusion.*  $(a + b)^{k+1} = a^{k+1} + C(k+1, 1)a^k b + \cdots + C(k+1, r)a^{k-r+1}b^r + \cdots + b^{k+1}$ .

*Proof.* Multiply both sides of the hypothesis by  $(a + b)$ . The left member becomes  $(a + b)^{k+1}$ . In the right member let us consider a typical term in the product, say the term involving  $b^r$ . This will be the sum of two terms, the first being the product of  $a$  by the term involving  $b^r$  in the hypothesis, and the second being the product of  $b$  by the term involving  $b^{r-1}$  in the hypothesis. These two products are  $C(k, r)a^{k-r+1}b^r$  and  $C(k, r-1)a^{k-r+1}b^r$  respectively. Hence their sum is

$$[C(k, r) + C(k, r-1)]a^{k-r+1}b^r.$$

By direct calculation,\* we find that  $C(k, r) + C(k, r-1) = C(k+1, r)$ . Hence the general term of the conclusion is  $C(k+1, r)a^{k-r+1}b^r$ . This completes the proof.

\* May be omitted without disturbing continuity.

\* See exercise 24, section 136, chapter XVII.



**145. The binomial series.** Consider the binomial expansion

$$(1) \quad (1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \dots + \frac{n(n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r}x^r + \dots$$

If  $n$  is a positive integer, the term involving  $x^n$  is the last non-zero term, since if  $r = n+1, n+2, \dots$  the coefficient contains the factor zero in the numerator. But if  $n$  is not a positive integer, the expression (1) does not terminate. It becomes what is known as an **infinite series** which has significance under circumstances which will be explained in chapter XXVII.

This binomial series was used by Isaac Newton (English, 1642–1727) for negative and fractional values of  $n$ . The correctness of the expression (1) under suitable conditions was established for all complex values of  $n$  by N. H. Abel (Norwegian, 1802–1829).

*Example 1.* Write and simplify the first four terms of the expansion of  $(1+x)^{-1/3}$ .

*Solution.*

$$\begin{aligned} (1+x)^{-1/3} &= 1^{-1/3} + \frac{(-\frac{1}{3})}{1} 1^{-1/3-1}x + \frac{(-\frac{1}{3})(-\frac{1}{3}-1)}{1 \cdot 2} 1^{-1/3-2}x^2 + \\ &\quad \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{1 \cdot 2 \cdot 3} 1^{-1/3-3}x^3 + \dots \\ &= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \dots \end{aligned}$$

*Example 2.* Find to the nearest hundredth the positive cube root of 29.

*Solution.* We write  $\sqrt[3]{29} = (27+2)^{1/3} = 3\left(1 + \frac{2}{27}\right)^{1/3}$ . By (1),

we have

$$\begin{aligned} \left(1 + \frac{2}{27}\right)^{1/3} &= 1 + \frac{1}{3}\left(\frac{2}{27}\right) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1 \cdot 2}\left(\frac{2}{27}\right)^2 \\ &\quad + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1 \cdot 2 \cdot 3}\left(\frac{2}{27}\right)^3 + \dots \\ &= 1 + 0.0247 - 0.0006 + 0.00002 - \dots = 1.02412 \\ &\quad \text{approximately.} \end{aligned}$$

Hence  $\sqrt[3]{29} = 3(1.02412) = 3.07$  to the nearest hundredth.



## EXERCISES

*Write the first four terms in the expansion of:*

1.  $\sqrt{1+x}$ . (Hint:  $\sqrt{1+x} = (1+x)^{1/2}$ .)

2.  $(1+x)^{-2}$ . 3.  $\sqrt[3]{1+x}$ . 4.  $\frac{1}{1+x}$ . 5.  $\frac{1}{1-x}$ . 6.  $\frac{1}{\sqrt{1+x}}$ .

*Using the binomial expansion, calculate to the nearest hundredth:*

7.  $\sqrt{17}$ . 8.  $\sqrt[3]{9}$ . 9.  $\sqrt[3]{65}$ . 10.  $(1.01)^{-6}$ .  
11.  $\sqrt{28}$ . 12.  $(1.02)^{-10}$ .

*By the binomial expansion, calculate to the nearest thousandth:*

13.  $\sqrt{50}$ . 14.  $\sqrt{48}$ . 15.  $\sqrt[3]{25}$ . 16.  $\sqrt[3]{28}$ .



## CHAPTER XXI

### Progressions

**146. Arithmetic progressions.** A sequence\* of numbers in which each term is obtained from the preceding one by addition of a constant number  $d$  is called an arithmetic progression. The number  $d$  is called the common difference. If  $a$  is the first term of an arithmetic progression with difference  $d$ , the first  $n$  terms are

$$(1) \quad a, a + d, a + 2d, a + 3d, \dots, a + (n - 1)d.$$

The  $n$ th term or last term  $l$  is

$$(2) \quad l = a + (n - 1)d.$$

*Example 1.* The first five terms of the arithmetic progression with first term equal to 3 and common difference equal to 2 are 3, 5, 7, 9, 11. The  $n$ th term is  $3 + (n - 1)2 = 2n + 1$ .

*Example 2.* Given that 4, 7, 10 are the first three terms of an arithmetic progression, find the 11th term.

*Solution.* Here  $d = 7 - 4 = 3$ . Hence the eleventh term is  $a + (11 - 1)d = 4 + 30 = 34$ .

Let  $s_n$  stand for the sum of the first  $n$  terms of the arithmetic progression (1). That is,

$$(3) \quad s_n = a + (a + d) + (a + 2d) + (a + 3d) + \dots + (l - d) + l.$$

Reversing the order of the terms, we have also

$$(4) \quad s_n = l + (l - d) + (l - 2d) + (l - 3d) + \dots + (a + d) + a.$$

Adding terms written under each other in (3) and (4), we obtain

$$(5) \quad 2s_n = (a + l) + (a + l) + (a + l) + (a + l) + \dots + (a + l) + (a + l)$$

where there are  $n$  parentheses in the right member. Hence,  $2s_n = n(a + l)$ , or

$$(6) \quad s_n = \frac{n}{2} (a + l).$$

\* See section 140, chapter XIX, for the general definition of a sequence.



Using (2), we can rewrite (6) as

$$(7) \quad s_n = \frac{n}{2} (2a + [n - 1]d).$$

*Example 3.* Find the sum of the first 20 terms of the arithmetic progression in example 1.

*First solution.* Here  $a = 3$ ,  $d = 2$ ,  $n = 20$ . Hence, from (7),

$$s_{20} = \frac{20}{2} (2 \cdot 3 + [20 - 1]2) = 440.$$

*Second solution.* The twentieth term is  $3 + [20 - 1]2 = 41$ . From (6),

$$s_{20} = \frac{20}{2} (3 + 41) = 440.$$

*Example 4.* Find the sum of the first 50 positive integral multiples of 3.

*Solution.* The positive integral multiples of 3 form an arithmetic progression with  $a = 3$ , and  $d = 3$ . Hence  $s_{50} = \frac{50}{2} (2 \cdot 3 + [50 - 1]3) = 3825$ .

In an arithmetic progression of  $n$  terms, the first and last ( $n$ th) terms are called the **extremes**, while those in between are called **arithmetic means**.

*Example 5.* Insert five arithmetic means between the numbers 4 and 22.

*Solution.* After the five arithmetic means are inserted, the number of terms in the arithmetic progression is  $n = 7$ . We have  $a = 4$ ,  $l = 22$ . From (2) we have  $22 = 4 + (7 - 1)d$ ; hence  $d = 3$ . Therefore the arithmetic means are 7, 10, 13, 16, 19.

*Note 1.* If we insert one arithmetic mean between  $a$  and  $b$ , we find  $d = \frac{b - a}{2}$ . Hence the arithmetic mean is  $a + \frac{b - a}{2} = \frac{a + b}{2}$ . The number  $\frac{a + b}{2}$  is often called the **arithmetic mean** (or **average**) of  $a$  and  $b$ . Generalizing this definition, the number  $\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$  is called the **arithmetic mean** (or **average**) of the numbers  $a_1, a_2, \dots, a_n$ .

*Note 2.* Further applications of arithmetic progressions will be found in chapter XXIV.



## EXERCISES

Find the common difference, the last term, and the sum of each of the following arithmetic progressions:

1. 3, 7, 11, . . . to ten terms.
2. 4, 7, 10, . . . to nineteen terms.
3. 10, 7, 4, . . . to eight terms.
4.  $-12, -9, -6, \dots$  to fifteen terms.
5.  $2, 3\frac{2}{3}, 5\frac{1}{3}, \dots$  to twelve terms.
6. 8, 6.6, 5.2, . . . to sixteen terms.
7. Find the sum of the first 80 positive integers divisible by 4.
8. Find the sum of the first 75 positive integers divisible by 5.
9. (a) How many even numbers lie between 11 and 65? (b) Find their sum.
10. (a) How many numbers divisible by 4 lie between 50 and 190? (b) Find their sum.
11. (a) How many numbers divisible by 6 lie between 75 and 190? (b) Find their sum.
12. (a) How many numbers divisible by 8 lie between 100 and 500? (b) Find their sum.
13. Insert five arithmetic means between 2.9 and 15.5.
14. Insert six arithmetic means between 4 and 25.
15. Insert seven arithmetic means between  $-5$  and 11.
16. Insert eight arithmetic means between 18 and  $-9$ .
17. Find the sum of the first  $n$  even positive integers.
18. Find the sum of the first  $n$  odd positive integers.
19. Find the sum of the first  $n$  positive integral multiples of 4.
20. Find the sum of the first  $n$  positive integral multiples of 5.

In each of the following problems, some of the quantities  $a, d, l, n, s_n$ , are given; find the remaining ones:

- |                                  |                                  |
|----------------------------------|----------------------------------|
| 21. $d = 3, n = 13, s_n = 286$ . | 22. $d = 2, n = 11, s_n = 165$ . |
| 23. $a = 7, l = 75, n = 18$ .    | 24. $a = 8, l = 88, n = 17$ .    |
| 25. $a = 5, l = 25, d = 2$ .     | 26. $a = 4, l = 40, d = 3$ .     |
| 27. $d = 5, n = 13, l = 63$ .    | 28. $d = -3, n = 15, l = -27$ .  |
| 29. $a = 6, s_n = 720, l = 74$ . | 30. $a = 7, s_n = 472, l = 52$ . |

Find a formula for:

- \*31.  $d$  in terms of  $a, s_n$ , and  $l$ .
- \*32.  $s_n$  in terms of  $n, l$ , and  $d$ .
- \*33.  $d$  in terms of  $s_n, l$ , and  $n$ .
- \*34.  $l$  in terms of  $d, n$ , and  $s_n$ .
- \*35.  $a$  in terms of  $d, n$ , and  $s_n$ .

\* May be omitted without disturbing continuity.



*Find the arithmetic mean of:*

36. 14 and 20.

37. 60, 72, 87.

38. 50, 75, 82, 95.

39. The first year a man is employed he saves \$100. In each succeeding year he saves \$50 more than the year before. How much has he accumulated at the end of 15 years?

40. A man pays a \$1000 debt by paying \$100 at the end of each year plus 5% interest on the amount unpaid during that year. Find his total payment.

41. A man receives a salary of \$1600 per annum with the understanding that his salary will increase by \$160 per year up to and including the tenth year. (a) What will his salary be for the tenth year? (b) What are his total earnings during the 10 years?

42. A body falls 16 feet during the first second, 48 feet during the second second, 80 feet during the third second, and so on in arithmetic progression. (a) How far does it fall during the tenth second? (b) How far does it fall during the first 10 seconds?

43. Show that if the  $n$ th term of a sequence is a linear function of  $n$ , say  $pn + q$ , where  $p$  and  $q$  are constants, then the sequence is an arithmetic progression. Find expressions for  $a$ ,  $d$ ,  $l$ , and  $s_n$  in terms of  $p$ ,  $q$ , and  $n$ .

**147. Geometric progressions.** A sequence of numbers in which each term is obtained from the preceding term by multiplication by a constant factor  $r$  is called a **geometric progression**. The constant factor  $r$  is called the **common ratio**. If  $a$  is the first term of a geometric progression, with common ratio  $r$ , the first  $n$  terms are

$$(1) \quad a, ar, ar^2, ar^3, \dots, ar^{n-1}.$$

The  $n$ th term or last term  $l$  is

$$(2) \quad l = ar^{n-1}.$$

*Example 1.* The first five terms of the geometric progression with first term equal to 3 and common ratio equal to 2 are 3, 6, 12, 24, 48. The  $n$ th term is  $3 \cdot 2^{n-1}$ .

*Example 2.* Given that 16, 8, 4 are the first three terms of a geometric progression, find the fifth term.

*Solution.* Here  $r = 8/16 = 1/2$ . Hence the fifth term is  $16 \cdot (\frac{1}{2})^4 = 1$ .

Let  $s_n$  stand for the sum of the first  $n$  terms of the geometric progression (1). That is,

$$(3) \quad s_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}.$$

Multiplying both members of (3) by  $r$ , we obtain



$$(4) \quad rs_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n.$$

Subtracting (4) from (3), we get

$$(5) \quad (1 - r)s_n = a - ar^n.$$

Dividing (5) by\*  $1 - r$ , we have the formula

$$(6) \quad s_n = \frac{a - ar^n}{1 - r}.$$

Using (2) we may rewrite (6) as

$$(7) \quad s_n = \frac{a - rl}{1 - r}.$$

*Example 3.* Find the sum of the first seven terms of the geometric progression whose first three terms are 3, 6, 12.

*First solution.* Here  $r = 6/3 = 2$ ,  $a = 3$ . Hence by (6),

$$s_7 = \frac{3 - 3 \cdot 2^7}{1 - 2} = \frac{3 - 3 \cdot 128}{-1} = 381.$$

*Second solution.* The last term is  $l = 3 \cdot 2^6 = 3 \cdot 64 = 192$ .

Hence by (7),  $s_7 = \frac{3 - 2 \cdot 192}{1 - 2} = 381.$

In a geometric progression of  $n$  terms, the first and last ( $n$ th) terms are called the **extremes**, while those in between are called **geometric means**. We shall assume that the common ratio  $r$  is real.

*Example 4.* Insert five geometric means between the numbers 3 and 192.

*Solution.* After the five geometric means are inserted, the number of terms in the geometric progression is  $n = 7$ . Hence, from (2),  $192 = 3r^6$ , or  $r^6 = 64$ . Hence  $r = \pm 2$ . Therefore the five geometric means are 6, 12, 24, 48, 96, or  $-6, 12, -24, 48, -96$ .

*Note 1.* Inserting one geometric mean between the numbers  $a$  and  $b$ , we find that if  $x$  is a geometric mean then  $\frac{x}{a} = \frac{b}{x}$ . Thus  $x = \pm \sqrt{ab}$ . Either quantity  $\pm \sqrt{ab}$  is called a **geometric mean** of  $a$  and  $b$ . Generalizing this definition,  $\sqrt[n]{a_1 a_2 a_3 \cdots a_n}$  is called a **geometric mean** of  $a_1, a_2, \dots, a_n$ .

*Note 2.* Further applications of geometric progressions will be found in chapter XXIV.

\* It is assumed that  $r \neq 1$ . If  $r = 1$ ,  $s_n = na$ .



## EXERCISES

Find the common ratio, the last term, and the sum of each of the following geometric progressions:

1. 2, 6, 18, . . . , to five terms.
2. 128, 64, 32, . . . , to six terms.
3. 3, -6, 12, . . . , to six terms.
4. 6, 2,  $2/3$ , . . . , to six terms.
5. 3,  $-3/2$ ,  $3/4$ , . . . , to five terms.
6. 4, -6, 9, . . . , to six terms.
7. Insert four geometric means between 5 and 160.
8. Insert four geometric means between 384 and 12.
9. Insert four geometric means between 162 and  $-2/3$ .
10. Insert three geometric means between 2 and 512. How many sets of answers are there?

In each of the following problems, some of the quantities  $a$ ,  $r$ ,  $n$ ,  $l$ ,  $s_n$  are given; find the remaining ones:

11.  $a = 512$ ,  $r = 1/2$ ,  $l = 16$ .
12.  $s_n = 252$ ,  $l = 128$ ,  $r = 2$ .
13.  $a = 36$ ,  $r = 1/3$ ,  $l = 4/9$ .
14.  $s_n = 222.222$ ,  $r = 10$ ,  $a = 0.002$ .
15. Find the seventh term of a geometric progression whose fourth term is 4 and common ratio is 2.
16. Find the sixth term of a geometric progression whose third term is -6 and common ratio -3.

Find a geometric mean of each set of numbers:

17. 2 and 32.
18. 3 and 12.
19.  $1/5$ , 4, 10.
20.  $\frac{1}{2}$ , 6, 9.
21. If each bacterium in a culture divides into 2 bacteria every hour, how many bacteria will be present at the end of 6 hours if there are 4 bacteria at the start?
22. A man deposits 10 cents in a toy bank at the beginning of the month and doubles the size of his deposit every month thereafter. How much has he in the bank at the end of a year?
23. Suppose a dropped ball always rebounds  $1/2$  the height it falls. If it is dropped from a height of 128 feet, how far has it travelled when it reaches the top of the 5th bounce?
24. At the end of each year the value of a certain machine depreciates by 20% of the value it had at the beginning of the year. If it was worth \$1000 new, what is its value at the end of 5 years?
25. A man receives a salary of \$1600 per annum for the first year and a 10% raise every year for 10 years. What is his salary during the 5th year?
26. If \$1000 is deposited in a bank at 2% interest compounded annually, what is the amount in the account at the end of 5 years?



27. Suppose an investment depreciates in value to 20% of the original value during the first year, and then depreciates during the second year to 80% of the value it had at the start of the second year. What steady annual rate of decrease would yield the same resulting value at the end of 2 years?

28. Suppose a house costing \$10000 depreciates 10% in value each year. What is it worth at the end of 6 years?

29. A tank full of alcohol is emptied of  $\frac{1}{3}$  of its contents and then filled up with water and mixed. If this is done 6 times, what fraction of the original volume of alcohol remains?

30. Show that the reciprocals of the terms of a geometric progression also form a geometric progression.

**148. Harmonic progressions.** A sequence of numbers is said to form a **harmonic progression** provided their reciprocals form an arithmetic progression.

*Example 1.* The numbers  $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots, \frac{1}{a+(n-1)d}$  form a harmonic progression, provided none of the denominators is zero.

The first and last terms of a harmonic progression of  $n$  terms are called the **extremes**, while the terms in between are called **harmonic means**. To insert  $k$  harmonic means between  $a$  and  $b$ , it is sufficient to find  $k$  arithmetic means between  $1/a$  and  $1/b$ ; the reciprocals of these arithmetic means will be the required harmonic means.

*Example 2.* Insert five harmonic means between  $1/4$  and  $1/22$ .

*Solution.* Five arithmetic means between 4 and 22 are 7, 10, 13, 16, 19. Hence  $1/7, 1/10, 1/13, 1/16, 1/19$  are the required harmonic means.

*Note.* If one harmonic mean  $H$  is inserted between  $a$  and  $b$ ,  $H$  is called the **harmonic mean** of  $a$  and  $b$ . Then  $\frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b}}{2}$

or  $H = \frac{2ab}{a+b}$ . Generalizing this definition the harmonic mean of  $a_1, a_2, \dots, a_n$  is the number  $H$  given by

$$\frac{1}{H} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}{n}.$$



## EXERCISES

1. Find the eighth term of the harmonic progression  $\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \dots$
2. Find the twelfth term of the harmonic progression  $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \dots$
3. Insert three harmonic means between  $1/4$  and  $1/24$ .
4. Insert four harmonic means between  $3/2$  and  $3/32$ .
5. Insert three harmonic means between  $1/14$  and  $-1/2$ .
6. Insert four harmonic means between  $-2/3$  and  $1/14$ .

Find the harmonic mean of each set of numbers:

7. 5, 2.

8. 6, 12.

9. 8, 3,  $1/2$ .

10.  $1/3$ , 6, 9, 12.

11. Show that if  $A$ ,  $G$ , and  $H$  are respectively the arithmetic, geometric, and harmonic means of two numbers  $a$  and  $b$ , then  $G^2 = AH$ .

12. Show that if a car travels a certain distance at the rate of  $x$  miles per hour and makes the return trip at the rate of  $y$  miles per hour, then the average rate for the entire journey is the harmonic mean of  $x$  and  $y$ .

**149. Infinite geometric series. Periodic decimals.** The sum of the terms of an unending geometric progression

$$(1) \quad S = a + ar + ar^2 + \dots + ar^{n-1} + \dots,$$

where the number of terms in the right member is infinite, is an example of a so-called **infinite series**. It is called a **geometric series**. If  $S_n$  is the sum of the first  $n$  terms, the sum of an infinite series is defined as  $S = \lim_{n \rightarrow \infty} S_n$ , provided this limit exists. See chapter

XXVII for a general discussion of infinite series.

If  $r$  is between  $-1$  and  $+1$ , the sum of the infinite geometric series (1) is given by the formula

$$(2) \quad S = \frac{a}{1-r}.$$

For,

$$(3) \quad S_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \cdot r^n.$$

But it can be proved that if  $-1 < r < 1$ , then  $\lim_{n \rightarrow \infty} r^n = 0$ ; for example,  $\lim_{n \rightarrow \infty} (\frac{1}{2})^n = 0$ , since successive powers of  $\frac{1}{2}$  clearly

approach zero as  $n$  increases, or  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots, \frac{1}{2^n}, \dots \rightarrow 0$ .

Hence the last term of (3) has zero as a limit as  $n$  increases indefinitely. Hence  $S = \frac{a}{1-r}$ .

**Example 1.** Find the sum  $S$  of the infinite geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}} + \dots$$



*Solution.* Here  $a = 1$ ,  $r = \frac{1}{2}$ . Hence  $S = \frac{1}{1 - \frac{1}{2}} = 2$ .

An unending decimal is called **periodic** if the same block of digits repeats itself endlessly. See section 14.

*Example 2.* Write in lowest terms the rational number which has the periodic decimal expression  $0.18181818 \dots$

*Solution.* The given decimal may be considered as an infinite geometric series

$$\frac{18}{100} + \frac{18}{100} \left( \frac{1}{100} \right) + \frac{18}{100} \left( \frac{1}{100} \right)^2 + \dots + \frac{18}{100} \left( \frac{1}{100} \right)^{n-1} + \dots$$

with  $a = 18/100$  and  $r = 1/100$ . By (2), its sum is

$$\frac{18/100}{1 - 1/100} = \frac{18/100}{99/100} = \frac{18}{99} = \frac{2}{11}.$$

It is clear, from example 2, that one could prove that every periodic decimal represents a rational number, which may be found by the method of example 2.

## EXERCISES

*Find the sum of each of the following infinite geometric series:*

1.  $1 + \frac{1}{3} + \frac{1}{9} + \dots$

2.  $8 + 4 + 2 + \dots$

3.  $1 - \frac{1}{2} + \frac{1}{4} - \dots$

4.  $3 - 0.3 + 0.03 - \dots$

5.  $3 + 2 + \frac{4}{3} + \dots$

6.  $4 - 3 + 9/4 - \dots$

*Find, in simplest form, the rational number which has the given periodic decimal expression, the repeating block of digits being written below only three times:*

7.  $0.666 \dots$

8.  $0.444 \dots$

9.  $0.181818 \dots$

10.  $0.343434 \dots$

11.  $0.1666 \dots$

12.  $3.252525 \dots$

13.  $0.230769230769230769 \dots$

14.  $0.142857142857142857 \dots$

15.  $2.090909 \dots$

16.  $3.272727 \dots$

17. Suppose a ball rebounds one-half the distance it falls. If it is dropped from a height of 40 feet, how far does it travel before\* coming to rest?

18. Suppose that each swing of a pendulum bob is 90% as long as the preceding swing. If the first swing is 20 inches long, how far does the bob travel before coming to rest?

\* Assume infinitely many bounces. This is a convenient idealization of actual reality which gives a very good approximation, such as is always made in applying mathematics to concrete situations.



19. Suppose a ball rebounds  $\frac{3}{4}$  the distance it falls. If it is dropped from a height of 40 feet, how far does it travel before coming to rest?

20. Suppose that each swing of a pendulum bob is 80% as long as the preceding swing. If the first swing is 20 inches long, how far does the bob travel before coming to rest?

\*150. Differences of  $r$ th order. Consider a sequence of numbers

$$(1) \quad a_0, a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$$

where  $a_n$  is the  $(n + 1)$ th term. The differences of first order are defined as the terms of the sequence

$$(2) \quad \Delta a_0 = a_1 - a_0, \Delta a_1 = a_2 - a_1, \Delta a_2 = a_3 - a_2, \dots, \\ \Delta a_n = a_{n+1} - a_n, \dots$$

The differences of second order are defined as the differences of first order of the sequence (2) of differences of the first order. In symbols, the differences of the second order are

$$(3) \quad \Delta^2 a_0 = \Delta a_1 - \Delta a_0, \Delta^2 a_1 = \Delta a_2 - \Delta a_1, \Delta^2 a_2 = \Delta a_3 - \Delta a_2, \\ \dots \Delta^2 a_n = \Delta a_{n+1} - \Delta a_n, \dots$$

In general, the differences of  $r$ th order are defined as the differences of first order of the sequence of differences of the  $(r - 1)$ th order. Thus,

$$(4) \quad \Delta^r a_0 = \Delta^{r-1} a_1 - \Delta^{r-1} a_0, \Delta^r a_1 = \Delta^{r-1} a_2 - \Delta^{r-1} a_1, \dots, \\ \Delta^r a_n = \Delta^{r-1} a_{n+1} - \Delta^{r-1} a_n, \dots,$$

for  $r = 2, 3, 4, \dots$ , and  $n = 0, 1, 2, \dots$ .

*Example 1.* From the sequence

$$1, 8, 27, 64, 125, \dots, n^3, \dots$$

we obtain the differences of

$$\begin{aligned} \text{1st order: } & 7, 19, 37, 61, \dots, (n+1)^3 - n^3 = 3n^2 + 3n + 1, \\ & \dots; \\ \text{2nd order: } & 12, 18, 24, \dots, 3(n+1)^2 + 3(n+1) + 1 - (3n^2 \\ & \quad + 3n + 1) = 6n + 6, \dots; \\ \text{3rd order: } & 6, 6, \dots, 6(n+1) + 6 - (6n + 6) = 6, \dots; \\ \text{all higher orders: } & 0, 0, \dots, 0, \dots \end{aligned}$$

By direct calculation, we obtain from the definitions (2), (3), and (4) above, the relations

\* May be omitted without disturbing continuity.



$$(5) \quad \left\{ \begin{array}{l} a_1 = a_0 + \Delta a_0 \\ a_2 = a_1 + \Delta a_1 = (a_0 + \Delta a_0) + (\Delta a_0 + \Delta^2 a_0) \\ \qquad \qquad \qquad = a_0 + 2\Delta a_0 + \Delta^2 a_0 \\ a_3 = a_0 + 3\Delta a_0 + 3\Delta^2 a_0 + \Delta^3 a_0 \\ \cdot \\ \cdot \\ \cdot \\ a_n = a_0 + C(n,1)\Delta a_0 + C(n,2)\Delta^2 a_0 + \cdots + C(n,n)\Delta^n a_0 \\ \cdot \\ \cdot \\ \cdot \end{array} \right.$$

which express the successive terms of the given sequence in terms of its first term and the first terms in the successive rows of differences. The relations (5) can be summed up as follows. For  $n = 1, 2, 3, \cdots$ , the  $(n+1)$ th term of the sequence (1) is given by

$$(6) \quad a_n = a_0 + n\Delta a_0 + \frac{n(n-1)}{2!} \Delta^2 a_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 a_0 + \cdots + \Delta^n a_0,$$

where the coefficients on the right are the coefficients in the binomial expansion of  $(x+y)^n$ . Formula (6) may be proved for all positive integral values of  $n$  by mathematical induction; this will not be done here.

Consider the sequence (1) and let  $s_k$  be the sum of its first  $k$  terms; that is

$$(7) \quad s_k = a_0 + a_1 + \cdots + a_{k-1}.$$

Then the sequence

$$(8) \quad s_0 = 0, s_1, s_2, s_3, \cdots, s_k, \cdots$$

is a sequence whose differences of first order are exactly the terms of the sequence (1), for

$$\Delta s_0 = s_1 - s_0 = a_0, \Delta s_1 = s_2 - s_1 = a_1, \cdots, \\ \Delta s_k = s_{k+1} - s_k = a_k, \cdots$$

Consequently the  $r$ th differences of the sequence (8) are the  $(r-1)$ th differences of the sequence (1). By (5), we may write

$$(9) \quad s_k = s_0 + C(k,1)\Delta s_0 + C(k,2)\Delta^2 s_0 + C(k,3)\Delta^3 s_0 + \cdots \\ + C(k,k)\Delta^k s_0.$$

Hence,



$$(10) \quad s_k = C(k,1)a_0 + C(k,2)\Delta a_0 + C(k,3)\Delta^2 a_0 + \cdots + C(k,k)\Delta^{k-1}a_0.$$

Note that in general the expression (10) for  $s_k$  has  $k$  terms.

**\*151. Arithmetic progressions of  $r$ th order.** A sequence for which all the differences of  $r$ th order are equal, and this is true for no lower order, is called an **arithmetic progression of  $r$ th order**.

For example, an arithmetic progression of first order is what we have formerly called an arithmetic progression. The sequence of example 1, section 150, is an arithmetic progression of third order.

For an arithmetic progression of  $r$ th order, all differences of  $(r+1)$ th and higher orders are zero. Hence (10), section 150, yields a terminating expression for the sum of the first  $n$  terms of such a progression. That is

$$(1) \quad s_n = C(n,1)a_0 + C(n,2)\Delta a_0 + C(n,3)\Delta^2 a_0 + \cdots + C(n, r+1)\Delta^r a_0$$

or,

$$(2) \quad s_n = na_0 + \frac{n(n-1)}{2!}\Delta a_0 + \frac{n(n-1)(n-2)}{3!}\Delta^2 a_0 + \cdots + \frac{n(n-1)\cdots(n-r)}{(r+1)!}\Delta^r a_0.$$

*Example 1.* In the sequence of example 1, section 150, we have  $a_0 = 1$ ,  $\Delta a_0 = 7$ ,  $\Delta^2 a_0 = 12$ ,  $\Delta^3 a_0 = 6$ ,  $\Delta^4 a_0 = 0 = \Delta^5 a_0 = \cdots$ . Hence the sum  $s_n$  of the first  $n$  terms of the given sequence is

$$s_n = n + \frac{n(n-1)}{2} \cdot 7 + \frac{n(n-1)(n-2)}{6} \cdot 12 + \frac{n(n-1)(n-2)(n-3)}{24} \cdot 6.$$

Or,

$$s_n = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4}.$$

*Example 2.* Find the sum of the first  $n$  terms of the sequence 1, 4, 9, 16, . . . ,  $n^2$ , . . . .

*Solution.* The given sequence and its successive differences may be arranged as follows:

Sequence: 1, 4, 9, 16, . . . ,  $n^2$ ,  $(n+1)^2$ ,  $(n+2)^2$ , . . .  
 1st differences: 3, 5, 7, . . . ,  $2n+1$ ,  $2n+3$ , . . .  
 2nd differences: 2, 2, . . . , 2, . . .

\* May be omitted without disturbing continuity.



Hence we have an arithmetic progression of 2nd order. The sum of the first  $n$  terms is

$$s_n = n + \frac{n(n-1)}{2} \cdot 3 + \frac{n(n-1)(n-2)}{6} \cdot 2 = \frac{2n^3 + 3n^2 + n}{6} \\ = \frac{n(n+1)(2n+1)}{6}$$

The illustrative examples suggest that the general or  $n$ th term of an arithmetic progression of  $r$ th order is a polynomial in  $n$  of degree  $r$ . We shall prove this and the converse.

**THEOREM 1.** *The general term of an arithmetic progression of  $r$ th order is a polynomial in  $n$  of degree  $r$ .*

*Proof.* By definition, all differences of order  $r+1$  or higher are zero; that is,  $\Delta^{r+1}a_0 = \Delta^{r+2}a_0 = \cdots = 0$ . Hence (6) of section 150 becomes

$$a_n = a_0 + n\Delta a_0 + \frac{n(n-1)}{2!} \Delta^2 a_0 + \cdots \\ + \frac{n(n-1) \cdots (n-r+1)}{r!} \Delta^r a_0.$$

The numerator of the last coefficient has  $r$  factors and hence  $a_n$  is a polynomial of degree  $r$  in the variable  $n$ .

**THEOREM 2.** *If  $a_n = f(n)$  where  $f(n) = c_0 n^r + c_1 n^{r-1} + \cdots + c_r$  is a polynomial of degree  $r$  in the variable  $n$ , then the sequence  $a_0, a_1, a_2, \dots, a_n, \dots$  is an arithmetic progression of  $r$ th order.*

*Proof.* By definition,

$$\Delta a_n = a_{n+1} - a_n = f(n+1) - f(n) \\ = c_0(n+1)^r + c_1(n+1)^{r-1} + \cdots + c_r - c_0 n^r - c_1 n^{r-1} - \cdots - c_r \\ = c_0 n^r + r c_0 n^{r-1} + \cdots + c_1 n^{r-1} + \cdots + c_r - c_0 n^r \\ - c_1 n^{r-1} - \cdots - c_r \\ = r c_0 n^{r-1} + \cdots$$

Hence  $\Delta a_n$  is a polynomial in  $n$  of degree  $r-1$ . It follows that  $\Delta^r a_n$  is a constant (a polynomial of degree zero) and hence  $\Delta^{r+1} a_n = \Delta^{r+2} a_n = \cdots = 0$ .

## EXERCISES

(a) Determine the order of each arithmetical progression; (b) find a formula in terms of  $n$  for the sum of the first  $n$  terms of each progression:

1.  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1).$

2.  $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2).$



$$3. 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2}.$$

$$4. 3 + 12 + 33 + \cdots + n(n^2 + 2).$$

$$5. 1 + 16 + 81 + \cdots + n^4.$$

$$6. 3 + 11 + 31 + \cdots + (n^3 + n + 1).$$

7. Find the total number of spherical shot piled in a pyramid with an equilateral triangle as base, having just one shot in the top layer (a) if there are  $n$  layers; (b) if there are 18 layers.

8. A grocer displays cans of soup in a pyramid, having a rectangular base, whose top layer consists of one row of 6 cans. Find the total number of cans (a) if there are  $n$  layers; (b) if there are 15 layers.

9. Find the total number of spherical shot piled in a pyramid with a square base, having just one shot in the top layer, (a) if there are  $n$  layers; (b) if there are 18 layers.

10. Find the total number of spherical shot piled in a pyramid with a square base, having 16 shot in the top layer if there are 20 layers.

(a) By formula (6) of section 150, find a possible  $n$ th term of the sequence whose first five terms are given. (b) Find another possible  $n$ th term yielding the same given first five terms. (c) Is there another  $n$ th term different from that formed in (a) whose degree is not greater than that of the answer to (a)? Explain. (Hint: consider corollary 2 of section 104.)

$$11. 1, 3, 7, 13, 21, \dots$$

$$12. 6, 11, 18, 27, 38, \dots$$

$$13. 1, 5, 19, 49, 101, \dots$$

$$14. 1, 1, 9, 31, 73, \dots$$

\*15. Prove, by mathematical induction, that formula (6) of section 150 is true for all positive integral values of  $n$ .

\* May be omitted without disturbing continuity.



## CHAPTER XXII

### Inequalities

**152. Inequalities.** If  $a$  and  $b$  are any real numbers, then one and only one of the three relations  $a > b$ ,  $a = b$ ,  $a < b$  holds. *In this chapter we consider only real numbers.* The statement  $a > b$ , read “ $a$  is greater than  $b$ ,” means that  $a - b$  is positive, or that  $a = b + k$  where  $k$  is positive. Graphically,  $a > b$  means that  $a$  lies to the right of  $b$  if they are plotted on the  $x$ -axis, or that  $a$  is above  $b$  if they are plotted on the  $y$ -axis. For example,  $-2 > -4$ ,  $0 > -2$ ,  $2 > -4$ . The symbol  $a \geq b$  means that  $a$  is greater than or equal to  $b$ , or  $a$  is not less than  $b$ . Similarly  $a \leq b$  means that  $a$  is less than or equal to  $b$ , or  $a$  is not greater than  $b$ . Of course,  $b < a$  means the same thing as  $a > b$ .

A statement involving one of the symbols  $>$ ,  $<$ ,  $\geq$ ,  $\leq$  is called an **inequality**. An inequality which involves only constants, or one which is true for all permissible values of the variables involved, is called an **absolute inequality**. If an inequality fails to be true for some permissible value of the variables involved, it is called a **conditional inequality**. To solve a conditional inequality means to find those values of the variables for which it is true.

*Examples.*  $a^2 \geq 0$ ,  $2 > -4$ ,  $\pi < 22/7$  are absolute inequalities. But  $x + 2 < 7$  is a conditional inequality which is true only if  $x < 5$ . Similarly  $x^2 < 9$  is conditional, being true only if  $-3 < x < 3$ .

The **absolute value** of the real number  $x$  is defined as  $x$  itself if  $x \geq 0$ , and as the positive number  $-x$  if  $x < 0$ . The absolute value of  $x$  is denoted by  $|x|$ .

*Examples.*  $|3| = 3$ ,  $|-3| = 3$ ,  $|0| = 0$ .

Two inequalities are said to have the **same sense** if their symbols for inequality point in the same direction; they are said to have **opposite senses** if their symbols for inequality point in opposite directions.

*Examples.*  $x > y$  and  $u > v$  have the same sense, as do  $y < x$  and  $v < u$ . But  $x < y$  and  $v > u$  have opposite senses.



**153. Properties of inequalities.** The following theorems concerning inequalities are intuitively plausible, and can be proved from the axioms of section 17, chapter I.

**THEOREM 1.** *The sense of an inequality is not changed if the same real number is added to (or subtracted from) both sides. That is, if  $a > b$ , then  $a + c > b + c$  (and  $a - c > b - c$ ).*

*Proof.* By hypothesis,  $a > b$ , or  $a - b = k$  where  $k$  is positive. That is,  $a = b + k$ . Adding  $c$  to both sides, we have  $(a + c) = (b + c) + k$ . Therefore  $(a + c) - (b + c) = k$  where  $k$  is positive. Hence  $a + c > b + c$ .

Since subtracting  $c$  is equivalent to adding  $-c$ , we have similarly  $a - c > b - c$ .

**THEOREM 2.** *The sense of an inequality is not changed if both sides are multiplied or divided by the same positive number. That is, if  $a > b$  and  $c > 0$ , then  $ac > bc$  and  $a/c > b/c$ .*

*Proof.* By hypothesis,  $a = b + k$  where  $k$  is positive. Multiplying both sides by  $c$ , we have  $ac = bc + kc$ . But  $kc$  is positive. Hence  $ac > bc$ .

Since dividing by  $c$ , is equivalent to multiplying by the positive number  $1/c$ , we have similarly  $a/c > b/c$ .

**THEOREM 3.** *The sense of an inequality is reversed if both sides are multiplied or divided by the same negative number  $c$ . That is, if  $a > b$  and  $c < 0$ , then  $ac < bc$  and  $a/c < b/c$ .*

*Proof.* By hypothesis,  $a = b + k$ , where  $k$  is positive. Then  $ac = bc + kc$  where  $kc$  is negative. Or, transposing,  $-kc = bc - ac$  where  $-kc$  is positive. Hence,  $bc > ac$ , or  $ac < bc$ . The proof of  $a/c < b/c$  is left to the student.

**THEOREM 4.** *If  $a > b$  and if  $ab > 0$ , then  $\frac{1}{a} < \frac{1}{b}$ .*

*Proof.* Suppose  $a$  and  $b$  are positive. Let  $A = \frac{1}{a}$ ,  $B = \frac{1}{b}$ .

Then  $A$  and  $B$  are positive. By hypothesis,  $a = b + k$  where  $k$  is positive. Multiplying both sides by  $AB$ , we get  $aAB = bBA + kAB$ , or  $B = A + kAB$ . But  $kAB$  is positive. Hence  $B > A$ ,

or  $\frac{1}{b} > \frac{1}{a}$ , or  $\frac{1}{a} < \frac{1}{b}$ . The case where  $a$  and  $b$  are both negative is

left to the student.

The proofs of the remaining theorems are left as exercises.

**THEOREM 5.** *If unequals are added to unequals in the same sense, the results are unequal in the same sense. That is, if  $a > b$  and  $c > d$ , then  $a + c > b + d$ .*



**THEOREM 6.** If  $a, b, c, d$  are positive and if  $a > b$  and  $c > d$ , then  $ac > bd$ .

**THEOREM 7.** If  $a$  and  $b$  are positive and  $a > b$ , then  $a^n > b^n$  and  $\sqrt[n]{a} > \sqrt[n]{b}$  where  $n$  is any positive integer.

**154. Solution of inequalities.** By the use of the properties of the preceding section we may verify absolute inequalities and solve conditional inequalities, much as we treat equalities.

**Example 1.** Find those values of  $x$  for which  $3x + 4 < x + 10$ .

**Solution.** If  $x$  is such that  $3x + 4 < x + 10$ , then subtracting  $x$  and 4 from both sides we have  $2x < 6$ . Dividing both sides by 2 we have  $x < 3$ . We have proved that if  $x$  satisfies  $3x + 4 < x + 10$  then  $x < 3$ . The converse "if  $x < 3$  then  $3x + 4 < x + 10$ " can now be proved by reversing each step in the above discussion, since each step is reversible. Hence  $3x + 4 < x + 10$  for all values of  $x < 3$ .

**Example 2.** Show that if  $x \neq y$  then  $x^2 + y^2 > 2xy$ .

**Solution.** If  $x$  and  $y$  satisfy  $x^2 + y^2 > 2xy$ , then they also satisfy  $x^2 - 2xy + y^2 > 0$  or  $(x - y)^2 > 0$ . But the latter inequality is true if  $x - y \neq 0$  or  $x \neq y$ . We have proved that if  $x^2 + y^2 > 2xy$  then  $x \neq y$ . Again each step is reversible and hence the converse proposition, which we had to prove, is true.

## EXERCISES

Find the values of  $x$  for which the following inequalities are true:

1.  $2x - 8 < 0$ .
2.  $3x - 9 > 0$ .
3.  $4x - 1 < x + 8$ .
4.  $2x - 1 < \frac{8 - x}{2}$ .
5.  $\frac{4x - 3}{2} > \frac{2 - x}{3}$ .
6.  $\frac{x + 3}{2} > \frac{2 - x}{-2}$ .
7.  $x^2 < 4$ .
8.  $16x^2 < 9$ .
9.  $x^2 > 16$ .
10.  $9x^2 > 25$ .
11.  $(x - 3)(x - 2) < 0$ . (Hint: the factors must have opposite signs.)
12.  $(x - 5)(x - 1) < 0$ .
13.  $(x - 2)(x - 4) > 0$ .
14.  $(x - 1)(x - 3) > 0$ .
15.  $(x - 1)(x + 2) < 0$ .
16.  $(x - 2)(x + 3) > 0$ .
17.  $x^2 - 5x + 6 < 0$ .
18.  $x^2 + x - 12 < 0$ .
19.  $(x + 1)(x - 3)(x - 4) < 0$ .
20.  $(x + 2)(x - 1)(x - 3) > 0$ .

Assuming that all letters represent positive numbers, prove the following inequalities:

21.  $a + b > \frac{4ab}{a + b}$  if  $a \neq b$ .

22.  $a + \frac{1}{a} > 2$  if  $a \neq 1$ .



$$23. \frac{a}{b} + \frac{b}{a} > 2 \quad \text{if } a \neq b.$$

$$24. a^3 + 3ab^2 > b^3 + 3a^2b \quad \text{if } a > b.$$

$$25. a^3 + b^3 > a^2b + ab^2 \quad \text{if } a \neq b.$$

26. If  $a \neq b$  and if  $A$ ,  $G$ , and  $H$  are respectively the arithmetic, geometric, and harmonic means of  $a$  and  $b$ , then  $A > G > H$ . (Hint: See exercise 11, section 148.)

If all letters represent any real numbers, prove that:

$$*27. |x| + |y| \geq |x + y|.$$

$$*28. |x| - |y| \leq |x - y|.$$

$$*29. |ab| = |a||b|.$$

\*30.  $(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$ . (Hint: the quadratic equation  $(b_1x - a_1)^2 + (b_2x - a_2)^2 = (b_1^2 + b_2^2)x^2 - 2(a_1b_1 + a_2b_2)x + (a_1^2 + a_2^2) = 0$  cannot have distinct real roots; therefore its discriminant must be  $\leq 0$ .)

\*31.  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$ . (Hint: use exercise 30.)

\*32.  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} \geq \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}$ .

155. Graphical significance of inequalities. Any conditional inequality involving one variable  $x$  can be written in the form  $f(x) > 0$  or  $f(x) \geq 0$ . Then the values of  $x$  for which  $f(x) > 0$  are those values of  $x$  for which the graph of  $y = f(x)$  lies above the  $x$ -axis.

*Example.* Find graphically the values of  $x$  for which  $x^2 > 2x + 3$ .

*Solution.* We have  $f(x) = x^2 - 2x - 3 > 0$ . The graph of  $y = x^2 - 2x - 3$  crosses the  $x$ -axis at  $x = -1$  and at  $x = 3$ . From the graph (Fig. 58) we see that  $y = f(x) > 0$  for  $x > 3$  and for  $x < -1$ .

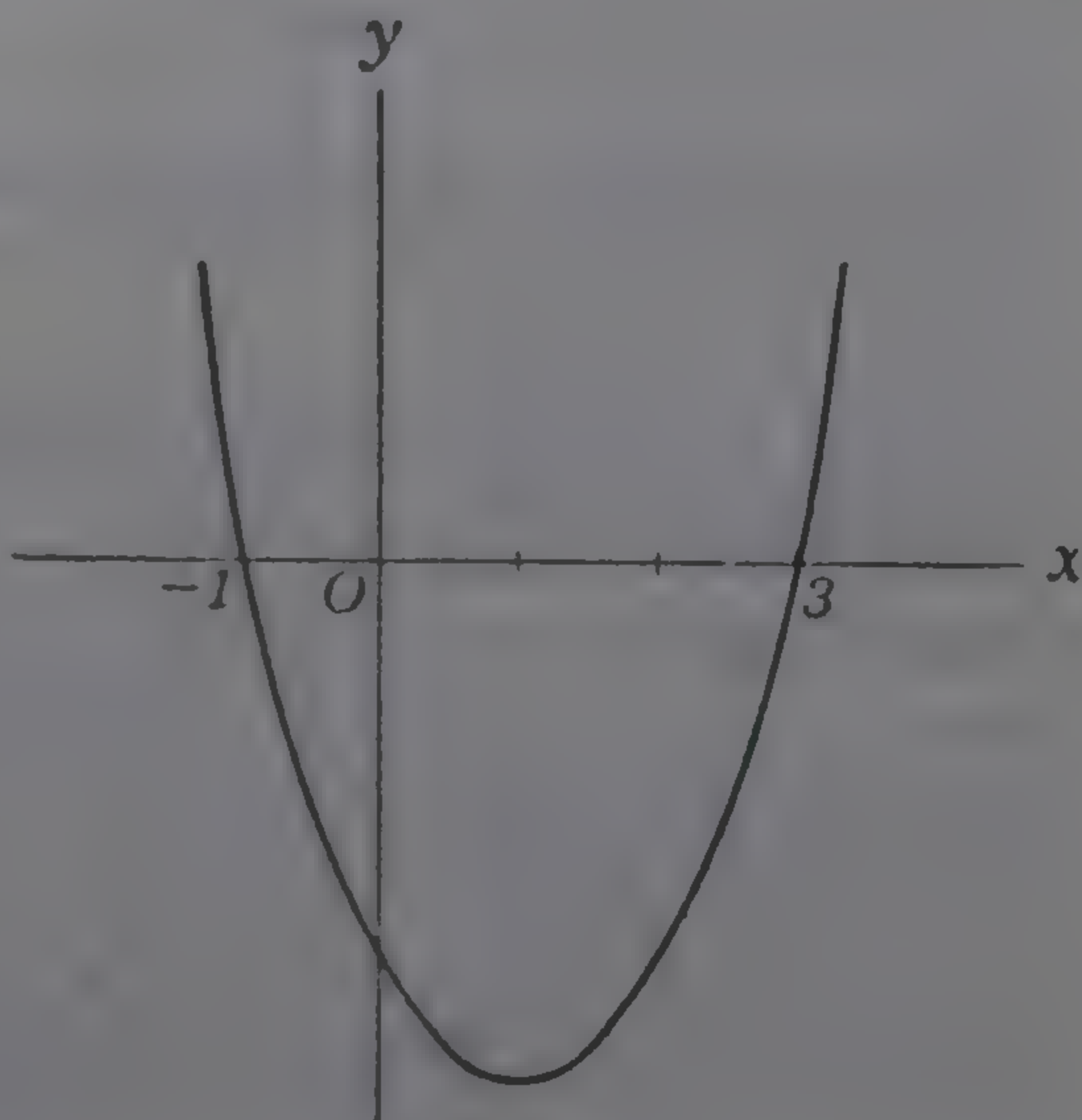


Fig. 58.

An inequality involving two variables,  $x$  and  $y$ , describes a region of the plane. Thus the straight line  $x + y = 3$  divides the rest of the plane into two regions, namely the region con-

\* May be proved without discussing continuity.



sisting of all points  $(x,y)$  such that  $x + y > 3$ , and the region consisting of all points  $(x,y)$  such that  $x + y < 3$ . (Fig. 59.) Similarly, the circle  $x^2 + y^2 = 4$  divides the rest of the plane into two regions, one for which  $x^2 + y^2 < 4$ , and the other for which  $x^2 + y^2 > 4$  (Fig. 60).

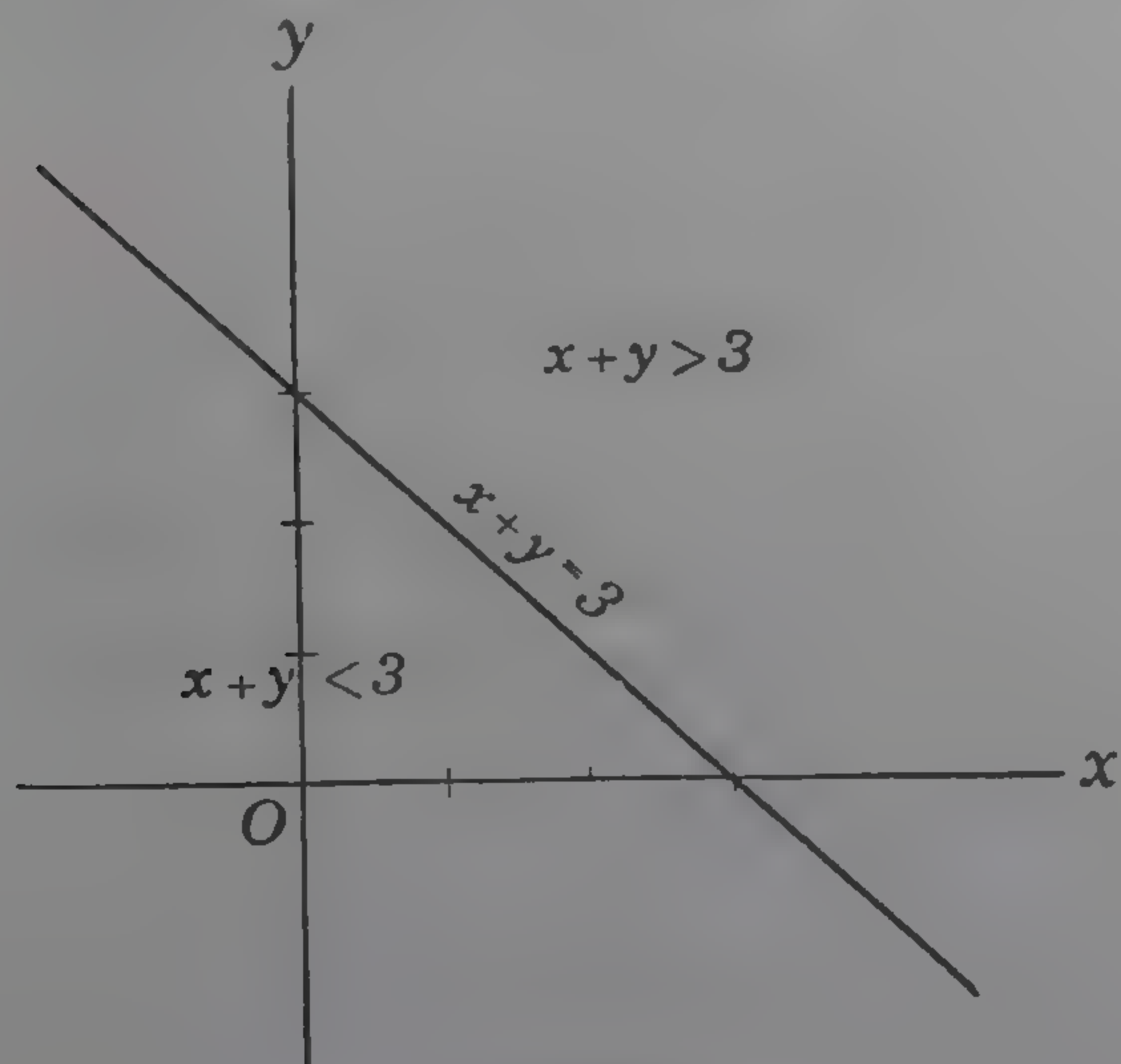


Fig. 59.

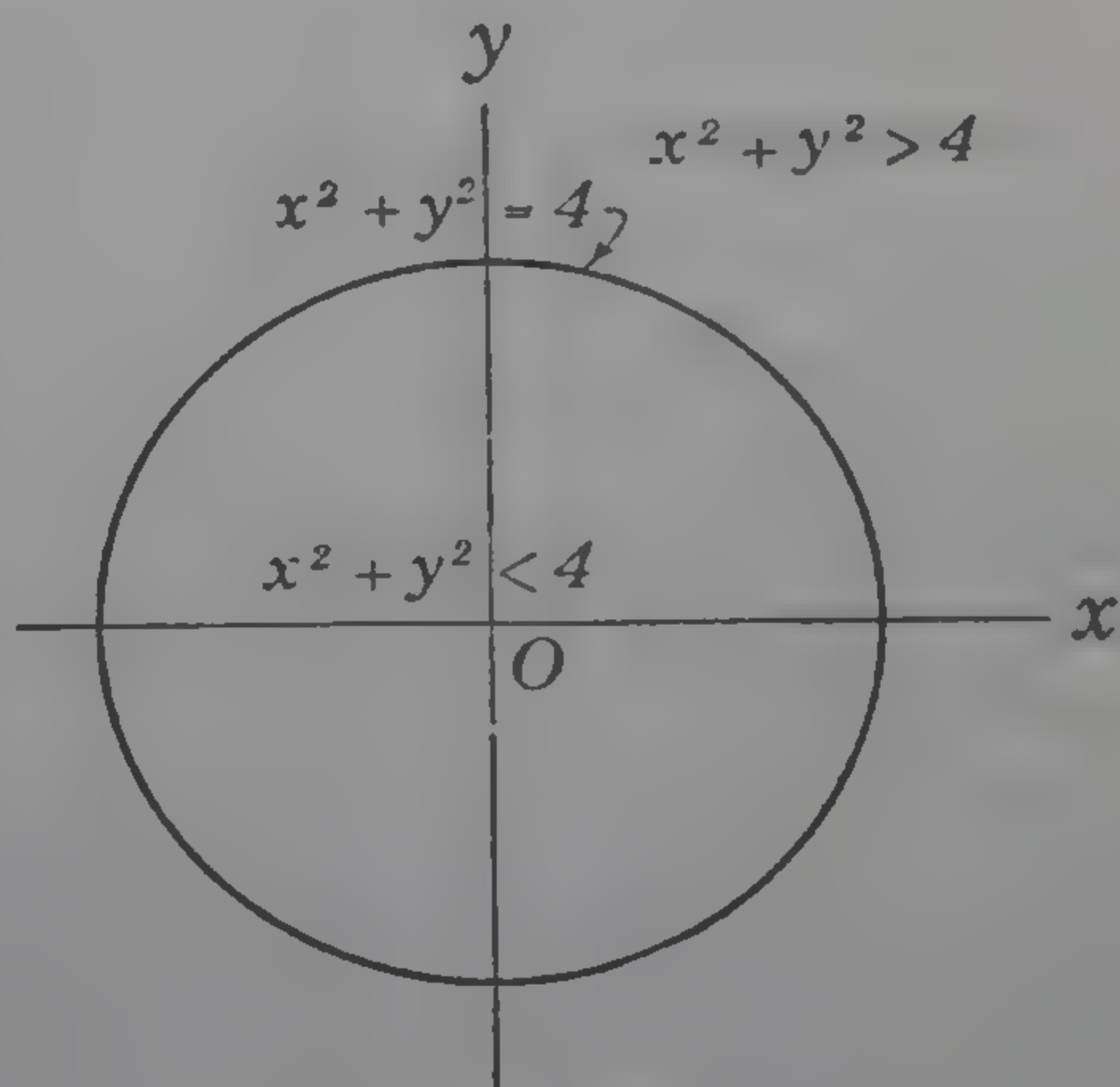


Fig. 60.

## EXERCISES

Find graphically the values of  $x$  for which the following inequalities are true:

1–20. Exercises 1–20, inclusive, of section 154.

21.  $x^2 + 3x - 4 > 0$ .    22.  $x^2 < 7x - 10$ .    23.  $x^2 < 3 - 2x$ .

24.  $x^2 > 10 - 3x$ .

Show on a graph the region described by the following inequalities or sets of inequalities:

25.  $x - y < 1$ .

26.  $2x + y < 3$ .

27.  $4y < x^2$ .

28.  $4y > x^2$ .

29.  $x^2 + y^2 < 1$ .

30.  $x^2 + y^2 > 1$ .

31.  $x^2 + y^2 < 25, x + y < 1$ .

32.  $y > x^2, x + y < 1$ .

33.  $x > y^2, x < y$ .

34.  $x + y < 2, y - x < 1, 2y - x + 2 > 0$ .

35.  $x^2 + y^2 < 1, y > x, x + y + 1 > 0$ .



## CHAPTER XXIII

### Logarithms

**156. Definition of logarithm.** We have noted (see chapter V) the simplifications of arithmetic due to the use of Hindu-Arabic notation and decimal fractions. In the 17th century, the computations required in astronomy and other sciences became extremely tedious, and further simplifications were sought. In answer to this need, the mathematicians of the 17th century produced logarithms, mechanical calculating machines, and slide rules. Slide rules are convenient but severely limited in accuracy. While calculating machines are much used today in office work, they are not available to everyone. Since they are expensive and not portable, they still do not replace completely the logarithms which will be explained here.\* Quite apart from numerical calculations, logarithms are essential in higher mathematics. *All numbers in this chapter are understood to be real.*

**DEFINITION.** If  $b$  is any positive number, different from 1, and if  $b^y = x$ , then the exponent  $y$  is called the **logarithm** of  $x$  to the **base**  $b$ . In symbols  $y = \log_b x$ . That is, the logarithm of  $x$  to the base  $b$  is that exponent  $y$  to which  $b$  must be raised in order to get  $x$ .

*Example.*  $3 = \log_2 8$  since  $2^3 = 8$ . Similarly  $2 = \log_3 9$ , since  $3^2 = 9$ .

*Note 1.* It can be proved that every positive real number  $x$  can be expressed as  $b^y$ , where  $y$  is a real number. We shall study here only logarithms of positive numbers.

*Exercise.* Why do we make the restriction that  $b \neq 1$ ?

*Note 2.* If we did not restrict  $b$  to be positive, we would introduce technical difficulties which would serve no useful purpose here. For example, if  $b$  were negative,  $b^2$  would be positive,  $b^3$  negative,  $b^{1/2}$  imaginary, etc.

### EXERCISES

*Express in logarithmic notation:*

- |                    |                        |                      |                     |
|--------------------|------------------------|----------------------|---------------------|
| 1. $5^2 = 25$ .    | 2. $3^4 = 81$ .        | 3. $10^2 = 100$ .    | 4. $25^{1/2} = 5$ . |
| 5. $8^{1/3} = 2$ . | 6. $8^{-1/3} = 1/2$ .  | 7. $5^{-2} = 1/25$ . | 8. $2^{-3} = 1/8$ . |
| 9. $5^0 = 1$ .     | 10. $10^{-1} = 1/10$ . |                      |                     |

\* For a brief explanation of calculating machines and slide rules, see M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, chapter VII, section 57.



*Express in exponential notation:*

- |                                 |                           |                            |
|---------------------------------|---------------------------|----------------------------|
| 11. $\log_2 16 = 4.$            | 12. $\log_{10} 1000 = 3.$ | 13. $\log_{10} 0.01 = -2.$ |
| 14. $\log_{10} 10 = 1.$         | 15. $\log_{10} 1 = 0.$    | 16. $\log_2 8 = 3.$        |
| 17. $\log_5 125 = 3.$           | 18. $\log_8 4 = 2/3.$     | 19. $\log_{10} 0.1 = -1.$  |
| 20. $\log_3 8 = 3 \frac{1}{2}.$ |                           |                            |

*Find the value of:*

- |                           |                              |                              |                           |
|---------------------------|------------------------------|------------------------------|---------------------------|
| 21. $\log_2 32.$          | 22. $\log_{10} 0.001.$       | 23. $\log_{27} 9.$           | 24. $\log_5 1.$           |
| 25. $\log_5 5.$           | 26. $\log_5 b.$              | 27. $\log_5 1.$              | 28. $\log_5 (b^3).$       |
| 29. $\log_{25} 125.$      | 30. $\log_9 27.$             | 31. $\log_{10} 10,000.$      | 32. $\log_{10} (10^2).$   |
| 33. $\log_{10} (10^5).$   | 34. $\log_5 (b^7).$          | 35. $\log_9 16.$             | 36. $\log_{16} 8.$        |
| 37. $\log_4 \frac{1}{2}.$ | 38. $\log_{16} \frac{1}{4}.$ | 39. $\log_{27} \frac{1}{3}.$ | 40. $\log_5 \frac{1}{5}.$ |

*Find the value of the unknown letter:*

- |                          |                                 |                           |
|--------------------------|---------------------------------|---------------------------|
| 41. $\log_5 16 = 2.$     | 42. $\log_6 x = -2.$            | 43. $\log_4 x = -3.$      |
| 44. $\log_8 64 = 2.$     | 45. $\log_{16} N = 1/4.$        | 46. $\log_{10} N = -1/2.$ |
| 47. $x = \log_9 81.$     | 48. $\log_x \frac{1}{16} = -2.$ | 49. $\log_x 32 = 5/2.$    |
| 50. $\log_{16} N = 3/2.$ |                                 |                           |

**157. Properties of logarithms.** The simplifications of arithmetic accomplished by logarithms are due to the three properties given in the following theorems.

**THEOREM 1.**  $\log_b (xw) = \log_b x + \log_b w.$  That is, the logarithm of a product is the sum of the logarithms of the factors.

**THEOREM 2.**  $\log_b (x/w) = \log_b x - \log_b w.$  That is, the logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.

**THEOREM 3.**  $\log_b (x^r) = r \log_b x.$  That is, the logarithm of a quantity raised to a power equals the exponent times the logarithm of the quantity.

*Note.* We write  $r \log_b x$  rather than  $\log_b x \cdot r$  because the latter expression would give rise to confusion as to whether it meant  $\log_b (x \cdot r)$  or  $(\log_b x) \cdot r.$

*Examples.* Suppose  $\log_{10} 2 = 0.3010$  and  $\log_{10} 3 = 0.4771$  approximately. Then, by theorem 1,  $\log_{10} 6 = \log_{10} (2 \cdot 3) = \log_{10} 2 + \log_{10} 3 = 0.3010 + 0.4771 = 0.7781$ , approximately. By theorem 2,  $\log_{10} (3/2) = \log_{10} 3 - \log_{10} 2 = 0.1761$ , approximately. By theorem 3,  $\log_{10} \sqrt[3]{2} = \log_{10} (2^{1/3}) = \frac{1}{3} \log_{10} 2 = \frac{1}{3}(0.3010) = 0.1003$ , approximately.

Since logarithms are exponents, the theorems above may be proved by translating from exponential language into logarithmic language the corresponding three laws of exponents, namely: (a)  $b^x b^y = b^{x+y}$ ; (b)  $b^x b^y = b^{x-y}$ ; (c)  $(b^x)^y = b^{xy}$ . Let



(1)  $y = \log_b x$  and  $u = \log_b w$

or,

(2)  $b^y = x$  and  $b^u = w$ .

*Proof of theorem 1.* By (2),  $xw = b^y b^u$ . Hence, by (a),  $xw = b^{y+u}$ . By definition, this says that  $\log_b (xw) = y + u$ . Substituting from (1), we get the theorem.

*Proof of theorem 2.* By (2),  $x/w = b^y/b^u$ . Hence, by (b),  $x/w = b^{y-u}$ . By definition,  $\log_b (x/w) = y - u$ . Substituting from (1), we have the theorem.

*Proof of theorem 3.* By (2),  $x^r = (b^y)^r$ . Hence, by (c),  $x^r = b^{yr}$ . By definition,  $\log_b (x^r) = yr$ . Substituting from (1), we get the theorem.

EXERCISES

Given  $\log_{10} 2 = 0.3010$ ,  $\log_{10} 3 = 0.4771$ , and  $\log_{10} 7 = 0.8451$ , find, without using tables, the value of:

1.  $\log_{10} 14$ .
2.  $\log_{10} 4$ .
3.  $\log_{10} (7/2)$ .
4.  $\log_{10} 9$ .
5.  $\log_{10} 12$ .
6.  $\log_{10} 20$ .
7.  $\log_{10} 0.2$ .
8.  $\log_{10} 18$ .
9.  $\log_{10} \sqrt{2}$ .
10.  $\log_{10} \sqrt[3]{7}$ .
11.  $\log_{10} (\sqrt[3]{6}/2)$ .
12.  $\log_{10} (7/\sqrt{14})$ .
13.  $\log_{10} \sqrt[3]{42}$ .
14.  $\log_{10} \sqrt{\frac{7}{3}}$ .
15.  $\log_{10} 5$ .
16.  $\log_{10} \left(\frac{21}{2\sqrt{3}}\right)$ .
17.  $\log_{10} \left(\frac{49}{3\sqrt[3]{2}}\right)$ .
18.  $\log_{10} \frac{9\sqrt[3]{7}}{\sqrt{21/2}}$ .
19.  $\log_{10} \frac{108}{7\sqrt[3]{7}}$ .
20.  $\log_{10} \frac{\sqrt[3]{24/7}}{98}$ .

**158. Common logarithms.** While any positive number  $b \neq 1$  can be used as a base for a system of logarithms, the base  $b = 10$  is most convenient for computation. Logarithms with the base 10 are called **common logarithms** or **Briggsian logarithms** (after H. Briggs, who first used them). Hereafter we write  $\log x$ , without any base indicated, to mean  $\log_{10} x$ . Thus we write  $\log 100 = 2$  because  $10^2 = 100$ ,  $\log 0.1 = -1$  because  $10^{-1} = 1/10 = 0.1$ , etc. The student should learn to construct the following rudimentary table of logarithms:

Table A

|          |                     |        |       |      |     |   |    |     |      |       |                     |
|----------|---------------------|--------|-------|------|-----|---|----|-----|------|-------|---------------------|
| $x$      | $\cdot \cdot \cdot$ | 0.0001 | 0.001 | 0.01 | 0.1 | 1 | 10 | 100 | 1000 | 10000 | $\cdot \cdot \cdot$ |
| $\log x$ | $\cdot \cdot \cdot$ | -4     | -3    | -2   | -1  | 0 | 1  | 2   | 3    | 4     | $\cdot \cdot \cdot$ |



This table suggests that *if  $m < n$  then  $\log m < \log n$* . This can be proved, but we shall assume it here. Hence we expect  $\log 532$  to be between 2 and 3 since 532 is between  $10^2$  and  $10^3$ . It can also be proved, and we shall assume here, that every positive number can be expressed as the product of a number between 1 and 10 and an integral power of 10. For example,

$$\begin{aligned} 53.2 &= 5.32(10^1), \\ 532 &= 5.32(10^2), \\ 5.32 &= 5.32(10^0), \\ 0.00532 &= 5.32(10^{-3}). \end{aligned}$$

By theorem 1, section 157, we have

$$\log 532 = \log 5.32 + \log (10^2).$$

By definition of logarithm, or by table A, above,

$$(1) \quad \log 532 = \log 5.32 + 2.$$

Similarly,

$$\begin{aligned} (2) \quad \log 53.2 &= \log 5.32 + \log (10^1) = \log 5.32 + 1, \\ (3) \quad \log 5.32 &= \log 5.32 + \log (10^0) = \log 5.32 + 0, \\ (4) \quad \log 0.00532 &= \log 5.32 + \log (10^{-3}) = \log 5.32 + (-3). \end{aligned}$$

Clearly we could write down the logarithm of any number if we only knew *the logarithms of numbers between 1 and 10*; the latter are called **mantissas**. A table of mantissas approximated to four decimal places (table III) is found at the back of the book. Mantissas are decimal fractions, since the logarithm of a number between 1 and 10 must be between 0 and 1.

From (1), (2), (3), and (4), we observe that *the mantissa depends only on the sequence of digits beginning with the first non-zero digit*; all numbers which have the same sequence of digits have the same mantissas. From the table we find that

$$\begin{aligned} \log 532 &= 2 + 0.7259 = 2.7259 \\ \log 53.2 &= 1 + 0.7259 = 1.7259 \\ \log 5.32 &= 0 + 0.7259 = 0.7259 \\ \log 0.00532 &= (-3) + 0.7259. \end{aligned}$$

When a logarithm is expressed so that its decimal part is written positively, the integral part of the logarithm is called its **characteristic**.

For example, the characteristic of  $\log 532$  is 2, and the characteristic of  $\log 0.00532$  is  $-3$ , not  $-2$  (although  $\log 0.00532 =$



$-2.2741 = -2 - 0.2741$ ). When a characteristic is negative, it is convenient to add and subtract a suitable multiple of 10; for example, we could write

$$\log 0.00532 = 7.7259 - 10 \text{ (adding and subtracting 10),}$$

or,

$$\log 0.00532 = 17.7259 - 20 \text{ (adding and subtracting 20).}$$

To find the logarithm of a number  $N$ , we determine the characteristic by inspection, and look up the mantissa in table III.

The following rule may be used to determine the characteristic.

*Rule 1.* Refer to table A above. If  $N$  lies between two entries in this table, the characteristic of  $\log N$  is the logarithm of the lower of the two entries. In other words, if  $N$  lies between  $10^k$  and  $10^{k+1}$ , ( $10^k < N < 10^{k+1}$ ), then the characteristic of  $\log N$  is  $k$ .

*Note 1.* The characteristic may also be determined by means of any one of the following rules:

*Rule 2.* Let  $N$  be written in decimal form. If  $N \geq 1$ , then the characteristic of  $\log N$  is one less than the number of digits in  $N$  to the left of the decimal point; if  $N < 1$ , and if the first non-zero digit of  $N$  is in the  $k$ th decimal place, then  $-k$  is the characteristic of  $\log N$ .

*Rule 3.* Express the number  $N$  in the standard form of section 65, chapter IX, that is, as a product of a number between 1 and 10 and an integral power of 10. Then the exponent of 10 is the characteristic of  $\log N$ .

*Rule 4.* Write  $N$  in decimal form. If moving the decimal point  $k$  places to the left will make the resulting number between 1 and 10, then the characteristic of  $\log N$  is  $k$ ; if moving the decimal point  $k$  places to the right will make the resulting number between 1 and 10, then the characteristic of  $\log N$  is  $-k$ .

The justification of these rules is left as an exercise.

*Example 1.* Find  $\log 630$ .

*Solution.* By rule 1, the characteristic is 2. In table III we find the first two digits 63 in the  $N$  column and the third digit 0 at the top of the page. In this column and row, we find the mantissa 0.7993. Hence  $\log 630 = 2.7993$ .

*Example 2.* Find  $\log 0.0234$ .

*Solution.* By rule 1, the characteristic is  $-2$ , or  $8 - 10$ . From table III, the mantissa corresponding to the digits 234 is 0.3692. Hence  $\log 0.0234 = 8.3692 - 10$ .



To find a number  $N$  whose logarithm is known, we look up the mantissa in table III, write the digits corresponding to it, and place the decimal point according to the characteristic.

*Example 3.* Find  $N$  if  $\log N = 1.5752$ .

*Solution.* From table III, the digits corresponding to the mantissa 0.5752 are 376. Since the characteristic is 1,  $N$  must be between  $10^1$  and  $10^2$ . Hence  $N = 37.6$ .

*Example 4.* Find  $N$  if  $\log N = 8.5752 - 10$ .

*Solution.* From table III, the digits are 376. Since the characteristic is  $8 - 10 = -2$ ,  $N$  must be between  $10^{-2} = 0.01$  and  $10^{-1} = 0.1$ . Hence  $N = 0.0376$ .

*Note 2.* Table III gives for three-digit numbers the corresponding mantissas correct to four decimal places. If we want the mantissa for a four-digit number or the digits corresponding to a mantissa not found exactly in the table, we may proceed in several ways: (a) we may take the nearest thing in the table as a sufficiently good approximation; (b) we may use a larger table giving mantissas correct to more than four decimal places for numbers of more than three digits; (c) we may use the process of interpolation by proportional parts described in the next section.

*Note 3.* If  $\log N = L$ , then  $N$  may be called the **antilogarithm** of  $L$ ; in symbols,  $N = \text{antilog } L$ .

*Note 4.* The statement  $\log 37.6 = 1.5752$  means that  $10^{1.5752} = 37.6$  approximately, or  $\sqrt[10000]{10^{15752}} = 37.6$  approximately. However, the tables are not constructed by means of this relation directly. How they are calculated will be indicated in section 163.

## EXERCISES

*Find  $\log N$  if  $N$  is equal to:*

- |          |            |            |              |          |
|----------|------------|------------|--------------|----------|
| 1. 124.  | 2. 12.4.   | 3. 0.0124. | 4. 387.      | 5. 39.6. |
| 6. 5.34. | 7. 0.0435. | 8. 0.467.  | 9. 0.000346. |          |
| 10. 306. | 11. 360.   | 12. 500.   | 13. 0.005.   |          |

*Find  $N$  if  $\log N$  is equal to:*

- |                  |             |                  |                  |
|------------------|-------------|------------------|------------------|
| 14. 2.5211.      | 15. 1.5211. | 16. 8.5211 - 10. | 17. 9.5211 - 10. |
| 18. 0.5211.      | 19. 1.6972. | 20. 0.7316.      | 21. 9.7364 - 10. |
| 22. 8.9400 - 10. | 23. 3.9717. | 24. 2.5888.      |                  |

25. Express the statement  $\log 862 = 2.9355$  in (a) exponential notation; (b) radical notation.

26. Express the statement  $\log 631 = 2.8000$  in (a) exponential notation; (b) radical notation.



**159. Interpolation by proportional parts.** Instead of taking the nearest entry in the table, we may obtain greater accuracy from a given table by the process of interpolation by proportional parts, as in the following examples.

*Example 1.* Find  $\log 36.14$ .

*Solution.* The characteristic is 1. From table III, we find that the mantissa for the digits 361 is 0.5575 and the mantissa for 362 is 0.5587. Since 36.14 is 0.4 of the way from 36.1 to 36.2, we take a mantissa 0.4 of the way from 0.5575 to 0.5587. The difference between 0.5575 and 0.5587 is 0.0012; this is called the **tabular difference**. We add 0.4 of the tabular difference, that is  $0.4(0.0012) = 0.00048 = 0.0005$  approximately, to 0.5575, obtaining 0.5580. Hence  $\log 36.14 = 1.5580$ .

The work may be arranged briefly as follows.

$$10 \left[ \begin{array}{c|c} \text{digits} & \text{mantissa} \\ \hline 4 \left[ \begin{array}{c} 3610 \\ 3614 \\ 3620 \end{array} \right] \begin{array}{c} 0.5575 \\ \\ 0.5587 \end{array} & x \end{array} \right] \begin{array}{l} 0.0012 = \text{tabular difference} \\ 0.4 \\ \hline 0.00048 = 0.0005 \end{array}$$

Then  $\frac{x}{0.0012} = \frac{4}{10}$ , or  $x = 0.00048 = 0.0005$  to the nearest ten-thousandth. Hence the mantissa to be put in the blank space is 0.5580. Thus,  $\log 36.14 = 1.5580$ .

*Example 2.* Find  $N$  if  $\log N = 9.5579 - 10$ .

*Solution.* The digits corresponding to the mantissas 0.5575 and 0.5587 are 3610 and 3620 respectively. We find that the mantissa 0.5579 is  $4/12$  of the way from 0.5575 to 0.5587. Hence we take the fourth digit to be  $4/12$  of the way between 10 and 20. Since  $4/12$  of 10 is  $3\frac{1}{3}$  or 3 to the nearest integer, the digits desired are 3613. Since the characteristic is  $9 - 10 = -1$ ,  $N = 0.3613$ .

The work may be arranged briefly as follows.

$$10 \left[ \begin{array}{c|c} \text{digits} & \text{mantissa} \\ \hline x \left[ \begin{array}{c} 3610 \\ \\ 3620 \end{array} \right] \begin{array}{c} 0.5575 \\ 0.5579 \\ 0.5587 \end{array} & 4 \end{array} \right] 12$$

Then  $\frac{x}{10} = \frac{4}{12}$ , or  $x = 3.3 \dots = 3$  to the nearest integer. Hence  $N = 0.3613$ .



*Note.* It can be proved that if  $p < N < q$  then  $\log p < \log N < \log q$ . But the assumption on which the above procedure is based states further that changes in logarithms are proportional to changes in the corresponding numbers. This is not true, but results obtained by using this assumption are sufficiently accurate for many practical purposes, especially when the difference between the successive values between which we are interpolating is sufficiently small. The significance of this assumption will be discussed in chapter XXVIII.

## EXERCISES

*Using interpolation by proportional parts, find  $\log N$  if  $N$  is equal to:*

- |             |            |           |            |             |
|-------------|------------|-----------|------------|-------------|
| 1. 241.2.   | 2. 29.85.  | 3. 2.957. | 4. 0.6944. | 5. 0.08513. |
| 6. 0.08866. | 7. 0.4267. | 8. 4.752. | 9. 4.015.  | 10. 103.8.  |

*Using interpolation by proportional parts, find  $N$  if  $\log N$  is equal to:*

- |                  |                   |                  |             |
|------------------|-------------------|------------------|-------------|
| 11. 2.1917.      | 12. 1.2688.       | 13. 0.5374.      | 14. 0.6815. |
| 15. 9.7995 — 10. | 16. 9.8135 — 10.  | 17. 8.9035 — 10. |             |
| 18. 8.9303 — 10. | 19. 8.31048 — 10. | 20. 3.46345.     |             |

**160. Significant figures.** When a number is written in the usual decimal form, the **significant figures** or **significant digits** are the digits beginning with the first non-zero digit and ending with the last digit written.

*Examples.* The numbers 341, 34.1, and 0.00341 all have three significant figures, namely 3, 4, and 1. But 3410, 341.0, 34.10, and 0.003410 all have four significant figures, namely 3, 4, 1, and 0.

In dealing with measurements or other approximate numbers, 34.10 means something different from 34.1. For 34.1 signifies a number known to be between 34.05 and 34.15; that is,  $34.05 \leq 34.1 < 34.15$ . But 34.10 signifies a number known to be between 34.095 and 34.105; that is,  $34.095 \leq 34.10 < 34.105$ . In other words, 34.1 is correct to the nearest tenth, while 34.10 is correct to the nearest hundredth. Or 34.1 signifies a number which is nearer to 34.1 than it is to 34.0 or to 34.2; while 34.10 is a number nearer to 34.10 than to 34.09 or to 34.11.

*Note.* If a number has zeros just preceding the decimal point and no digits after the decimal point, such as 341,000., the number of significant figures is six according to the definition above. If the number is not known to six-digit accuracy, the proper number of significant digits must be specifically stated. One



way to do that is to use powers of ten as in section 65, chapter IX. Thus if 341,000 is known to only three significant digits, it may be written as  $341 \times 10^3$ ; if it is known to four significant digits, it may be written as  $3,410 \times 10^2$ .

A number is said to be **rounded off** to four significant figures if it is replaced by the number of four significant digits to which it is nearest. If the fifth digit is a 5 we make the convention that the fourth digit is to be increased by one. A similar definition of rounding off applies in general.

*Example.* The number 3.14159265 is written as 3.142 rounded off to four significant figures, as 3.14 rounded off to three significant figures, and as 3.1416 rounded off to five significant figures.

In calculating with approximate numbers, the answer should be rounded off to the number of decimal places justified by the data. In general, the answer cannot be expected to be more accurate than the least accurate number in the data. However, it is advisable to carry one more place throughout the computation and do the rounding off at the end.

*Example.* The sum of the approximate numbers 3.43 and 1.124 should be written as 4.55 rather than as 4.554. Actually, all that is known, since  $3.425 \leq 3.43 < 3.435$  and  $1.1235 \leq 1.124 < 1.1245$ , is that the sum  $s$  is such that  $4.5485 \leq s < 4.5595$ . Hence, rounding off to three significant figures,  $4.55 \leq s < 4.56$ . Therefore  $s = 4.55$  rounded off to three significant figures gives somewhat more accuracy than is justified by the data. The answer 4.554 would give a false illusion of greater accuracy.

If the numbers entering a computation are considered to be exact, then, of course, all places may be retained.

## EXERCISES

*Round off the number:*

1. 3.206 to three significant figures.
2. 14.164 to four significant figures.
3. 14.195 to four significant figures.
4. 1.3204 to four significant figures.
5. 4.9997 to four significant figures.
6. 2.71828 to five significant figures.

*How many significant figures are there in:*

7. 0.0003.
8. 0.0230.
9.  $32 \times 10^6$ .
10.  $3.20 \times 10^{-4}$ .



11. Explain the difference in meaning between the approximate numbers 3.2 and 3.20.

12. Explain the difference in meaning between the approximate numbers 4.70 and 4.700.

161. Computation with logarithms. The simplifications in computation obtained by use of logarithms are due to theorems 1, 2, and 3 of section 157. The following examples will suffice to make the method clear.

*Note.* No more than four-figure accuracy is usually justified when using a four-place table of logarithms, even if the data of the problem are exact.

*Example 1.* Calculate  $\frac{(8.34)(65.2)}{376}$ .

*Solution.* Let  $N$  be the result. By theorems 1 and 2, section 157,  $\log N = \log \frac{(8.34)(65.2)}{376} = \log 8.34 + \log 65.2 - \log 376$ .

Now,

$$\begin{aligned}\log 8.34 &= 0.9212 \\ \log 65.2 &= 1.8142 \\ \log 8.34 + \log 65.2 &= \underline{2.7354} \\ \log 376 &= \underline{2.5752} \\ \log N &= 0.1602\end{aligned}$$

Hence  $N = 1.45$  approximately. Since the data are given to three significant figures, so is the answer.

*Example 2.* Find  $\sqrt[3]{473.0}$ .

*Solution.* Let  $x = \sqrt[3]{473.0} = (473.0)^{1/3}$ . By theorem 3, section 157,  $\log x = \frac{1}{3} \log 473.0 = \frac{1}{3}(2.6749) = 0.8916$ . Interpolating for the fourth significant figure, we get  $x = 7.792$  approximately.

*Example 3.* If  $P$  dollars is invested at an interest rate of  $r$  (expressed as a decimal) compounded  $n$  times, the amount is given by the formula\*

$$(1) \quad A_n = P(1 + r)^n.$$

If \$100 is invested as a trust fund for a child at 4% per annum interest compounded annually, how much will it amount to in 21 years?

*Solution.* From (1), we have  $A_{21} = 100(1.04)^{21}$ . Hence

$$\begin{aligned}\log A_{21} &= \log 100 + 21 \log 1.04 \\ &= 2 + 21(0.0170) = 2.3570.\end{aligned}$$

Hence  $A_{21} = \$227.50$  approximately.

\* This formula will be discussed in detail in chapter XXIV.



*Example 4.* If the interest in example 3 is compounded semi-annually, what will the amount be?

*Solution.* Interest at 4% per annum compounded semi-annually means that 2% interest is compounded twice a year. Hence in 21 years, interest is compounded forty-two times. Hence  $A_{42} = 100(1.02)^{42}$ . Thus

$$\begin{aligned}\log A_{42} &= \log 100 + 42 \log 1.02 \\ &= 2 + 42(0.0086) = 2.3612.\end{aligned}$$

Hence  $A_{42} = \$229.70$  approximately.

*Note 1.* The time-saving property of logarithms becomes obvious if you consider how long it would take to do example 4 by ordinary arithmetic, multiplying 100 by 1.02 forty-two times in succession. Other simple applications will be found in the exercises.

*Note 2.* The incorrect formula  $\log(x + y) = \log x + \log y$  appears plausible at first glance. But if true, we would need no table of logarithms, for all logarithms would be equal to 0. For  $\log 1 = 0$ . Hence  $\log 2 = \log(1 + 1)$  would be, according to this wrong formula,  $\log 1 + \log 1$  or  $0 + 0 = 0$ ;  $\log 3 = \log(2 + 1) = \log 2 + \log 1 = 0 + 0 = 0$ , and so on. The resemblance of this incorrect formula to the correct distributive law is no more than superficial, for we are here not *multiplying* by the word *log*. Actually, of course,  $\log x + \log y = \log xy$ .

*Note 3.* To compute logarithmically with negative numbers, compute the result obtained by ignoring all minus signs, and then determine the sign of the result by the usual rule of signs.

## EXERCISES

*Compute by means of logarithms:*

- |                                                     |                                                                   |                                       |
|-----------------------------------------------------|-------------------------------------------------------------------|---------------------------------------|
| 1. $\frac{(34.2)(1.57)}{31.3}$                      | 2. $\sqrt[3]{45.2}$                                               | 3. $\sqrt[5]{117}$                    |
| 4. $112(1.04)^{16}$                                 | 5. $21.3/27.2$                                                    | 6. $\frac{(627)(32.4)}{(3.26)(65.8)}$ |
| 7. $\sqrt[3]{673} \times 10^2$                      | 8. $\sqrt[5]{963} \times 10^2$                                    | 9. $376(1.06)^{21}$                   |
| 10. $\frac{(87.6)(47.4)}{53.7}$                     | 11. $\sqrt[3]{0.00857}$                                           | 12. $\sqrt[5]{0.0468}$                |
| 13. $\frac{86.42}{2.560 \times 3.120}$              | 14. $\frac{\sqrt{435.7}}{8.420 \times (0.1780)^{-3}}$             |                                       |
| 15. $\frac{(23.54)(0.8765)}{1.760 \sqrt[3]{6.380}}$ | 16. $\frac{(5.363)(2.466) \sqrt[3]{12.30}}{0.08620 \sqrt{62.30}}$ |                                       |



17.  $(-43.27)(732.2).$

19.  $\frac{47.20(1.025)^{10}}{(6.520)(3.480)}.$

21.  $\sqrt[3]{\frac{4.23}{0.0842 \times 6.94}}.$

\*23.  $\frac{\sqrt{45.2} - (1.37)^5}{(1.63)^7 + 4.25}.$

18.  $\frac{(-2.345)(-6.921)}{-3.427}.$

20.  $\sqrt{\frac{4.27 \times 0.328}{0.560 \times 2.01}}.$

22.  $\sqrt[3]{\frac{-425.3}{(4.692)(-32.68)}}.$

\*24.  $\frac{\sqrt{(42.3)(16.1)} - \sqrt{8.24}}{(1.03)^6 + 1}.$

Use formula (1),\* above, in each of the following problems:

25. If \$100 is invested at 4% interest compounded quarterly, how much will it amount to in 21 years?

26. If \$350 is deposited in a bank at 2% interest compounded semi-annually, what will the amount be in 6 years?

27. It is desired to have a sum of \$1000 in the bank 10 years from now. If the bank pays 4% interest compounded annually, how much should we deposit now?

28. If the \$24 which the Indians received in 1626 for Manhattan had been deposited in a bank paying 4% interest compounded annually, what would it have amounted to in 1946?

29. How long will it take a sum of money to become doubled if it is invested at 4% compounded annually?

30. If an investment of \$75 will amount to \$100 in 10 years, what rate of interest, compounded annually, is being paid?

31. The area of the surface of a sphere is  $4\pi r^2$  where  $r$  is the radius. Taking  $\pi = 3.14$  and assuming that the earth is a sphere of radius 3960 miles, find the area of the earth's surface.

32. The volume of a sphere is  $\frac{4\pi r^3}{3}$  where  $r$  is the radius. Using the data of exercise 31, find the volume of the earth.

33. If a person bets 2 cents on April 1st and doubles his bet each day thereafter, how much will his bet be on April 30th?

34. The volume of a right circular cylinder is  $\pi r^2 h$  where  $r$  is the radius and  $h$  is the height. Taking  $\pi = 3.14$ , find the volume of a right circular cylinder with  $r = 4.32$  inches and  $h = 17.7$  inches.

35. The period  $T$ , measured in seconds, of a simple pendulum (that is, the time required for a complete oscillation) of length  $k$  is given by the formula  $T = 2\pi \sqrt{k/g}$ . If  $k = 3.26$  feet,  $g = 32.2$ ,  $\pi = 3.14$ , find  $T$ .

36. Using the formula and data of exercise 35, find the length of a simple pendulum whose period is 1 second.

37. Suppose that an automobile costing \$1000 depreciates at the rate of 20% per year; that is, its value at the end of each year is 80% of its value at the beginning of that year. Find its value at the end of (a) 5 years; (b) 10 years.

\* May be omitted without disturbing continuity.

\* Further applications to financial problems will be found in the next chapter.



38. If  $a, b, c$  are the lengths of the sides of a triangle, the area  $K$  of the triangle is given by the formula  $K = \sqrt{s(s-a)(s-b)(s-c)}$  where  $s = \frac{1}{2}(a+b+c)$ . Find the area of a triangle with sides 472.3 feet, 617.8 feet, and 392.4 feet, respectively.

39. Using the formula of exercise 38, find the area of a triangle whose sides are 515.6 inches, 742.2 inches, and 469.6 inches, respectively.

40. If  $a, b, c$  are the lengths of the sides of a triangle, the radius  $r$  of the inscribed circle is given by the formula  $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ ,

where  $s = \frac{1}{2}(a+b+c)$ . Find the radius of the inscribed circle for a triangle whose sides are 312.5 feet, 434.7 feet, and 522.1 feet, respectively.

41. Find the geometric mean of the numbers 469, 72.5, 3.42, 0.658, and 0.0566. (*Hint*: See note 1, section 147, for the definition of geometric mean.)

42. In a polytropic atmosphere, temperature  $T$  and pressure  $p$  are related by the equation  $\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}$  where  $T_0$  is the temperature at pressure  $p_0$ . Show that  $k = \frac{\log p - \log p_0}{\log p - \log p_0 - \log T + \log T_0}$ .

43. From the equation  $\log p - \log p_0 = \frac{1}{lR} [\log (T_0 - lz) - \log T_0]$ , which occurs in the study of the atmosphere, show that

$$p = p_0 \left(1 - \frac{lz}{T_0}\right)^{1/lR}.$$

**\*162. Cologarithms.** By the cologarithm of  $N$  is meant the logarithm of the reciprocal of  $N$ ; in symbols,

$$\text{colog } N = \log \frac{1}{N} = -\log N.$$

Hence  $\text{colog } N$  is found by subtracting  $\log N$  from 0 which may be written as  $10 - 10$  for convenience.

*Example 1.* Since  $\log 376 = 2.5752$ ,  $\text{colog } 376 = 7.4248 - 10$ .

Since division by  $N$  is equivalent to multiplication by  $1/N$ , we may add  $\text{colog } N$  instead of subtracting  $\log N$ , whenever the latter operation is indicated.

*Example 2.* Do example 1, section 161, by means of cologarithms.

*Solution.* As before,  $\log N = \log 8.34 + \log 65.2 - \log 376$ . Hence  $\log N = \log 8.34 + \log 65.2 + \text{colog } 376$ . Now,

\* This section may be omitted without disturbing the continuity of the chapter.



$$\begin{aligned}\log 8.34 &= 0.9212 \\ \log 65.2 &= 1.8142 \\ \text{colog } 376 &= 7.4248 - 10 \\ \log N &= \frac{10.1602 - 10}{1} = 0.1602.\end{aligned}$$

Therefore,  $N = 1.45$  approximately.

## EXERCISES

*Find the cologarithm of:*

1. 2.03.      2. 47.6.      3. 0.891.      4. 0.0645.      5. 558.3.  
6. 0.3546.

*Work the following exercises, using cologs:*

7. Exercise 1, section 161.      8. Exercise 6, section 161.  
9. Exercise 13, section 161.      10. Exercise 14, section 161.  
11. Show that  $\log(x^{-r}) = r \text{ colog } x$ .

**163. Natural logarithms. Computation of tables.** Theoretically any positive number except 1 can be used as a base for a system of logarithms, although we have so far used only the base 10. Logarithms with any base can be converted into logarithms with any other base merely by multiplying by a suitable constant factor. For, suppose  $y = \log_b x$  and  $z = \log_a x$ . Then

$$(1) \quad b^y = a^z,$$

since both are equal to  $x$ . Taking the logarithm with the base  $a$  of both members of (1), we get  $y \log_a b = z$  or,

$$(2) \quad \log_b x \cdot \log_a b = \log_a x.$$

Hence the logarithm of any number  $x$  with the base  $a$  can be obtained from the logarithm of  $x$  with the base  $b$  by multiplying by the constant  $\log_a b$ . The number  $\log_a b$  is called the **modulus** of the system of logarithms to the base  $a$  with respect to the system of logarithms to the base  $b$ .

Aside from common logarithms (with the base 10), the only system in actual use is the system of **natural logarithms** whose base is the irrational number  $e = 2.71828 \dots$ . Natural logarithms are natural for reasons which become clear only when one studies the calculus. For example, it can be shown by calculus that

$$(3) \quad \log_e \frac{1+x}{1-x} = 2 \left[ x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right], \text{ for } -1 < x < 1.$$



The right member of (3) is an infinite series (see chapter XXVII). For example, for  $x = 1/3$ , we obtain from (3)

$$(4) \quad \log_e 2 = 2 \left[ \frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \cdots \right].$$

Clearly the successive terms of the series get small rapidly, and it is plausible and true that a good approximation is obtained by neglecting all but the first few terms. Thus from (4) we get

$$(5) \quad \log_e 2 = 2[0.333333 + 0.012345 + 0.000823 + \cdots] \\ = 0.693002$$

which is correct to the nearest thousandth. By (2),

$$(6) \quad \log_{10} x = \log_e x \cdot \log_{10} e.$$

The number

$$(7) \quad M = \log_{10} e = 0.434294 \cdots$$

Hence, using (5), (6), and (7), we have

$$(8) \quad \log_{10} 2 = 0.301 \cdots$$

to the nearest thousandth. This, with technical modifications, is the way tables of logarithms are computed.

*Note 1.* Logarithms were invented in the 17th century by J. Napier (Scottish, 1550–1617) and independently by J. Bürgi (Swiss, 1552–1632). Their systems were closely related to the system of natural logarithms. Strangely enough, the invention of logarithms preceded that of exponents. Common logarithms are credited to Napier and H. Briggs (English, 1556–1631). The first table of common logarithms of numbers was laboriously calculated by Briggs and A. Vlacq (Dutch, c. 1600–1667).

*Note 2.* The natural logarithm  $\log_e x$  is often written as  $\ln x$ .

## EXERCISES

Using (3) and (7), find (a) the natural logarithm and (b) the common logarithm of each of the following numbers to the nearest thousandth:

1. 3. (Hint: take  $x = \frac{1}{2}$ .)      2.  $5/3$ .      3.  $3/2$ .      4. 5.

Using (7), convert each of the following common logarithms into the corresponding natural logarithms, or vice versa:

5.  $\log_{10} x = 0.6021$ .      6.  $\log_{10} x = 0.9031$ .  
7.  $\log_e x = 1.936$ .      8.  $\log_e x = 2.193$ .

164. Exponential and logarithmic functions and equations. The exponential function  $y = b^x$  and the logarithmic function  $y =$



$\log_b x$ , where  $b$  is positive and different from 1, occur frequently in applied mathematics. Their graphs are given in Figs. 61 and 62 for  $b > 1$ .

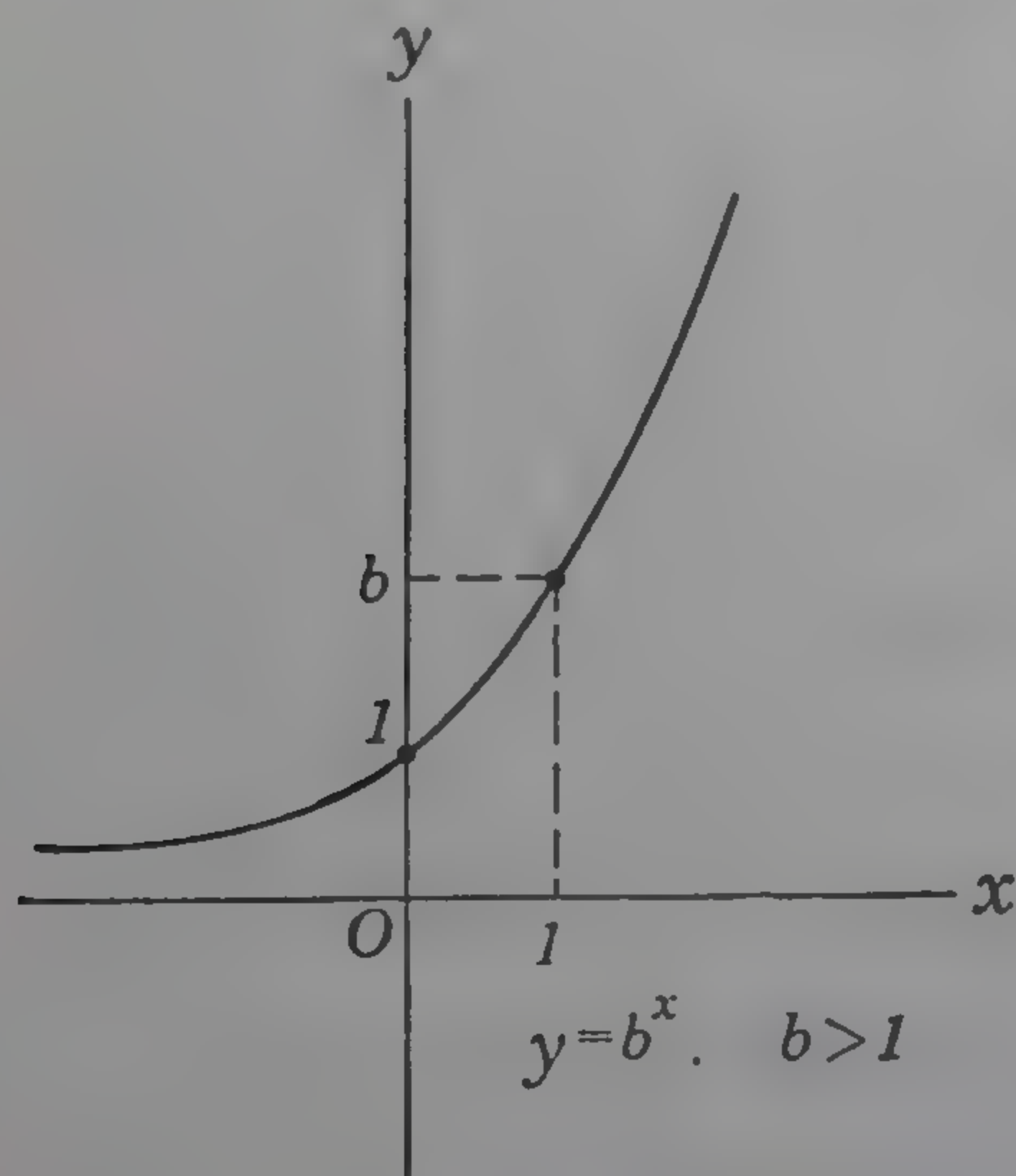


Fig. 61.

Equations involving exponential functions (that is, in which the unknown occurs as an exponent) can sometimes be solved by taking logarithms, as in the following examples.

*Example 1.* Solve the equation  $3^{x+1} = 5(2^x)$ .

*Solution.* Taking the logarithm of both sides of the given equation, we obtain  $(x + 1) \log 3 = \log 5 + x \log 2$ . Hence  $x = \frac{\log 5 - \log 3}{\log 3 - \log 2}$ , or  $x = 0.2219/0.1761$ .

Therefore  $\log x = \log 0.2219 - \log 0.1761 = 0.1004$ . Hence  $x = 1.26$

approximately.

*Example 2.* The number  $n$  of bacteria in a certain culture at the end of  $t$  hours is given by  $n = 100 \times 10^{0.243t}$ . Find (a) the number of bacteria at  $t = 0$ ; (b) at  $t = 3$ ; (c) the number of hours required for the number of bacteria to double itself.

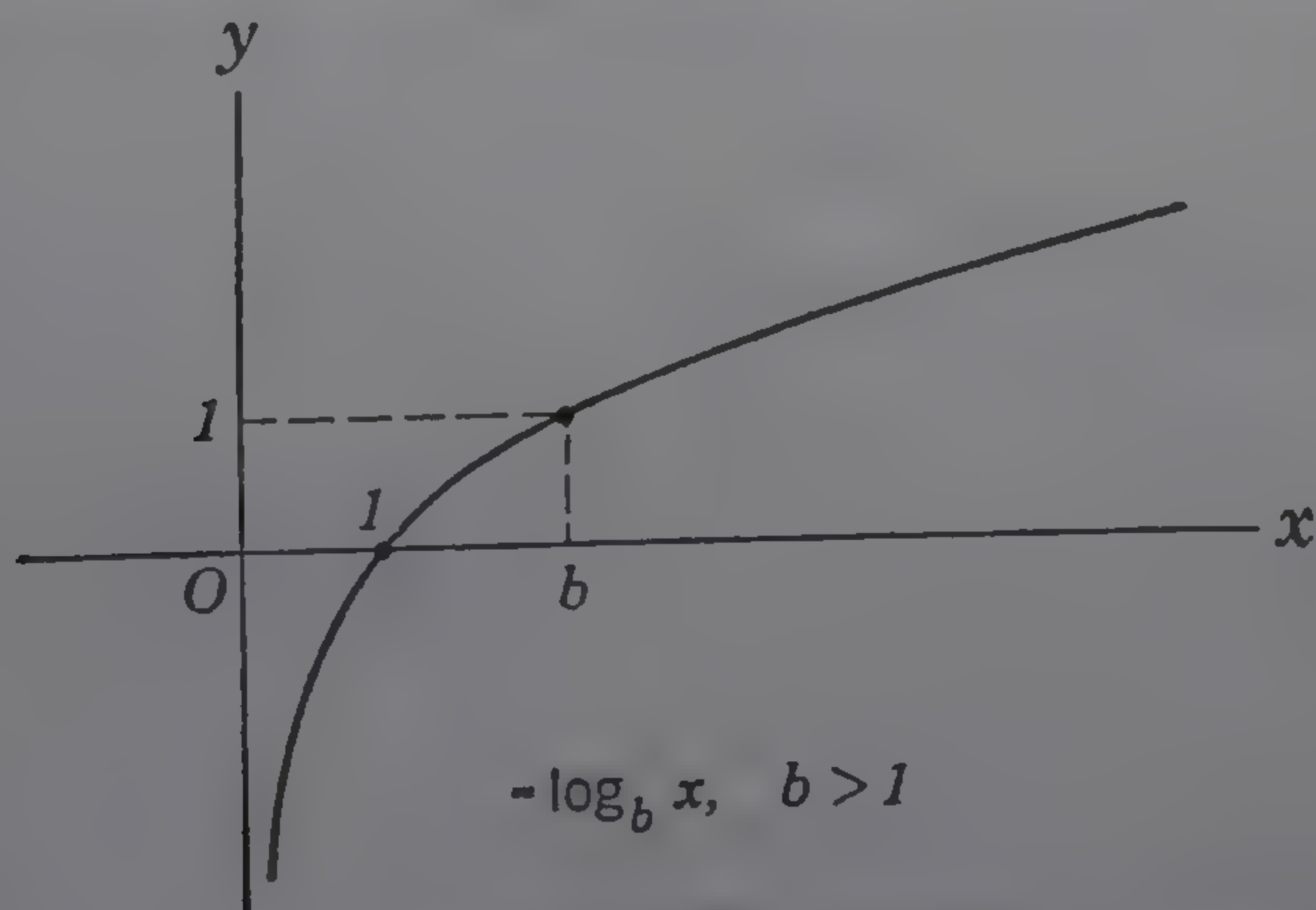


Fig. 62.

*Solution.* (a) At  $t = 0$ ,  $n = 100 \times 10^0 = 100$ . (b) At  $t = 3$ ,  $n = 100 \times 10^{0.729}$ . Now,  $\log n = \log 100 + 0.729 \log 10 = 2.729$ . Hence,  $n = 536$  approximately. (c)  $t$  must satisfy the equation  $200 = 100 \times 10^{0.243t}$ . Hence  $\log 200 = \log 100 +$



$0.243t \log 10$ . Hence  $2.301 = 2 + 0.243t$ , or  $t = \frac{0.301}{0.243} = 1.24$  hours, approximately.

## EXERCISES

1. Plot the graph of  $y = \log_{10} x$  from  $x = 0.01$  to  $x = 20$ .

2. Plot the graph of  $y = 10^x$  from  $x = -2$  to  $x = 1.5$ .

Solve each of the following equations for  $x$  approximately:

3.  $5^x = 10$ . 4.  $6^x = 18$ . 5.  $3^{2x+1} = 2(5^x)$ . 6.  $5^{3x-1} = 6(2^{x+1})$ .

7.  $\log x^2 - \log \frac{2x}{5} = 2.641$ . 8.  $\log 3x^2 - \log \frac{6x}{5} = 1.463$ .

Solve each of the following formulas for  $n$ :

9.  $A = P(1 + r)^n$ . 10.  $S = \frac{a - ar^n}{1 - r}$ . 11.  $s = \frac{(1 + i)^n - 1}{i}$ .

12.  $a = \frac{1 - (1 + i)^{-n}}{i}$ . 13.  $pv^n = c$ .

14. Show that if  $x$  takes a succession of values in geometric progression, then  $y = \log_b x$  ( $b \neq 1$ ) takes correspondingly a succession of values in arithmetic progression.

15. Show that if  $x$  takes a succession of values in arithmetic progression, then  $y = b^x$  ( $b \neq 1$ ) takes correspondingly a succession of values in geometric progression.

16. The number  $n$  of bacteria in a certain culture at the end of  $t$  hours is given by  $n = 1000 \times 10^{0.146t}$ . Find (a) the number present at  $t = 2$ ; (b) the time required for the number to be doubled.

17. Suppose radium decomposes in such a way that, of  $x$  milligrams of it,  $y$  milligrams will remain at the end of  $t$  centuries where  $y = x(0.97)^t$ . How much will remain of 2 milligrams after 1000 years?

18. If  $pv^n = 4.654$  and  $p = 10$  when  $v = 3$ , find  $n$ .

19. Suppose a radioactive substance decomposes so that the amount  $y$  remaining of an original amount  $x$  after  $t$  years is given by  $y = xe^{-rt}$  where  $e = 2.71828 \dots$ . If  $x = 2$  grams,  $r = 0.035$ , and  $t = 100$ , find  $y$ .

20. Suppose the number  $y$  of bacteria present in a certain culture after  $t$  hours is given by  $y = xe^{rt}$  where  $x = 100$ ,  $r = 0.25$ ,  $e = 2.71828 \dots$ . (a) If  $t = 24$ , find  $y$ . (b) Find the value of  $t$  at which there will be 300 bacteria present.



## CHAPTER XXIV

### Mathematics of Investment

Since few people are able to follow Polonius' advice\* to Laertes, problems concerning loans, investments, annuities, etc., arise in everyday life. Such problems may be solved by applying certain principles of elementary algebra already taken up.

**165. Simple interest and simple discount.** Money paid by one party for the use of another's money is called **interest**. The parties concerned may be individuals, companies, banks, governments, etc. Rates of interest are usually stated in percent per annum. For example, if Mr. Borrower borrows \$100 from Mr. Lender for one year with the understanding that at the end of that period he will repay \$106, we say that the rate of interest is 6% per annum. We shall write rates of interest as decimals for the purpose of actual computation. Thus 6% will be written as 0.06.

When interest is to be proportional to the length of time, it is called **simple interest**. The sum loaned is called the **principal**. The sum to be repaid at the end of the period is called the **amount**.

*Example 1.* Mr. B borrows \$100 from Mr. L for 2 years at simple interest of 6% per annum. Then the interest will be \$12.

*If the principal  $P$  dollars is invested at simple interest at the rate  $r$  per annum, the interest at the end of  $n$  years will be*

$$(1) \qquad I = Prn,$$

*and the amount will be*

$$(2) \qquad A = P + Prn = P(1 + rn).$$

*Example 2.* Mr. B borrows \$200 from Mr. L for  $2\frac{1}{2}$  years at simple interest of 6% per annum. Then the interest is  $\$200(0.06)(2.5) = \$30$ . The amount is \$230.

*Example 3.* Mr. B borrows \$1000 from Mr. L with the understanding that he will pay at the end of each month \$100 on the principal and simple interest at 6% per annum on the principal outstanding during each month. Find the total amount to be paid.

\* "Neither a borrower nor a lender be." (W. Shakespeare, *Hamlet*.)



*Solution.* There will be 10 payments. The interest due with the first payment is  $\$1000(0.06)(1/12) = \$5$ ; with the second payment, the interest due is  $\$900(0.06)(1/12) = \$4.50$ ; with the third payment, the interest due is  $\$800(0.06)(1/12) = \$4.00$ ;  $\dots$ ; with the tenth payment, the interest due is  $\$100(0.06)(1/12) = \$0.50$ . The total interest is the sum of the arithmetic progression  $\$0.50 + \$1.00 + \dots + \$5.00 = \frac{1}{2}(\$0.50 + \$5.00) = \$27.50$ . The total amount paid is therefore  $\$1027.50$ .

Often confused with simple interest is the concept of simple discount. For example, if Mr. B borrows  $\$100$  from a banker for one year at  $6\%$ , he may be given only  $\$94$  and is expected to repay  $\$100$  at the end of the year. Here the  $6\%$  is called a **simple discount rate**. Actually, the interest received by the banker on his investment is more than  $6\%$ , for he receives  $\$6$  interest plus the principal at the end of a year, but the principal invested is only  $\$94$ . Hence the rate of interest is  $6/94 = 0.0638$  approximately.

### EXERCISES

Find (a) the simple interest and (b) the amount on a loan of:

- $\$500$  at  $6\%$  per annum for 2 years.
- $\$800$  at  $4\%$  per annum for 6 months.
- $\$750$  at  $2\frac{1}{2}\%$  per annum for 15 months.
- $\$600$  at  $4\frac{1}{4}\%$  per annum for 10 months.
- A man borrows  $\$250$  for 1 year at a simple discount rate of  $5\%$  per annum. (a) How much does he actually receive? (b) What rate of interest is he actually paying?
- A man borrows  $\$300$  for 6 months at a simple discount rate of  $6\%$  per annum. (a) How much does he actually receive? (b) What rate of interest is he actually paying?
- A man borrows  $\$1000$  for 18 months at a simple discount rate of  $4\frac{1}{2}\%$  per annum. (a) How much does he actually receive? (b) What rate of simple interest is he actually paying?
- A man borrows some money at a simple discount rate of  $6\%$  per annum for 1 year. If he wishes to receive  $\$150$  in cash, how much must he borrow?
- A man borrows some money at a simple discount rate of  $6\%$  per annum for 8 months. If he wishes to receive  $\$200$  in cash, how much must he borrow?

10. Show that if  $i$  is the actual simple interest rate paid when money is borrowed for 1 year at a simple discount rate of  $d$ , then (a)  $i = \frac{d}{1-d}$ ,

(b)  $d = \frac{i}{1+i}$ , where both  $i$  and  $d$  are rates expressed as decimals.



11. A loan of \$1200 is to be repaid in monthly installments of \$100 plus simple interest at the rate of 6% per annum on the principal outstanding during each month. Find the total amount paid.

12. A man borrows \$500 for 10 months, and agrees to pay back \$50 per month plus simple interest at the rate of 6% per annum on the principal outstanding during each month. Find the total amount paid.

13. Solve the formula  $A = P(1 + rn)$  for (a)  $P$ ; (b)  $r$ ; (c)  $n$ .

14. A mortgage of \$4500 is to be repaid in monthly installments of \$50 plus simple interest at the rate of 6% per annum on the principal outstanding during each month. Find the total amount paid.

**166. Compound interest and compound discount.** If Mr.  $B$  borrows \$100 from Mr.  $L$  for two years at simple interest of 6%, it is usually understood that at the end of the first year he is to pay \$6 interest and at the end of the second year \$6 more plus the \$100 principal. If at the end of the first year  $B$  fails to pay the \$6 interest,  $L$  would be justified in saying that for the second year he has loaned \$6 more to  $B$ . Thus he would demand, at the end of the two years

$$\$100(1.06) + \$100(1.06)(0.06) = \$100(1.06)^2 = \$112.36.$$

In general, if the interest due is added to the principal, or converted into principal, at stated intervals of time, and thereafter itself earns interest as new principal, then the sum by which the original principal has been increased at the end of any time is called **compound interest**. A common illustration is a savings account where the depositor seldom withdraws the interest he is entitled to but prefers to let it be converted into principal, which in turn will draw interest thenceforth. The time between successive additions of interest to principal (or conversions of interest into principal) is called the **conversion period** or **interest period**. At the end of each conversion period the new principal, consisting of the original principal plus the compound interest, is called the **compound amount**. The interest rate is usually stated in percent per annum, but the conversion period may be shorter than a year, usually 6 or 3 months. In the latter cases, interest is said to be compounded semi-annually or quarterly, as the case may be. Interest at 6% per annum compounded semi-annually means that 3% interest is added every 6 months.

Suppose  $P$  dollars is invested at compound interest at a rate  $r$  per conversion period; as usual,  $r$  will be expressed as a decimal. Then at the end of the first period the compound amount will be



$$A_1 = P + Pr = P(1 + r).$$

At the end of the second period the amount will be

$$A_2 = A_1 + A_1r = A_1(1 + r) = P(1 + r)^2.$$

At the end of the third period the amount will be

$$A_3 = A_2 + A_2r = A_2(1 + r) = P(1 + r)^3.$$

Clearly at the end of the  $n$ th conversion period the compound amount will be

$$(1) \quad A = A_n = P(1 + r)^n.$$

The basic formula (1) connects four quantities, namely  $A$ ,  $P$ ,  $r$ , and  $n$ . If any three of them are known, the fourth may be found.

*Example 1.* If \$100 is invested at 6% per annum compounded semi-annually, what will it amount to in 3 years?

*Solution.* The rate per conversion period is 3%. Hence, by (1), the amount at the end of 3 years or 6 conversion periods will be

$$(2) \quad A_6 = 100(1.03)^6 \text{ dollars.}$$

To evaluate this quantity by multiplying 100 six times by 1.03 is extremely tedious. This calculation may be avoided by the following three methods.

*First method.* From table IV, we find that  $(1.03)^6 = 1.1941$  approximately. Hence  $A_6 = \$119.41$  approximately.

*Second method.* If table IV is not available, we may use logarithms as follows. From (2), and table III, we find

$$\log A_6 = \log 100 + 6 \log 1.03 = 2 + 6(0.0128) = 2.0768.$$

Hence,  $A_6 = \$119.40$  approximately.

*Third method.* If no tables are available, we may calculate  $(1.03)^6$  by the binomial theorem, as follows.

$$\begin{aligned} (1 + 0.03)^6 &= 1^6 + 6 \cdot 1^5 \cdot (0.03) + 15 \cdot 1^4(0.03)^2 + 20 \cdot 1^3(0.03)^3 \\ &\quad + 15 \cdot 1^2(0.03)^4 + 6 \cdot 1 \cdot (0.03)^5 + (0.03)^6 \\ &= 1 + 0.18 + 0.0135 + 0.00054 + 0.00001215 + \dots \\ &= 1.1941 \text{ approximately.} \end{aligned}$$

Hence  $A_6 = \$119.41$  approximately.

*Note.* For financial purposes, approximations to the nearest hundredth of a dollar are all that are required. But the four- or five-place tables included in this book may be insufficient to yield such accuracy. More extensive tables may be obtained. The



binomial theorem method, of course, may be carried out to any desired accuracy. Calculating machines are useful, if available.

The quantity  $P$  is called the **present value** of  $A_n$  dollars due at the end of  $n$  conversion periods at the interest rate  $r$  per conversion period. Clearly

$$(3) \quad P = A_n \cdot \frac{1}{(1 + r)^n} = A_n(1 + r)^{-n}.$$

*Example 2.* How much must be deposited in a trust fund for a child at 4% compounded annually, if it is desired to have \$1000 in the fund at the end of 10 years?

*Solution.* By (3), the present value is

$$P = 1000(1.04)^{-10}.$$

From table V,  $(1.04)^{-10} = 0.67556$ . Hence  $P = \$675.56$  approximately.

*Note.* Logarithms or the binomial theorem may also be used as in the second and third methods in example 1 above.

To **discount** a sum  $A$  for  $n$  conversion periods, at the rate  $r$  per conversion period, means to find the present value  $P$  of  $A$  at a time  $n$  periods before the payment  $A$  is due. The difference  $A - P$  is called the **discount**. The present value of \$1 at the rate  $r$  per period for  $n$  periods is called **compound discount**; its value is  $(1 + r)^{-n}$ .

*Example 3.* Mr. X holds a note from Mr. Y promising to pay \$1000 at a date 10 years from now. Mr. X sells this note to Mr. Z. If discount is reckoned at 4% compounded annually, Mr. Z will pay Mr. X \$675.56 approximately for the note. (See example 2.)

## EXERCISES

1. Find the amount if \$500 is invested for 7 years at 4% per annum compounded (a) annually; (b) semi-annually.
2. Find the amount if \$300 is invested for 10 years at 5% per annum compounded (a) annually; (b) semi-annually.
3. Find the amount at the end of 21 years if \$100 is invested at 6% per annum compounded (a) annually; (b) semi-annually.
4. Find the amount at the end of 12 years if \$250 is invested at 7% per annum compounded (a) annually; (b) semi-annually.
5. How much must be deposited in a bank now in order to have \$500 at the end of 10 years if interest is paid at the rate of 4% per annum compounded (a) annually; (b) semi-annually.



6. How much must be invested now in order to have \$750 at the end of 5 years if interest is paid at the rate of 5% per annum compounded (a) annually; (b) semi-annually.

7. A note promising to pay \$600 in 3 years is sold. How much should be paid for it if discount is reckoned at 7% per annum compounded (a) annually; (b) semi-annually?

8. A note promising to pay \$800 in 5 years is sold. How much should be paid for it if discount is reckoned at 5% per annum compounded (a) annually; (b) semi-annually.

9. How many years will it take for a deposit of \$100 to amount to \$253.16 if interest is paid at the rate of  $3\frac{1}{2}\%$  per annum compounded annually?

10. How many years will it take for a deposit of \$250 to amount to \$475 if interest is paid at the rate of  $2\frac{1}{2}\%$  per annum compounded annually?

11. How many years will it take for a deposit of \$100 to amount to \$196 if interest is paid at the rate of 4% per annum compounded semi-annually?

12. How many years will it take for an investment to double itself if interest is paid at the rate of 2% per annum compounded annually?

13. How many years will it take for an investment to triple itself if interest is paid at the rate of  $3\frac{1}{2}\%$  per annum compounded annually?

14. A bond costing \$75 will be redeemed in 10 years for \$100. If interest is compounded annually, what rate of interest is being paid?

15. A deposit of \$100 will amount to \$266.58 in 25 years. If interest is compounded annually, what rate of interest is being paid?

16. A deposit of \$100 will amount to \$163.86 in 10 years. If interest is compounded semi-annually, what rate of interest is being paid?

17. A deposit of \$500 will amount to \$1048.80 in 15 years. Find the interest rate if interest is compounded semi-annually.

18. A deposit of \$500 will amount to \$671.95 in 5 years. If interest is compounded semi-annually, find the interest rate.

**167. Effective and nominal rates.** The interest rate  $i$  per annum compounded  $m$  times per year stated in describing an arrangement of compound interest is called the **nominal rate**. The **effective rate** is the rate  $j$  which, if compounded annually, would yield the same interest. For interest compounded once a year the nominal rate and the effective rate are the same. Equating the amounts at the end of one year under both arrangements, we have

$$P(1 + j) = P \left( 1 + \frac{i}{m} \right)^m.$$

Hence

$$(1) \quad 1 + j = \left( 1 + \frac{i}{m} \right)^m$$



or

$$(2) \quad j = \left(1 + \frac{i}{m}\right)^m - 1.$$

*Example.* Find the effective rate if the nominal rate is 8% compounded quarterly.

*Solution.* We have  $1 + j = (1.02)^4 = 1.0824$  from table IV. Hence the effective rate is 8.24% approximately.

### EXERCISES

Find the effective rate if interest is compounded semi-annually at the nominal rate of:

- |                  |                 |                 |
|------------------|-----------------|-----------------|
| 1. 4% per annum. | 2. 5% per annum | 3. 6% per annum |
| 4. 7% per annum. | 5. 8% per annum | 6. 9% per annum |

Find the effective rate if interest is compounded quarterly at the nominal rate of:

- |                   |                 |                  |
|-------------------|-----------------|------------------|
| 7. 4% per annum.  | 8. 5% per annum | 9. 6% per annum. |
| 10. 8% per annum. |                 |                  |

**168. Annuities.** Many business transactions are arranged so that equal payments are made at regular intervals of time. Such a series of payments is called an **annuity**. The interval of time at the end\* of which payments are to be made is called the **payment period**. Although the word *annuity* suggests annual payments, the payment period of an annuity may be of any length of time. Familiar examples of annuities are installment buying, premiums for insurance, amortizing a mortgage, etc. The term of the annuity is the time between the beginning of the first payment period and the end of the last. Each payment is called **rent** and is denoted by  $R$ . The sum of the payments made in one year is called the **annual rent**.

The **amount** of an annuity is the total amount which would be accumulated at the end of the term if each payment were invested at a given rate of compound interest at the time of payment. We treat only the case where the conversion period of the compound interest coincides with the payment period of the annuity. The amount of an annuity which pays \$1 at the end of each payment period for  $n$  periods is denoted by  $s_n$ , read "s angle n." The **present value** of an annuity is the sum of the present values of all the payments at the beginning of the term. The present value

\* If payments are made at the beginning of the payment periods, slight adjustments must be made in the subsequent formulas. The same principles, however, may be applied.



of an annuity which pays \$1 at the end of each payment period for  $n$  periods is denoted by  $a_{\overline{n}|i}$ , read “ $a$  angle  $n$ .”

The amount or the present value of an annuity can be determined by working out a schedule of the compound amount or present value of all payments, and adding, but if the number of payments is large this is very tedious. Instead we shall calculate formulas for  $s_{\overline{n}|i}$  and  $a_{\overline{n}|i}$  when compound interest is computed at the rate  $i$  per period.

At the end of the term, the last payment of \$1 will have just been made and will amount of \$1. The payment before the last will have drawn interest for one period and will amount to  $(1 + i)$ . The payment before that, the  $(n - 2)$ th payment, will amount to  $(1 + i)^2$ . And so on. The first payment will have drawn interest for  $n - 1$  periods and will amount to  $(1 + i)^{n-1}$ . Hence

$$s_{\overline{n}|i} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}.$$

This is a geometric progression with first term 1 and common ratio  $1 + i$ . By section 147, chapter XXI, the sum is therefore

$$(1) \quad s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i}.$$

Values of this quantity for various values of  $n$  and  $i$  are given in table VI.

If the rent, or periodic payment, is \$ $R$  (instead of \$1), it is easily seen that the amount of the annuity is

$$(2) \quad R \cdot s_{\overline{n}|i} \text{ dollars.}$$

To find  $a_{\overline{n}|i}$ , we compute the present values of the payments at the beginning of the term and add. The present value of the first payment of \$1, which is due at the end of the first period, is  $(1 + i)^{-1}$ . The present value of the second payment is  $(1 + i)^{-2}$ , and so on. The present value of the last ( $n$ th) payment is  $(1 + i)^{-n}$ . Hence

$$a_{\overline{n}|i} = (1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-n}.$$

This is a geometric progression with first term  $(1 + i)^{-1}$  and common ratio  $(1 + i)^{-1}$ . By section 147, chapter XXI, we find

$$(3) \quad a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i}.$$

Values of this quantity are given in table VII.



If the rent, or periodic payment, is  $\$R$  instead of  $\$1$ , it is easily seen that the present value of the annuity is

$$(4) \quad R \cdot a_{\overline{n}|i} \text{ dollars.}$$

*Example.* Find the amount and present value of an annuity paying  $\$100$  every 6 months for 10 years, interest being compounded semi-annually at 6% per annum.

*First solution.* The rate  $i$  is 0.03, and the number of periods is  $n = 20$ . From tables VI and VII,  $s_{\overline{20}|0.03} = 26.8704$  and  $a_{\overline{20}|0.03} = 14.8775$ . Hence the amount is  $\$2687.04$  approximately and the present value is  $\$1487.75$  approximately.

*Second solution.* If tables VI and VII are not available, logarithms may be used as follows. We have

$$(5) \quad s_{\overline{20}|0.03} = \frac{(1.03)^{20} - 1}{0.03}.$$

By logarithms we find  $(1.03)^{20} = 1.8061$  approximately. Hence  $s_{\overline{20}|0.03} = \frac{0.8061}{0.03} = 26.87$  approximately. Hence the amount is  $\$2687$  approximately. A similar method can be used for the present value.

*Third solution.* In (5), the quantity  $(1.03)^{20} = (1 + 0.03)^{20}$  can be evaluated by means of the binomial theorem, as in section 166, example 1, third method.

As before, answers obtained by use of tables will be approximate because of the limitations of the tables. More extensive tables are available.

## EXERCISES

*Solve the following problems using the methods above rather than formulas.*

1. Find (a) the amount, and (b) the present value of an annuity of  $\$300$  per year for 10 years at  $3\frac{1}{2}\%$  per annum.

2. Find (a) the amount and (b) the present value of an annuity of  $\$250$  per year for 25 years at  $4\frac{1}{2}\%$  per annum.

3. A man deposits  $\$200$  in a bank on the first day of each year for 20 years. If the bank pays interest at  $2\frac{1}{2}\%$  per annum, compounded annually, what is the amount at the end of the 20th year? (*Hint: note that payments are made at the beginning, not at the end, of periods. See footnote on page 372.*)

4. A man deposits  $\$300$  in a bank on the first day of each year for 25 years. If the bank pays interest at  $3\%$  per annum, compounded annually, what is the amount at the end of the 25th year?



5. A man deposits \$200 every 6 months in a bank paying interest at 4% per annum compounded semi-annually. How much is in his account just after he makes his 30th deposit?

6. A man deposits \$150 every 6 months in a bank paying interest at 5% per annum compounded semi-annually. How much is in his account just after he makes his 40th deposit?

7. A man wishes to make equal annual deposits at the beginning of each year for 30 years so as to accumulate \$10,000 at the end of the 30th year. If the bank pays interest at 2% per annum, compounded annually, what must his annual deposit be?

8. A machine costing \$5000 will wear out in 20 years. What equal annual deposits must be made at the end of each year to pay for its replacement if interest is paid at 3% per annum compounded annually?

9. Show that if a debt (such as a mortgage) of  $D$  dollars, at an interest rate  $i$  per annum compounded annually, is paid off (amortized) in  $n$  years by equal annual payments of  $E$  dollars, then  $E = D/a_{\overline{n}|i}$  or  $E \cdot a_{\overline{n}|i} = D$ .

10. A man pays off a \$10000 mortgage, interest and principal, by equal payments, made at the end of each year for 15 years. Find his annual payment if interest is paid at 6% per annum compounded annually. Make a schedule showing the part of each payment which goes for interest and principal respectively. (*Hint*: Use the result of exercise 9.)

11. A man pays off a \$5000 mortgage, interest and principal, by equal payments, made at the end of each year for 10 years. Find his annual payment if interest is paid at 6% per annum compounded annually.

12. A mortgage of \$6000 is paid off, interest and principal, by equal payments, made at the end of each year, for 20 years. Find the annual payment if interest is paid at 5% per annum compounded annually.

13. A man pays premiums of \$23.24 for an insurance policy at the beginning of each year for 20 years. Just after paying the 20th premium he surrenders his policy, receiving \$327.58 from the company. What was the actual total cost of being insured if money is worth 3%?\* (*Hint*: The answer will be the amount of the annuity of which the premium is the periodic payment minus the cash received at surrender of the policy.)

14. A man pays premiums of \$164.50 for an insurance policy at the beginning of each year for 20 years. Just after paying the 20th premium, he surrenders the policy, receiving \$562.15 from the company. What was the actual total cost of being insured if money is worth 4%?

\* This phrase means that interest is to be computed at the rate of 3% per annum compounded annually.



## CHAPTER XXV

### The Euclidean Algorithm

**169. The Euclidean algorithm for positive integers.** Recall\* that to divide a positive integer  $a$  by a positive integer  $b$ , where  $b \leq a$ , means to find a positive integer  $q$ , the quotient, and a non-negative integer  $r$  such that

$$(1) \quad a = bq + r \text{ where } 0 \leq r < b.$$

If  $r = 0$ , we say  $b$  is a factor of  $a$ , or  $a$  is exactly divisible by  $b$ , or  $a$  is a multiple of  $b$ .†

It can be proved that for a given  $a$  and  $b$ , the  $q$  and  $r$  are uniquely determined.

**THEOREM 1.** Any common factor  $c$  of  $a$  and  $b$  is a factor of  $r$ . Also any common factor  $d$  of  $b$  and  $r$  is a factor of  $a$ .

*Proof.* By hypothesis,  $a = cx$  and  $b = cy$  where  $x$  and  $y$  are positive integers. Then by (1),

$$cx = cyq + r$$

or

$$r = c(x - yq).$$

Since  $x - yq$  is a non-negative integer,  $c$  is a factor of  $r$ . This completes the proof of the first statement in the theorem.

If  $b = du$  and  $r = dv$  then  $a = bq + r = duq + dv = d(uq + v)$ . This proves the second statement in the theorem.

It follows from theorem 1 that the highest common factor (H.C.F.) of  $a$  and  $b$  is also the highest common factor of  $b$  and  $r$ .

Given two positive integers  $a$  and  $b$  ( $a > b$ ), their H.C.F. can be found by the following process, known as the Euclidean algorithm. Divide  $a$  by  $b$ , obtaining

$$(2) \quad a = bq_1 + r_1 \quad (0 \leq r_1 < b).$$

If  $r_1 > 0$ , divide  $b$  by  $r_1$ , obtaining

$$(3) \quad b = r_1q_2 + r_2 \quad (0 \leq r_2 < r_1).$$

\* See section 43, chapter V.

† The definitions and procedures of this section are easily extended to all integers, positive, negative, or zero.



If  $r_2 > 0$ , divide  $r_1$  by  $r_2$ , obtaining

$$(4) \quad r_1 = r_2 q_3 + r_3 \quad (0 \leq r_3 < r_2).$$

If  $r_3 > 0$ , divide  $r_2$  by  $r_3$ , obtaining

$$(5) \quad r_2 = r_3 q_4 + r_4 \quad (0 \leq r_4 < r_3),$$

and so on until a remainder  $r_{n+1}$  equal to 0 is reached:

$$(6) \quad r_{n-1} = r_n q_{n+1}.$$

**THEOREM 2.** *The last non-zero remainder  $r_n$  is the H.C.F. of  $a$  and  $b$ .*

*Proof.* By theorem 1, (2) implies that the H.C.F. of  $a$  and  $b$  is equal to the H.C.F. of  $b$  and  $r_1$ . Similarly (3) implies that the H.C.F. of  $b$  and  $r_1$  is the H.C.F. of  $r_1$  and  $r_2$ . And so on until we conclude that the H.C.F. of  $a$  and  $b$  is equal to the H.C.F. of  $r_{n-1}$  and  $r_n$ . But since  $r_n$  is a factor of  $r_{n-1}$  by (6), this H.C.F. is  $r_n$  itself.

*Example 1.* Find the H.C.F. of 66 and 180.

*Solution.* The successive divisions may be arranged compactly, as follows.

$$\begin{array}{r}
 2 \\
 \hline
 66 \overline{)180} \\
 \underline{132} \quad 1 \\
 48 \overline{)66} \\
 \underline{48} \quad 2 \\
 18 \overline{)48} \\
 \underline{36} \quad 1 \\
 12 \overline{)18} \\
 \underline{12} \quad 2 \\
 6 \overline{)12} \\
 \underline{12} \\
 0.
 \end{array}$$

Therefore 6 is the H.C.F. of 66 and 180.

Several consequences of the Euclidean algorithm are worth noting.

**THEOREM 3.** *If  $h$  is the H.C.F. of  $a$  and  $b$ , there exist integers  $s$  and  $t$  such that  $h = sa + tb$ .*

*Proof.* To simplify the discussion, suppose  $n = 4$  in (6), so that  $h = r_4$ . Then from (2) we have

$$r_1 = a - bq_1.$$



From (3),

$$r_2 = b - r_1q_2 = b - (a - bq_1)q_2 = (-q_2)a + (1 + q_1q_2)b.$$

From (4),

$$\begin{aligned} r_3 &= r_1 - r_2q_3 = (a - bq_1) - [(-q_2)a + (1 + q_1q_2)b]q_3 \\ &= (1 + q_2q_3)a + (-q_1 - q_3 - q_1q_2q_3)b. \end{aligned}$$

From (5),

$$\begin{aligned} r_4 = r_2 - r_3q_4 &= (-q_2 - q_4 - q_2q_3q_4)a \\ &\quad + (1 + q_1q_2 + q_1q_4 + q_3q_4 + q_1q_2q_3q_4)b. \end{aligned}$$

Since the parentheses on the right represent integers, we have  $r_4 = sa + tb$ , which was to have been proved. The proof for any value of  $n$  is similar.

**DEFINITION 1.** *If a positive integer greater than 1 has no factors except itself and 1, it is called a **prime** or a **prime number**.*

*Examples.* The first few primes are 2, 3, 5, 7, 11, . . . .

**DEFINITION 2.** *Two different positive integers are **relatively prime** if they have no common factor except 1.*

*Examples.* 4 and 9 are relatively prime. So are 10 and 21; and 6 and 35. But 6 and 10 are not relatively prime because they have the common factor 2.

Clearly, if two positive integers are relatively prime, their H.C.F. is 1, and conversely.

**THEOREM 4.** *If  $a$  and  $b$  are relatively prime but  $a$  is a factor of  $bc$ , then  $a$  is a factor of  $c$ .*

*Proof.* The H.C.F. of  $a$  and  $b$  is 1. Hence, by theorem 3, there exist integers  $s$  and  $t$  such that

$$1 = sa + tb.$$

Multiplying by  $c$ , we have

$$(7) \quad c = sac + tbc.$$

By hypothesis, there exists an integer  $x$  such that  $bc = ax$ . Substituting in (7), we have

$$c = sac + tax = a(sc + tx).$$

Hence  $a$  is a factor of  $c$ .

Theorem 4 is useful in the theory of equations; see section 107, chapter XV. Compare also theorem 2, section 6, chapter I.

## EXERCISES

*By means of the Euclidean algorithm, find the H.C.F. of each of the following pairs of numbers:*



1. 60, 80.

2. 60, 75.

3. 24, 36.

4. 70, 75.

5. 86, 75.

6. 153, 189.

7. 156, 306.

8. 1881, 3675.

9. Prove that if  $p$  is prime and  $p$  is a factor of  $ab$ , where  $a$  and  $b$  are positive integers, then either  $p$  is a factor of  $a$  or  $p$  is a factor of  $b$ .

10. Prove that if  $a, b, c$  are positive integers,  $a$  and  $b$  being relatively prime, and if  $a$  and  $b$  are both factors of  $c$ , then  $ab$  is a factor of  $c$ .

\*11. Prove that the  $q$  and  $r$  in (1) above are uniquely determined. That is, if  $a = bq + r$ ,  $0 \leq r < b$ , and  $a = bq' + r'$ ,  $0 \leq r' < b$ , then  $q = q'$  and  $r = r'$ .

\*12. Prove that any positive integer can be expressed as a product of prime factors in one and only one way apart from the order of the factors. (*Hint*: use the theorem of exercise 9.)

**170. Euclidean algorithm for polynomials.** We shall confine ourselves to polynomials with real coefficients, and factors with real coefficients, although this restriction is not essential. Recall that if  $a$  and  $b$  are polynomials, to divide  $a$  by  $b$  means to find polynomials  $q$  and  $r$  such that  $a = bq + r$  and such that  $r$  has lower degree than  $b$ . If  $r = 0$ ,  $b$  and  $q$  are factors of  $a$ . We shall consider as **essentially the same** polynomials which differ only by a constant factor. For example, we shall regard  $x + 2$ ,  $2x + 4$ ,  $3x + 6$ ,  $\frac{1}{2}x + 1$ , etc., as essentially the same. Thus we shall say that  $x^2 - 1 = (x - 1)(x + 1) = (2x - 2)(\frac{1}{2}x + \frac{1}{2}) = (3x - 3)(\frac{1}{3}x + \frac{1}{3})$  are essentially the same factorization. If we merely interpret the letters  $a, b, q_1, r_1, q_2, r_2, \dots$  as polynomials instead of integers, and  $<$  as referring to degree, it can be seen that the Euclidean algorithm will yield an H.C.F.\* of the polynomials  $a$  and  $b$ . That is, theorems 1 and 2 of the preceding section and their proofs apply to polynomials as well as to integers.

*Example 1.* Find an H.C.F. of  $x^4 - 3x^2 + 2$  and  $x^4 + x^3 - x - 1$ .

*Solution.* By the Euclidean algorithm,

$$\begin{array}{r}
 \begin{array}{r}
 1 \\
 x^4 - 3x^2 + 2 \overline{) x^4 + x^3 - x - 1} \\
 \underline{x^4 \phantom{+ 3x^3} - 3x^2 \phantom{+ 3x} + 2} \\
 x^3 + 3x^2 - x - 3
 \end{array}
 \quad
 \begin{array}{r}
 x - 3 \\
 x^4 - 3x^3 + 3x^2 - 3x + 2 \\
 \underline{x^4 + 3x^3 - x^2 - 3x} \\
 - 3x^3 - 2x^2 + 3x + 2 \\
 \underline{- 3x^3 - 9x^2 + 3x + 9} \\
 7x^2 - 7
 \end{array}
 \quad
 \begin{array}{r}
 \frac{1}{7}x + \frac{1}{7} \\
 x^3 + 3x^2 - x - 3 \\
 \underline{x^3 \phantom{+ 3x^2} - x} \\
 3x^2 - 3 \\
 \underline{3x^2} \\
 0
 \end{array}
 \end{array}$$

\* May be omitted without disturbing continuity.

\* An H.C.F. is a common factor of  $a$  and  $b$  which is exactly divisible by any common factor of  $a$  and  $b$ . All H.C.F.'s are essentially the same.



Hence  $x^2 - 1$  (which is essentially the same as  $7x^2 - 7$ ) is an H.C.F. of the given polynomials.

Similarly theorem 3 of the preceding section and its proof may be extended to polynomials as follows.

**THEOREM.** *If  $h(x)$  is an H.C.F. of two polynomials  $a(x)$  and  $b(x)$ , then there exist polynomials  $s(x)$  and  $t(x)$  such that  $h(x) = s(x)a(x) + t(x)b(x)$ .*

A polynomial may be called **prime** or **irreducible\*** if it has no factors except constants (which are all essentially the same as 1) and polynomials essentially the same as itself. Two polynomials are called **relatively prime** if they have no common factors except constants (which are all essentially the same as 1). Clearly, if two polynomials are relatively prime, their H.C.F. is essentially the same as 1. Hence we have the following immediate corollary of the above theorem.

**COROLLARY.** *If polynomials  $a(x)$  and  $b(x)$  are relatively prime, then there exist polynomials  $s(x)$  and  $t(x)$  such that  $1 = s(x)a(x) + t(x)b(x)$ .*

*Note.* By corollary 1, section 106, chapter XV, every polynomial (with real coefficients) can be expressed as a product of prime factors (with real coefficients) of the first or second degree.

## EXERCISES

*By means of the Euclidean algorithm, find a H.C.F. of each of the following pairs of polynomials:*

1.  $x^2 - 9, x^2 - 6x + 9.$
2.  $x^2 - 4, x^2 - 5x + 6.$
3.  $x^2 - x - 2, x^2 + x - 6.$
4.  $x^2 - 6x + 9, x^2 + 2x - 15.$
5.  $2x^3 - x^2 + 6x - 3, 8x^3 - 2x^2 + 17x - 9.$
6.  $x^4 + x^3 + 4x^2 + 3x + 3, 2x^3 + 5x^2 + 6x + 15.$

\* More precisely, prime or irreducible over the domain of real coefficients.



## CHAPTER XXVI

### Partial Fractions

**171. The theorem and its use.** To add two or more rational expressions, as

$$\frac{3}{x-2} + \frac{2}{x-3} = \frac{-5x-13}{x^2-5x+6},$$

is a matter of routine. For certain purposes, particularly in connection with a more advanced subject called *integral calculus*, it is advantageous to be able to start with the sum and find out the so-called **partial fractions** of which it is the sum. This inverse process of “un-adding” the sum is more difficult than the direct process of addition, just as factoring is harder than multiplying and chemical analysis is harder than making compounds.

Every rational function or fractional expression can be expressed as a quotient of two polynomials. We confine ourselves to expressions of the form  $N(x)/D(x)$  where  $N(x)$  and  $D(x)$  are polynomials with real coefficients. If the degree of the numerator  $N(x)$  is less than the degree of the denominator  $D(x)$ , the expression is called a **proper fraction**; otherwise, **improper**. Every improper fraction can be reduced to the sum of a polynomial and a proper fraction, for if the degree of the numerator is equal to or greater than the degree of the denominator, we can divide  $N(x)$  by  $D(x)$  obtaining a quotient polynomial  $Q(x)$  and a remainder polynomial  $R(x)$  whose degree is less than that of  $D(x)$ . Thus

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

where  $R(x)/D(x)$  is a proper fraction. For example,

$$\frac{x^3 - 4x^2 + 6x - 7}{x^2 - 5x + 6} = x + 1 + \frac{5x - 13}{x^2 - 5x + 6}.$$

Hence we confine ourselves in what follows to proper fractions.

By corollary 1, section 106, chapter XV, the denominator  $D(x)$  can be expressed as a product of linear and quadratic factors, with real coefficients, which cannot be factored further\* into linear

\* That is, they are prime. Factors differing only by a constant factor are regarded as essentially the same. See section 170, chapter XXV.



factors with real coefficients. We suppose this done; these are the factors referred to throughout, below. Then every proper fraction can be resolved into partial fractions in accordance with the following theorem, whose proof will be given in the following section.

**THEOREM.** Every proper fraction  $N(x)/D(x)$  can be resolved into an algebraic sum of partial fractions, as follows:

*I. If a linear factor  $(ax + b)$  occurs once as a factor of  $D(x)$ , there is a partial fraction of the form  $\frac{A}{ax + b}$ , where  $A$  is a constant;*

*II. If a linear factor  $(ax + b)$  occurs  $k$  times as a factor of  $D(x)$ , there are  $k$  partial fractions of the form*

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k},$$

*where  $A_1, A_2, \dots, A_k$  are constants;*

*III. If a quadratic factor\*  $(ax^2 + bx + c)$  occurs once as a factor of  $D(x)$ , there is a partial fraction of the form  $\frac{Cx + K}{ax^2 + bx + c}$  where  $C$  and  $K$  are constants;*

*IV. If a quadratic factor  $(ax^2 + bx + c)$  occurs  $k$  times as a factor of  $D(x)$ , there are  $k$  partial fractions of the form*

$$\frac{C_1x + K_1}{ax^2 + bx + c} + \frac{C_2x + K_2}{(ax^2 + bx + c)^2} + \cdots + \frac{C_kx + K_k}{(ax^2 + bx + c)^k},$$

*where  $C_1, K_1, C_2, K_2, \dots, C_k, K_k$  are constants.*

The process of determining the unknown constants will be illustrated in the following examples. In every case the result may be checked by adding the partial fractions to get the original expression.

*Case I. When the factors of the denominator  $D(x)$  are all linear and distinct.†*

*Example 1. Resolve  $\frac{9x^2 - 9x + 6}{(x - 1)(2x - 1)(x + 2)}$  into partial fractions.*

*Solution.* By *I* of the theorem, we may write

$$(1) \quad \frac{9x^2 - 9x + 6}{(x - 1)(2x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{2x - 1} + \frac{C}{x + 2}.$$

\* More precisely, a prime or irreducible quadratic factor; that is, with  $b^2 - 4ac < 0$ . In fact the theorem as stated does not require that the factors be irreducible. But it is often advantageous to use irreducible factors.

† That is, no two are essentially the same.



This is an identity; that is, it is true for all values of  $x$  except possibly for the values  $x = 1, \frac{1}{2}$ , and  $-2$ , which make the denominator vanish. Hence, multiplying both members of (1) by the least common denominator  $(x - 1)(2x - 1)(x + 2)$ , we get

$$(2) \quad 9x^2 - 9x + 6 = A(2x - 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(2x - 1).$$

We may determine the values of the constants  $A, B, C$  by either of two methods.

*First method.* Rewriting (2) we get

$$(3) \quad 9x^2 - 9x + 6 = (2A + B + 2C)x^2 + (3A + B - 3C)x + (-2A - 2B + C).$$

Since (2) is true for all values of  $x$  except possibly  $x = 1, \frac{1}{2}$ , and  $-2$ , it follows from corollary 2, section 104, chapter XV, that (2) is an identity. Hence we may equate coefficients of like powers of  $x$  in (3) obtaining

$$(4) \quad \begin{cases} 2A + B + 2C = 9 \\ 3A + B - 3C = -9 \\ -2A - 2B + C = 6. \end{cases}$$

Solving the system (4), we get  $A = 2, B = -3, C = 4$ . Hence, from (1), we have

$$(5) \quad \frac{9x^2 - 9x + 6}{(x - 1)(2x - 1)(x + 2)} = \frac{2}{x - 1} - \frac{3}{2x - 1} + \frac{4}{x + 2}.$$

*Second method.* Since (2) is true for all values of  $x$  except possibly  $x = 1, \frac{1}{2}, -2$ , it follows from corollary 2, section 104, chapter XV, that (2) is an identity and hence true for all values of  $x$  including  $x = 1, \frac{1}{2}$ , and  $-2$ . Substituting  $x = 1$  in (2), we find that  $6 = 3A$ , or  $A = 2$ . Substituting  $x = \frac{1}{2}$  in (2), we find that  $15/4 = -5B/4$ , or  $B = -3$ . Substituting  $x = -2$  in (2), we find that  $60 = 15C$ , or  $C = 4$ . Hence we have (5) as before.

## EXERCISES

*Resolve into partial fractions:*

1.  $\frac{x + 12}{(2x - 1)(x + 2)}$
2.  $\frac{5x - 5}{(x - 3)(3x + 1)}$
3.  $\frac{-x - 12}{2x^2 - x - 6}$
4.  $\frac{4x - 1}{x^2 + 3x - 10}$
5.  $\frac{2x^2 + 10x + 2}{(x + 1)(x - 2)(x + 3)}$
6.  $\frac{5x^2 - 5x + 6}{(2x + 1)(x - 1)(x + 1)}$
7.  $\frac{x^4 + 5x^3 + 7x^2 + 7x + 13}{x^2 + 5x + 6}$



$$\begin{array}{ll} 8. \frac{2x^3 + x^2 - 3x - 5}{x^2 - x - 2} & 9. \frac{5x^3 + 36x^2 + 77x + 48}{(x+1)(x^2 + 5x + 6)} \\ 10. \frac{2x^3 - 2x^2 - 4x - 14}{(x-1)(x^2 - x - 6)} & \end{array}$$

*Case II.* When all the factors of  $D(x)$  are linear but some are repeated.

*Example 2.* Resolve  $\frac{x^2 - 3x + 1}{(x-1)^2(x-2)}$  into partial fractions.

*Solution.* By *I* and *II* of the theorem, we may write

$$(6) \quad \frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Multiplying both members by the L.C.D., we obtain

$$(7) \quad x^2 - 3x + 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2,$$

or,

$$(8) \quad x^2 - 3x + 1 = (A+C)x^2 + (-3A+B-2C)x + (2A-2B+C).$$

*First method.* Since (8) is an identity, we may equate coefficients of like powers, obtaining

$$(9) \quad \begin{cases} A + C = 1 \\ -3A + B - 2C = -3 \\ 2A - 2B + C = 1. \end{cases}$$

Solving (9), we get  $A = 2$ ,  $B = 1$ ,  $C = -1$ . Hence

$$(10) \quad \frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{x-2}.$$

*Second method.* Since (7) is an identity, we may substitute  $x = 1$  in (7), obtaining  $-1 = -B$ , or  $B = 1$ . Similarly substituting  $x = 2$  in (7), we get  $-1 = C$ . Substituting these values for  $B$  and  $C$  and substituting for  $x$  any value different from 1 or 2, say  $x = 3$ , in (7), we get  $1 = 2A + 1 - 4$ , or  $A = 2$ . Hence we have (10) as before.

## EXERCISES

*Resolve into partial fractions:*

$$\begin{array}{ll} 11. \frac{3x+5}{(x+1)^2} & 12. \frac{2x-7}{x^2-6x+9} \\ 13. \frac{2x^2-9x+11}{x^3-6x^2+12x-8} & 14. \frac{x^2-x-3}{(x+1)^3} \end{array}$$



$$15. \frac{5x^2 + 18x + 14}{(2x + 1)(x^2 + 6x + 9)}.$$

$$17. \frac{5x^3 - 6x^2 - 3x - 1}{(x + 1)^2(x - 2)^2}.$$

$$19. \frac{6x^3 - 29x^2 - 51x + 278}{(2x + 4)(x - 4)^2}.$$

$$16. \frac{5x^2 - 24x + 13}{(3x - 1)(x - 2)^2}.$$

$$18. \frac{2x^3 - 3x + 4}{x^4 + 4x^3 + 4x^2}.$$

$$20. \frac{x^4 + 4x^3 + 8x^2 + 5x}{(x - 1)(x + 2)^2}.$$

*Case III.* When  $D(x)$  has (irreducible) quadratic factors, none repeated.

*Example 3.* Resolve  $\frac{4x^2 + x + 1}{(x^2 + 1)(x - 1)}$  into partial fractions.

*Solution.* By *I* and *III* of the theorem, we may write

$$(11) \quad \frac{4x^2 + x + 1}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1}.$$

Hence,

$$(12) \quad 4x^2 + x + 1 = (Ax + B)(x - 1) + C(x^2 + 1)$$

or,

$$(13) \quad 4x^2 + x + 1 = (A + C)x^2 + (-A + B)x + (-B + C).$$

*First method.* Equating coefficients in (13), we get

$$(14) \quad \begin{cases} A + C = 4 \\ -A + B = 1 \\ -B + C = 1. \end{cases}$$

Solving (14), we get  $A = 1$ ,  $B = 2$ ,  $C = 3$ . Hence

$$(15) \quad \frac{4x^2 + x + 1}{(x^2 + 1)(x - 1)} = \frac{x + 2}{x^2 + 1} + \frac{3}{x - 1}.$$

*Second method.* Since (12) is an identity, we may set  $x$  equal to any three values in (12). Substituting  $x = 1$ , we get at once  $6 = 2C$ , or  $C = 3$ . Substituting  $C = 3$ , and any value of  $x$ , say  $x = 0$ , in (12) we get  $1 = -B + 3$ , or  $B = 2$ . Substituting  $C = 3$ ,  $B = 2$ , and any other value for  $x$ , say  $x = 2$ , in (12), we get  $19 = (2A + 2) + 15$ , or  $A = 1$ . Hence, we have (15) as before.

## EXERCISES

*Resolve into partial fractions:*

$$21. \frac{x^2 - 6x + 2}{(x - 1)(x^2 + x + 1)}.$$

$$22. \frac{5x^2 - 8x + 8}{(2x - 1)(x^2 - x + 2)}.$$

$$23. \frac{5x^2 + 1}{x^3 + 1}.$$

$$24. \frac{x^2 - 9x + 2}{x^3 - 8}.$$

$$25. \frac{5x^3 + 2x + 9}{x^3 + x^2 + 3x}.$$



$$26. \frac{4x^2 - 6x + 4}{3x^3 + 3x}.$$

$$27. \frac{3x^2 - 2x + 5}{(x^2 + 1)(2x^2 - x + 3)}.$$

$$28. \frac{-2x^3 + 2x^2 - 6x + 11}{(x^2 + x + 2)(x^2 - x + 3)}.$$

$$29. \frac{x^5 + x^4 + 4x^3 + 7x^2 - 2x + 19}{(x^2 + 3)(x^2 - x + 2)}.$$

$$30. \frac{3x^4 + 10x^3 + 19x^2 + 28x + 21}{(x + 1)^2(x^2 + 4)}.$$

$$31. \frac{4x^4 + 7x^3 + 5x^2 + x - 6}{x^4 + x^3 + 2x^2}.$$

$$32. \frac{2x^4 - 5x^3 + 14x^2 - 13x + 10}{(x - 1)^2(x^2 + 3)}.$$

*Case IV.* When  $D(x)$  has repeated (irreducible) quadratic factors.

*Example 4.* Resolve  $\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x - 1)(x^2 + x + 1)^2}$  into partial fractions.

*Solution.* By *I* and *IV* of the theorem, we may write

$$(16) \quad \frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x - 1)(x^2 + x + 1)^2} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1} + \frac{Dx + E}{(x^2 + x + 1)^2}.$$

Multiplying both members of (16) by the L.C.D., we obtain

$$(17) \quad 4x^4 + 3x^3 + 6x^2 + 5x = A(x^2 + x + 1)^2 + (Bx + C)(x - 1)(x^2 + x + 1) + (Dx + E)(x - 1),$$

or,

$$(18) \quad 4x^4 + 3x^3 + 6x^2 + 5x = (A + B)x^4 + (2A + C)x^3 + (3A + D)x^2 + (2A - B - D + E)x + (A - C - E).$$

*First method.* Equating coefficients in (18), we obtain

$$(19) \quad \begin{cases} A + B = 4 \\ 2A + C = 3 \\ 3A + D = 6 \\ 2A - B - D + E = 5 \\ A - C - E = 0. \end{cases}$$

Solving (19), we obtain  $A = 2$ ,  $B = 2$ ,  $C = -1$ ,  $D = 0$ ,  $E = 3$ .



Hence,

$$(20) \quad \frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{2}{x-1} + \frac{2x-1}{x^2+x+1} + \frac{3}{(x^2+x+1)^2}.$$

*Second method.* Since (17) is an identity, we may substitute any five values for  $x$ . Setting  $x = 1$  in (17) we get  $18 = 9A$ , or  $A = 2$ . Using  $A = 2$ , and setting  $x = 0, -1, 2, -2$  successively in (17), we get

$$(21) \quad \begin{cases} 0 = 2 - C - E & (x = 0) \\ 2 = 2 + 2B - 2C + 2D - 2E & (x = -1) \\ 122 = 98 + 14B + 7C + 2D + E & (x = 2) \\ 54 = 18 + 18B - 9C + 6D - 3E & (x = -2). \end{cases}$$

Solving (21), we get  $B = 2, C = -1, D = 0, E = 3$ , and hence (20) as before.

## EXERCISES

*Resolve into partial fractions:*

$$33. \quad \frac{x^2 + 2x - 2}{(x^2 + 1)^2}.$$

$$34. \quad \frac{x^3 + 3x - 7}{(x^2 + x + 2)^2}.$$

$$35. \quad \frac{3x^4 + 15x^2 + 3x + 18}{x(x^2 + 3)^2}.$$

$$36. \quad \frac{2x^4 + 5x^2 + 5x + 1}{(x-1)(x^2 + 2)^2}.$$

$$37. \quad \frac{3x^5 + 2x^4 + 6x^3 + 3x + 5}{(x^2 + 1)^2(x^2 + 2)^2}.$$

$$38. \quad \frac{x^5 + 7x^4 + 14x^3 + 22x^2 + 16x + 12}{x^2(x^2 + x + 2)^2}.$$

$$39. \quad \frac{3x^4 - 2x^3 + 4x^2 - 3x + 2}{(x-1)(x^2 + 1)^2}.$$

$$40. \quad \frac{3x^4 + 10x^2 - x + 6}{(x+1)(x^2 + 2)^2}.$$

**\*172. Proof of the theorem.** We shall indicate briefly a proof of the theorem that every rational expression  $N(x)/D(x)$ , where  $N(x)$  and  $D(x)$  are polynomials with real coefficients, can be expressed as the sum of a polynomial and partial fractions of the forms described in the theorem of the preceding section. The proof will be broken up into a number of auxiliary theorems.

**THEOREM 1.** A rational expression (fraction)  $N(x)/D(x)$ , whose denominator  $D(x) = a(x)b(x)$  where  $a(x)$  and  $b(x)$  are relatively prime polynomials, can be written as the sum of two rational expressions (fractions) with denominators  $a(x)$  and  $b(x)$  respectively.

\* Chapter XXV is required for this section.



*Proof.* By the corollary of section 170, chapter XXV, there exist polynomials  $s(x)$  and  $t(x)$  such that

$$(1) \quad 1 = s(x)a(x) + t(x)b(x).$$

Multiplying both members of (1) by  $\frac{N(x)}{a(x)b(x)}$ , we get\*

$$(2) \quad \frac{N}{ab} = \frac{Ns}{b} + \frac{Nt}{a}.$$

This proves theorem 1.

**THEOREM 2.** *If  $D(x) = [c(x)]^p$  where  $c(x)$  is a polynomial and  $p$  is a positive integer, then the rational expression  $N(x)/D(x)$  can be written as a sum of a polynomial and several rational expressions whose denominators are powers of  $c(x)$  with exponents  $\leq p$  and whose numerators have lower degree than  $c(x)$ .*

*Proof.* Dividing  $N(x)$  by  $c(x)$ , we obtain a quotient  $Q_0(x)$  and a remainder  $R_0(x)$  whose degree is less than that of  $c(x)$ ; thus we write

$$(3) \quad N = cQ_0 + R_0.$$

Dividing  $Q_0$  by  $c$ , we get similarly

$$(4) \quad Q_0 = cQ_1 + R_1$$

where the degree of  $R_1$  is less than that of  $c$ . Substituting (4) into (3), we obtain

$$(5) \quad N = c^2Q_1 + cR_1 + R_0.$$

Dividing  $Q_1$  by  $c$ , we get  $Q_1 = cQ_2 + R_2$ , and substituting in (5), we get

$$(6) \quad N = c^3Q_2 + c^2R_2 + cR_1 + R_0.$$

After  $p$  such steps, we have finally

$$(7) \quad N = c^pQ_{p-1} + c^{p-1}R_{p-1} + c^{p-2}R_{p-2} + \cdots + cR_1 + R_0.$$

Dividing both sides by  $D = c^p$ , we have

$$(8) \quad \frac{N}{D} = Q_{p-1} + \frac{R_{p-1}}{c} + \frac{R_{p-2}}{c^2} + \cdots + \frac{R_1}{c^{p-1}} + \frac{R_0}{c^p}.$$

This completes the proof of theorem 2.

*Proof of the theorem of section 171.* From corollary 1, section 106, chapter XV, we see that the denominator  $D(x)$  of any rational expression can be expressed as a product of the form

\* For brevity, we write  $N, a, b, \dots$  instead of  $N(x), a(x), b(x), \dots$  respectively.



$$(9) \quad D(x) = a_0(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_h)^{m_h}(x^2 + u_1x + v_1)^{n_1} \cdots (x^2 + u_kx + v_k)^{n_k}$$

where the discriminants of the quadratic factors are negative. The distinct factors in (9) are all prime or irreducible and hence certainly relatively prime. We may apply theorem 1 with  $a(x) = (x - r_1)^{m_1}$  and  $b(x)$  equal to the product of the remaining factors in (9), obtaining

$$\frac{N}{D} = \frac{A_1(x)}{(x - r_1)^{m_1}} + \frac{B(x)}{b(x)}.$$

Applying theorem 1 similarly to  $B(x)/b(x)$ , we get

$$\frac{N}{D} = \frac{A_1(x)}{(x - r_1)^{m_1}} + \frac{A_2(x)}{(x - r_2)^{m_2}} + \frac{C(x)}{c(x)}.$$

Continuing in this way, we express  $N/D$  as a sum of rational expressions whose denominators are  $(x - r_1)^{m_1}$ ,  $(x - r_2)^{m_2}$ ,  $\cdots$ ,  $(x^2 + u_kx + v_k)^{n_k}$ , respectively. To each of these rational expressions we may apply theorem 2. This completes the proof.



## CHAPTER XXVII

### Infinite Series

**173. Introduction.** Dividing 1 by  $1 - x$  by the usual process,\* we obtain the expression

$$(1) \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots$$

where the dots indicate that the process does not terminate. What can (1) mean? If  $x = \frac{1}{2}$  is substituted in (1), we get

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots$$

or,

$$(2) \quad 2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

Now the right member of (2) does seem to add up to 2 in the sense that (Fig. 63) if you add up enough of its terms you can get an



Fig. 63.

answer as close as you please to 2; for example, the sum of the first four terms is  $1 + \frac{7}{8}$  while the sum of the first five terms is  $1 + \frac{15}{16}$ . However, substituting  $x = 1$  in (1), we get the senseless result

$$(3) \quad \frac{1}{0} = 1 + 1 + 1 + 1 + \cdots + 1 + \cdots;$$

and substituting  $x = 2$  in (1), we get the false result

$$(4) \quad -1 = 1 + 2 + 4 + 8 + \cdots + 2^{n-1} + \cdots$$

Hence it is clear that the right member of (1) does not always have a sense and that the equation (1) is not always correct.

\* The expression (1) can also be obtained by expanding  $(1 - x)^{-1}$  by the binomial theorem.



Expressions like the right members of (1), (2), (3), or (4) are called infinite series. In general, an infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

where the number of terms is infinite. Only series whose terms are real numbers will be considered here. We shall now examine more precisely under what circumstances such an expression has a meaning, and shall indicate some uses for them. Infinite series are among the most powerful tools in advanced mathematics, both pure and applied.

**174. Limit of an infinite sequence.** By an infinite sequence is meant an unending succession of terms  $s_1, s_2, s_3, \dots, s_n, \dots$ . The subscript  $n$  indicates the position of the term  $s_n$  in the sequence. The term  $s_n$  is called the  *$n$ th term* or *general term*. An infinite sequence may be defined by stating its general term.

*Example 1.* The first four terms of the sequence whose  $n$ th term is  $1/n$  are  $1/1, 1/2, 1/3, 1/4, \dots$ .

*Example 2.* The first four terms of the sequence whose  $n$ th term is  $2 - \frac{1}{10^n}$  are  $1.9, 1.99, 1.999, 1.9999, \dots$ .

*Example 3.* The first four terms of the sequence whose  $n$ th term is  $2n$  are  $2, 4, 6, 8, \dots$ .

**DEFINITION.** The number  $S$  will be called the *limit of the sequence*  $s_1, s_2, s_3, \dots, s_n, \dots$  provided that, given any positive number  $h$ , no matter how small, there exists a term  $s_N$  of the sequence such that every succeeding term  $s_m, m > N$ , lies between  $S - h$  and  $S + h$  (that is  $|s_m - S| < h$  for all  $m > N$ ). If there is such a limit  $S$ , we call the sequence convergent, and we write  $\lim_{n \rightarrow \infty} s_n = S$ .

In this case, we say that as  $n$  increases indefinitely,  $s_n$  approaches  $S$  as a limit. A sequence which has no limit is called divergent.

It can be shown that if a sequence has a limit, the value of the limit is uniquely determined.

*Example 4.* The limit of the sequence in example 1 is 0. For, given any positive  $h$ , we have only to take  $N > \frac{1}{h}$ , and clearly  $0 - h < \frac{1}{m} < 0 + h$  for all  $m > N$ . We write  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  (see Fig. 64).

It is intuitively clear that the limit of the sequence in example 2 is 2 (Fig. 64), and that the sequence of example 3 is divergent.



The following theorems are proved in more advanced books. We assume them here.

**THEOREM 1.** If  $\lim_{n \rightarrow \infty} s_n = S$ , and  $\lim_{n \rightarrow \infty} t_n = T$ , then

$$(a) \lim_{n \rightarrow \infty} (s_n + t_n) = S + T$$

$$(b) \lim_{n \rightarrow \infty} (s_n - t_n) = S - T$$

$$(c) \lim_{n \rightarrow \infty} (s_n \cdot t_n) = ST$$

$$(d) \lim_{n \rightarrow \infty} \frac{s_n}{t_n} = \frac{S}{T} \text{ provided } T \neq 0.$$



Fig. 64.

**THEOREM 2.\*** If  $s_n \leq s_{n+1}$  for all  $n = 1, 2, 3, \dots$ , and if there exists a number  $B$  such that  $s_n \leq B$  for all  $n = 1, 2, 3, \dots$ , then the sequence  $s_1, s_2, s_3, \dots, s_n, \dots$  has a limit  $S \leq B$ .

**THEOREM 3.** If  $|r| < 1$ , then  $\lim_{n \rightarrow \infty} r^n = 0$ .

**THEOREM 4.** If  $s_n = c$ , where  $c$  is any constant, for every  $n = 1, 2, 3, \dots$ , then  $\lim_{n \rightarrow \infty} s_n = c$ .

**THEOREM 5.** If  $s_n$  increases beyond all bounds as  $n \rightarrow \infty$ , then  $\lim_{n \rightarrow \infty} \frac{c}{s_n} = 0$ , where  $c$  is any constant.

**Example 5.** Show that  $\lim_{n \rightarrow \infty} \frac{n^2 + 3n + 4}{2n^2 - n - 5} = \frac{1}{2}$ .

**Solution.** Using theorems 1, 4, and 5,

$$\lim_{n \rightarrow \infty} \frac{n^2 + 3n + 4}{2n^2 - n - 5} = \lim_{n \rightarrow \infty} \frac{1 + \frac{3}{n} + \frac{4}{n^2}}{2 - \frac{1}{n} - \frac{5}{n^2}} = \frac{1}{2}.$$

**THEOREM 6.** If  $|s_n| \geq nc$ , where  $c$  is any positive constant, then the sequence  $s_1, s_2, s_3, \dots, s_n, \dots$  diverges.

## EXERCISES

- (a) Write the first four terms of the sequence whose  $n$ th term is given;  
 (b) using the theorems above, find its limit:

1.  $3 + \left(\frac{1}{10}\right)^n$

2.  $2 - \left(\frac{1}{2}\right)^n$

3.  $2 + \left(\frac{-1}{2}\right)^n$

\* This theorem may be taken as one of the fundamental axioms of the system of real numbers.



4.  $\frac{n+1}{2n}$ . (*Hint: divide numerator and denominator by  $n$ .*)
5.  $\frac{n^2+1}{3n^2-n+2}$       6.  $\frac{n}{n+1}$       7.  $\frac{n!n}{(n+1)!}$
8.  $\frac{2n^2-3n+2}{n^2+n+1}$       9.  $\frac{2^n-1}{2^n+1}$       10.  $\frac{n^2-n+2}{2n^2+1}$
11.  $\frac{n^2+2n+1}{2n+3}$       12.  $\frac{3^n-1}{3^n+1}$

**175. Convergence of infinite series.** Consider the infinite series

$$(1) \quad u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

Let  $s_1 = u_1$ ,  $s_2 = u_1 + u_2$ ,  $s_3 = u_1 + u_2 + u_3$ ,  $\cdots$ ,  $s_n = u_1 + u_2 + \cdots + u_n$ ,  $\cdots$ . The numbers  $s_1, s_2, \cdots, s_n, \cdots$  are called the **partial sums** of the series (1).

**DEFINITION.** If the sequence of partial sums has the limit  $S$ , that is, if  $\lim_{n \rightarrow \infty} s_n = S$ , then the series (1) is said to have the sum  $S$  and is said to be **convergent**, or to **converge to  $S$** . A series which is not convergent is called **divergent**.

*Note.* The “sum” of an infinite series is not a sum in the ordinary sense, but is the limit of such sums.

*Example 1.* Consider the series  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$ . Here  $s_1 = \frac{1}{2}$ ,  $s_2 = \frac{3}{4}$ ,  $s_3 = \frac{7}{8}$ ,  $\cdots$ ,  $s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$ . Hence  $\lim_{n \rightarrow \infty} s_n = 1$ . The series converges to the “sum” 1.

**THEOREM 1.** If the series (1) converges, then  $\lim_{n \rightarrow \infty} u_n = 0$ .

*Proof.* Suppose the series (1) converges to  $S$ . Then  $\lim_{n \rightarrow \infty} s_n = S$  and  $\lim_{n \rightarrow \infty} s_{n-1} = S$ . But  $s_n = s_{n-1} + u_n$  or  $u_n = s_n - s_{n-1}$ . Hence  $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = S - S = 0$ .

*Note.* The converse of theorem 1 is false, as will be seen in the next section.

**COROLLARY 1.** If  $\lim_{n \rightarrow \infty} u_n \neq 0$ , then the series (1) does not converge.

*Proof.* An immediate consequence of theorem 1.



**THEOREM 2.** *The geometric series  $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$  converges to  $\frac{a}{1-r}$  if  $|r| < 1$  and diverges if  $|r| \geq 1$ .*

*Proof.* By the formula for geometric progressions, we have  $s_n = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} r^n$ . By the theorems of section 174, if  $|r| < 1$ , we have  $\lim_{n \rightarrow \infty} s_n = \frac{a}{1 - r}$ . By corollary 1, above, the series diverges if  $|r| \geq 1$ , for then  $\lim_{n \rightarrow \infty} ar^{n-1} \neq 0$ .

### EXERCISES

1. Show that the series  $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n} + \cdots$  converges to  $\frac{1}{2}$ .

2. Show that the series  $1 + 0.1 + 0.01 + \cdots + \left(\frac{1}{10}\right)^{n-1} + \cdots$  converges to  $10/9$ .

3. Show that the series  $1 + 2 + 3 + \cdots + n + \cdots$  diverges.

4. Show that the series  $1 - 1 + 1 - \cdots + (-1)^{n-1} + \cdots$  diverges.

5. Show that the series  $1 + \frac{3}{2} + \frac{9}{4} + \cdots + \left(\frac{3}{2}\right)^{n-1} + \cdots$  diverges.

**176. The comparison tests.** Consider two series

$$\begin{aligned} (1) & \quad a_1 + a_2 + a_3 + \cdots + a_n + \cdots, \\ (2) & \quad b_1 + b_2 + b_3 + \cdots + b_n + \cdots, \end{aligned}$$

with non-negative terms.

**THEOREM 1.** *If  $b_n \leq a_n$  for all  $n = 1, 2, 3, \cdots$ , and if the series (1) converges, then the series (2) converges.*

*Proof.* Let  $s_n = a_1 + a_2 + \cdots + a_n$  and  $t_n = b_1 + b_2 + \cdots + b_n$ . Then, by hypothesis,  $\lim_{n \rightarrow \infty} s_n = S$  exists. But since

the terms of (1) and (2) are non-negative, we have  $t_n \leq s_n \leq S$  and  $t_n \leq t_{n+1}$  for all  $n = 1, 2, 3, \cdots$ . Hence by theorem 2, section 174,  $\lim_{n \rightarrow \infty} t_n = T \leq S$  exists.

**THEOREM 2.** *If  $b_n \geq a_n$  for all  $n = 1, 2, 3, \cdots$ , and if the series (1) diverges, then the series (2) diverges.*

*Proof.* If (2) converged, then theorem 1 would imply that (1) converged, contrary to hypothesis.



*Note.* In applying these theorems to test convergence of series, any finite number of terms at the beginning of the series may be omitted, for if the series  $u_{k+1} + u_{k+2} + u_{k+3} + \cdots$  converges to  $U$ , then the series  $u_1 + u_2 + u_3 + \cdots + u_k + u_{k+1} + \cdots$  converges to  $u_1 + u_2 + \cdots + u_k + U$ .

Any series known to be convergent or divergent may be used, according to the above theorems, to test the convergence or divergence of certain other series.

*Example 1.* The series

$$(3) \quad 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{n^n} + \cdots$$

is convergent, for, comparing it with the convergent geometric series  $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots$  with  $r = \frac{1}{2}$ , we find that  $\frac{1}{n^n} \leq \frac{1}{2^{n-1}}$  since  $n^n \geq 2^{n-1}$  for all  $n = 1, 2, 3, \cdots$ . Hence the series (3) is convergent by theorem 1.

In addition to the geometric series, the series

$$(4) \quad 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

known as the  $p$ -series will be found often useful for comparison. The special case of the  $p$ -series with  $p = 1$ , that is, the series

$$(5) \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is known as the harmonic series.

**THEOREM 3.** The  $p$ -series (4) is convergent for  $p > 1$  and divergent for  $p \leq 1$ .

*Proof.* *Case 1.*  $p > 1$ . We write (4) as

$$(6) \quad 1 + \left( \frac{1}{2^p} + \frac{1}{3^p} \right) + \left( \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right) \\ + \left( \frac{1}{8^p} + \cdots + \frac{1}{15^p} \right) + \cdots$$

where there are 2 terms in the first parentheses, 4 terms in the next, 8 terms in the next,  $\cdots$ ,  $2^k$  terms in the  $k$ th parentheses. Now

$$1 \leq 1, \\ \frac{1}{2^p} + \frac{1}{3^p} \leq \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}},$$



$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \leq \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{4^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^2,$$

$$\frac{1}{8^p} + \cdots + \frac{1}{15^p} \leq \frac{1}{8^p} + \cdots + \frac{1}{8^p} = \frac{8}{8^p} = \frac{1}{8^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^3,$$

and so on. The terms on the right form a geometric series

$$(7) \quad 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots + \left(\frac{1}{2^{p-1}}\right)^{n-1} + \cdots$$

with ratio  $r = \frac{1}{2^{p-1}}$ . Since  $p > 1$ ,  $r < 1$  and (7) converges to a sum  $S$ . Because of the above inequalities, the sum  $s_n$  of any number of terms of (4) is  $\leq S$ . Hence  $\lim_{n \rightarrow \infty} s_n$  exists, and (4) converges.

*Case 2.*  $p = 1$ . We write (5) as

$$(8) \quad 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

$$\begin{aligned} \text{Now,} \quad & 1 + \frac{1}{2} > 1 \\ & \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ & \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \\ & \frac{1}{9} + \cdots + \frac{1}{16} > \frac{1}{16} + \cdots + \frac{1}{16} = \frac{1}{2}, \end{aligned}$$

and so on. Hence for any  $n$ ,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}.$$

Since the right member increases beyond any bound as  $n$  increases, (5) diverges.

*Case 3.*  $p < 1$ . Since  $p < 1$ ,  $\frac{1}{n^p} \geq \frac{1}{n}$  because  $n^p \leq n$ . Hence by comparison with the harmonic series (5), (4) diverges in this case.

## EXERCISES

*Prove that the following series converge:*

$$1. \quad \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \cdots + \frac{1}{n \cdot 3^n} + \cdots$$



$$2. \frac{1}{1+2} + \frac{1}{1+2^2} + \frac{1}{1+2^3} + \cdots + \frac{1}{1+2^n} + \cdots$$

$$3. \frac{1}{3 \cdot 1^2} + \frac{1}{3 \cdot 2^2} + \frac{1}{3 \cdot 3^2} + \cdots + \frac{1}{3 \cdot n^2} + \cdots$$

$$4. \frac{1}{1+1^3} + \frac{1}{2+2^3} + \frac{1}{3+3^3} + \cdots + \frac{1}{n+n^3} + \cdots$$

$$5. \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

*Prove that the following series diverge:*

$$6. \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$$

$$7. \frac{1}{1 - \frac{1}{2}} + \frac{1}{2 - \frac{1}{3}} + \frac{1}{3 - \frac{1}{4}} + \cdots + \frac{1}{n - \frac{1}{n+1}} + \cdots$$

$$8. 2 + \frac{3}{2} \cdot \frac{5}{4} + \frac{4}{3} \cdot \left(\frac{5}{4}\right)^2 + \cdots + \frac{n+1}{n} \left(\frac{5}{4}\right)^{n-1} + \cdots$$

*Write the first three terms of the series whose  $n$ th term is given and test for convergence or divergence:*

$$9. \frac{1}{(2n-1)^2}$$

$$10. \frac{1}{n!}$$

$$11. \frac{1}{(2n)^3}$$

$$12. \frac{1}{(2n-1)!}$$

$$13. \frac{n}{n+1}$$

$$14. \frac{2}{\sqrt[3]{n}}$$

$$15. \frac{1}{2n\sqrt{n}}$$

$$16. \frac{1}{\sqrt{n} - \frac{1}{n+1}}$$

$$17. \frac{1}{2^n + n}$$

$$18. \frac{1}{(2n)!}$$

$$19. \frac{n^2 + n}{2n^2 + n + 1}$$

$$20. \frac{\sqrt{n}}{n^2}$$

$$21. \frac{n(n+1)}{(n+2)(n+3)}$$

$$22. \frac{1}{n^2 + 1}$$

$$23. \frac{n+1}{n^2 + 1}$$

$$24. \frac{1}{3^n + 1}$$

**\*177.** The decimal expression for a real number. We shall show that every real number can be expressed as a decimal (see section 14, chapter I).

**THEOREM.** If  $r$  is any real number such that  $0 < r < 1$ , then  $r$  is the sum of an infinite series

$$(1) \quad \frac{k_1}{10} + \frac{k_2}{10^2} + \frac{k_3}{10^3} + \cdots + \frac{k_n}{10^n} + \cdots$$

where  $k_n$  is an integer between 0 and 9 inclusive.

\* May be omitted without disturbing continuity.



*Proof.* Let  $l_1$  be the smallest integer such that  $l_1 \geq 10r$ . Let  $k_1 = l_1 - 1$ . Then

$$\frac{k_1}{10} < r \leq \frac{k_1 + 1}{10}.$$

Let  $l_2$  be the smallest integer such that  $l_2 \geq 10^2r - 10k_1$ . Let  $k_2 = l_2 - 1$ . Then

$$\frac{k_1}{10} + \frac{k_2}{10^2} < r \leq \frac{k_1}{10} + \frac{k_2 + 1}{10^2}.$$

Similarly, determine the integer  $k_n$  such that

$$\frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_n}{10^n} < r \leq \frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_n + 1}{10^n}.$$

The series (1) is convergent, since  $\frac{k_n}{10^n} \leq \frac{9}{10^n}$  and the series

$$\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \cdots$$

is a convergent geometric series with ratio  $\frac{1}{10}$ . Since

$$r - \frac{1}{10^n} \leq \frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_n}{10^n} < r,$$

we have  $\left| \frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_n}{10^n} - r \right| \leq \frac{1}{10^n}$ . But  $\lim_{n \rightarrow \infty} \frac{1}{10^n} = 0$ .

Hence  $r = \lim_{n \rightarrow \infty} \left( \frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_n}{10^n} \right),$

or  $r = \frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_n}{10^n} + \cdots$

The series (1) is the decimal expression of  $r$ , and is usually written as

$$(2) \quad 0.k_1k_2 \cdots k_n \cdots$$

Thus,  $\frac{1}{3} = 0.333 \cdots = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} + \cdots$

**COROLLARY.** Every real number has a decimal expression.

*Proof.* Every real number can be expressed as the sum of an integer plus a number  $r$  such that  $0 \leq r < 1$ .

*Example.*  $\sqrt{2} = 1.4142 \cdots = 1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + \cdots$



*Note.* The decimal expression of a real number is not always unique. Thus

$$\frac{1}{4} = 0.25000 \cdots = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \cdots + \frac{0}{10^n} + \cdots$$

and

$$\frac{1}{4} = 0.24999 \cdots = \frac{2}{10} + \frac{4}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^n} + \cdots$$

It can be shown that it is unique unless there is a term after which the digits are all zeros or all nines.

178. The ratio tests. Let

$$(1) \quad u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots$$

be a series of positive terms. We shall consider only series for which the ratio  $u_{n+1}/u_n$  approaches a limit as  $n \rightarrow \infty$ . Let

$$(2) \quad R = \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n}.$$

**THEOREM.** (a) If  $R < 1$ , the series (1) converges; (b) if  $R > 1$ , the series (1) diverges; (c) if  $R = 1$ , the test fails to tell us whether or not the series (1) converges.

*Proof of (a).* Consider any number  $r$  such that  $R < r < 1$ . From (2) it follows that there exists a number  $N$  such that, for all  $n \geq N$ , we have  $u_{n+1}/u_n < r$ . That is,

$$\begin{aligned} u_{N+1}/u_N &< r, \quad \text{or } u_{N+1} < u_N r, \\ u_{N+2}/u_{N+1} &< r, \quad \text{or } u_{N+2} < u_{N+1} r < u_N r^2, \\ u_{N+3}/u_{N+2} &< r, \quad \text{or } u_{N+3} < u_{N+2} r < u_N r^3, \end{aligned}$$

and so on. Hence the terms of the series

$$(3) \quad u_{N+1} + u_{N+2} + u_{N+3} + \cdots$$

are respectively less than the corresponding terms of the series

$$(4) \quad u_N r + u_N r^2 + u_N r^3 + \cdots$$

But (4) is a convergent geometric series with  $r < 1$ . Hence (3) converges by the comparison test, and (1) converges by the note in section 176.

*Proof of (b).* Consider any number  $r$  such that  $1 < r < R$ . By (2), there exists a number  $N$  such that for all  $n \geq N$  we have  $u_{n+1}/u_n > r > 1$ . Hence  $u_{n+1} > u_n$  for all  $n \geq N$  and we cannot have  $\lim_{n \rightarrow \infty} u_n = 0$ . Hence by corollary 1, section 175, the series diverges.



*Proof of (c).* Consider the divergent harmonic series  $1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots$ . Here  $R = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ . Consider the convergent  $p$ -series with  $p = 2$ ,  $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$ . Here also,  $R = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 2n + 1} = 1$ . This proves (c).

*Example 1.* The series  $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$  converges since  $R = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1$ .



## EXERCISES

By means of the ratio tests, decide the convergence or divergence of each of the following series, applying other tests only if the ratio tests fail:

1.  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$
2.  $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$
3.  $3 + \frac{9}{2} + \frac{27}{3} + \cdots + \frac{3^n}{n} + \cdots$
4.  $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$
5.  $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots$
6.  $\frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{4} + \frac{3 \cdot 4}{8} + \cdots + \frac{n(n+1)}{2^n} + \cdots$
7.  $1 + \frac{1}{3!} + \frac{1}{5!} + \cdots + \frac{1}{(2n-1)!} + \cdots$
8.  $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \cdots + \frac{n!}{2^n} + \cdots$
9.  $\frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \cdots + \frac{1 \cdot 3 \cdots (2n-1)}{3 \cdot 6 \cdots (3n)} + \cdots$
10.  $3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots + \frac{3^n}{n!} + \cdots$
11.  $2 + \frac{2^2}{2^2} + \frac{2^3}{3^2} + \cdots + \frac{2^n}{n^2} + \cdots$
12.  $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots + \frac{1}{(2n)!} + \cdots$



$$13. \frac{1}{2 \cdot 1} + \frac{1}{2 \cdot 4} + \frac{1}{2 \cdot 9} + \cdots + \frac{1}{2n^2} + \cdots$$

$$14. \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(2n-1) \cdot 2n} + \cdots$$

179. Series whose terms are all negative. Such a series obviously converges to  $-S$  if and only if the series of positive terms formed by reversing all the signs converges to  $S$ .

180. Alternating series. By an alternating series is meant a series whose successive terms alternate in sign.

**THEOREM.** *The alternating series\**

$$(1) \quad u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n + \cdots$$

converges if (a)  $\lim_{n \rightarrow \infty} u_n = 0$ , and (b)  $u_{n+1} < u_n$  for all  $n = 1, 2, 3, \dots$ .

*Proof.* If  $2m$  is any even positive integer, then the sum  $S_{2m}$  of the first  $2m$  terms of (1) may be written in two ways:

$$(2) \quad S_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m});$$

$$(3) \quad S_{2m} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - u_{2m}.$$

By hypothesis (b), each quantity in parentheses in (2) and (3) is positive. Therefore, (2) implies that  $S_{2m}$  increases as  $m$  increases; and (3) implies that  $S_{2m} < u_1$  for all  $m$ . By theorem 2, section 174,  $S_{2m}$  approaches a limit  $S$  as  $m$  increases indefinitely.

Let  $S_{2m+1}$  be the sum of an odd number of terms of (1). Then  $S_{2m+1} = S_{2m} + u_{2m+1}$ . Therefore

$$\lim_{m \rightarrow \infty} S_{2m+1} = \lim_{m \rightarrow \infty} S_{2m} + \lim_{m \rightarrow \infty} u_{2m+1} = S + 0 = S$$

by the first paragraph of the proof, and hypothesis (a).

Since the partial sums  $S_n$  approach the limit  $S$  whether  $n$  is even or odd, we have  $\lim_{n \rightarrow \infty} S_n = S$ . This completes the proof.

**COROLLARY.** *Under the hypotheses of the theorem, the difference between a partial sum  $S_n$  and the sum  $S$  of the series is not greater than the next term  $u_{n+1}$  in absolute value. That is,  $|S - S_n| \leq u_{n+1}$ .*

*Proof.* By arguments like those above, we have

$$|S - S_n| = u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5}) - \cdots \leq u_{n+1}.$$

*Example.* The series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \frac{1}{n}$

\* The terms  $u_1, u_2, u_3, \dots$  are all positive.



$+ \dots$  converges since (a)  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  and (b)  $\frac{1}{n+1} < \frac{1}{n}$  for all  $n = 1, 2, 3, \dots$ .

**181. Absolute convergence.** A series

$$(1) \quad u_1 + u_2 + u_3 + \dots + u_n + \dots$$

with some positive and some negative terms is said to **converge absolutely** if the series

$$(2) \quad |u_1| + |u_2| + |u_3| + \dots + |u_n| + \dots$$

of the absolute values of its terms converges.

**THEOREM.** *If a series converges absolutely, it converges. That is, if (2) converges, so does (1).*

*Proof.* Let  $S_n = u_1 + u_2 + \dots + u_n$ . Let  $P_n$  be the sum of the positive terms in the right member and let  $N_n$  be the sum of the absolute values of the negative terms in the right member. Then

$$(3) \quad S_n = P_n - N_n.$$

Let  $s_n = |u_1| + |u_2| + \dots + |u_n|$ . Then

$$(4) \quad s_n = P_n + N_n.$$

Since (2) converges to a sum  $s$ , we have  $s_n = P_n + N_n \leq s$ . Therefore,  $P_n \leq s$  and  $N_n \leq s$ . By theorem 2, section 174,  $P_n$  approaches a limit  $P$  and  $N_n$  approaches a limit  $N$  as  $n$  increases indefinitely. Therefore

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (P_n - N_n) = P - N.$$

This completes the proof.

A series which converges but does not converge absolutely is called **conditionally convergent**.

*Example.* The series of the example of section 180 is conditionally convergent since the absolute values of its terms form the divergent harmonic series.

**182. The general ratio tests.** Let  $u_1 + u_2 + u_3 + \dots + u_n + \dots$  be any series of real terms, and let

$$R = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right|.$$

We consider only series for which this limit exists.

**THEOREM.** (a) *If  $R < 1$ , the series converges absolutely.* (b) *If  $R > 1$ , the series diverges.* (c) *If  $R = 1$ , the test fails to decide whether or not the series converges.*



*Proof of (a).* From  $R < 1$  it follows that the series  $|u_1| + |u_2| + \cdots + |u_n| + \cdots$  converges. Hence the given series converges absolutely.

The proofs of (b) and (c) are like those of the theorem of section 178, and are left to the reader.

## EXERCISES

*Test for convergence and, if convergent, state whether absolutely or conditionally convergent:*

$$1. \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots + \frac{(-1)^{n+1}}{2^n} + \cdots$$

$$2. 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!} + \cdots$$

$$3. 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{2n-1} + \cdots$$

$$4. \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \cdots + \frac{(-1)^{n+1}}{n(n+1)} + \cdots$$

$$5. 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots + \frac{(-1)^{n+1}}{\sqrt{n}} + \cdots$$

$$6. 1 - \frac{1}{3!} + \frac{1}{5!} - \cdots + \frac{(-1)^{n+1}}{(2n-1)!} + \cdots$$

$$7. \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \cdots + \frac{(-1)^{n+1}n}{n+1} + \cdots$$

$$8. 1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^{n+1}}{(2n-2)!} + \cdots$$

$$9. \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{(-1)^{n+1}}{1+n} + \cdots$$

$$10. 1 - \frac{3}{2!} + \frac{3^2}{3!} - \cdots + \frac{(-1)^{n+1}3^{n-1}}{n!} + \cdots$$

**183. Power series.** If  $a_0, a_1, a_2, \dots, a_n, \dots$  are constants, then

$$(1) \quad a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + \cdots,$$

and

$$(2) \quad a_0 + a_1(x-h) + a_2(x-h)^2 + \cdots + a_{n-1}(x-h)^{n-1} + \cdots$$

where  $h$  is a constant, are called **power series in  $x$**  or in  $x-h$  respectively. The general ratio test may be used to determine values of  $x$  for which a power series converges.



*Example.* Find all (real) values of  $x$  for which the series  $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots$  converges.

*Solution.* Since

$$R = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} |x| \frac{n}{n+1} = |x| \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = |x|,$$

we see that the series converges absolutely when  $|x| < 1$ , or  $-1 < x < 1$ , and diverges when  $|x| > 1$ , or  $x > 1$  and  $x < -1$ . The ratio test fails when  $|x| = 1$ . Hence we must investigate the convergence of the series when  $x = \pm 1$  separately. For  $x = 1$ , we find the series  $1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$  which we know converges conditionally (see section 180). For  $x = -1$ , we have the series  $-1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} - \cdots$  which is the negative of the divergent harmonic series and hence diverges (see section 179). Hence the series converges for  $-1 < x \leq 1$ , and diverges for all other values.

## EXERCISES

Find all (real) values of  $x$  for which the series converges:

1.  $1 + x + x^2 + \cdots + x^{n-1} + \cdots$
2.  $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!} + \cdots$
3.  $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \cdots$
4.  $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots$
5.  $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n+1}x^{2n-1}}{2n-1} + \cdots$
6.  $1 - \frac{x}{2} + \frac{x^2}{2^2} - \cdots + \frac{(-1)^{n+1}x^{n-1}}{2^{n-1}} + \cdots$
7.  $2x + 2^2x^2 + 2^3x^3 + \cdots + 2^nx^n + \cdots$
8.  $\frac{x}{1 \cdot 2} - \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} - \cdots + \frac{(-1)^{n+1}x^n}{n(n+1)} + \cdots$
9.  $x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \cdots + \frac{x^n}{\sqrt{n}} + \cdots$
10.  $x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$



$$11. (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots + \frac{(-1)^{n+1}(x-1)^n}{n} + \dots$$

$$12. \frac{x-2}{1} + \frac{(x-2)^2}{2\sqrt{2}} + \frac{(x-2)^3}{3\sqrt{3}} + \dots + \frac{(x-2)^n}{n\sqrt{n}} + \dots$$

13. Show that the binomial series  $1 + kx + \frac{k(k-1)}{2!}x^2 + \dots + \frac{k(k-1)\dots(k-n+1)}{n!}x^n + \dots$ , where  $k$  is any real number, converges for  $-1 < x < 1$  and diverges\* for  $|x| > 1$ .

184. The use of series in computation. Many useful functions can be expressed as power series. For example, if  $x$  is the number of radians† in an angle, it can be shown, by more advanced methods, that

$$(1) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \dots$$

Since the series on the right converges for all  $x$ , and in particular for  $x = 1$ , we may compute the value of the sine of 1 radian by taking the first three terms of this series:

$$\begin{aligned} \sin 1 &= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots \\ &= 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \dots \\ &= 0.8417 \text{ approximately.} \end{aligned}$$

By the corollary of section 180, the error committed is less than  $1/5040$ , since we used only the first three terms. Since  $1/5040$  is less than 0.0005, the answer is correct to the nearest thousandth. See also section 163.

Infinite series are much used in the construction of tables and have many other uses in advanced mathematics, both pure and applied.

## EXERCISES

1. It can be shown that if  $x$  is the number of radians in an angle, then  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!} + \dots$ . Find the value of the cosine of 1 radian to the nearest thousandth.

\* The behavior of this series at  $x = \pm 1$  depends on the value of  $k$ , and requires much more advanced methods of study. A complete investigation of the binomial series when  $k$  is any complex number was made by N. H. Abel (Norwegian, 1802-1829), about 1826. See section 145.

† One radian is equal to  $180/\pi$  degrees.



2. Using the series (1) above, find to the nearest thousandth the value of the sine of  $1/2$  radian.

3. Using the series of exercise 1, find to the nearest thousandth the value of the cosine of  $0.3$  radian.

4. Using the first five terms of the binomial series (exercise 14, section 183), find the value of  $(1.02)^{-10}$ .

5. The value of  $e$ , the base of the system of natural logarithms, can be obtained from the series  $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \cdots$ . Calculate it using the first six terms, and round off the answer to two decimal places.

6. The value of  $\pi$  can be obtained from the series

$$\pi = 16 \left[ \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \cdots \right] \\ - 4 \left[ \frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \cdots \right]$$

Calculate  $\pi$ , using the first three terms in each bracket, and round off the answer to three decimal places.



## CHAPTER XXVIII

### Interpolation and Curve Fitting

**185. Linear interpolation.** Suppose  $y$  is some unknown function of  $x$ : that is,  $y = f(x)$ . Suppose further that we know the values of  $y$  for certain values of  $x$ :

|            |       |       |       |          |       |
|------------|-------|-------|-------|----------|-------|
| $x$        | $x_0$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ |
| $y = f(x)$ | $y_0$ | $y_1$ | $y_2$ | $\cdots$ | $y_n$ |

The graph of  $y = f(x)$  is then drawn through the points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $\dots$ ,  $(x_n, y_n)$ .

The act of estimating the values of  $y$  for values of  $x$  between those actually known or calculated or plotted is called **interpolation**.

A rough method of interpolation is to read from the graph sketched through the known points (Fig. 65).

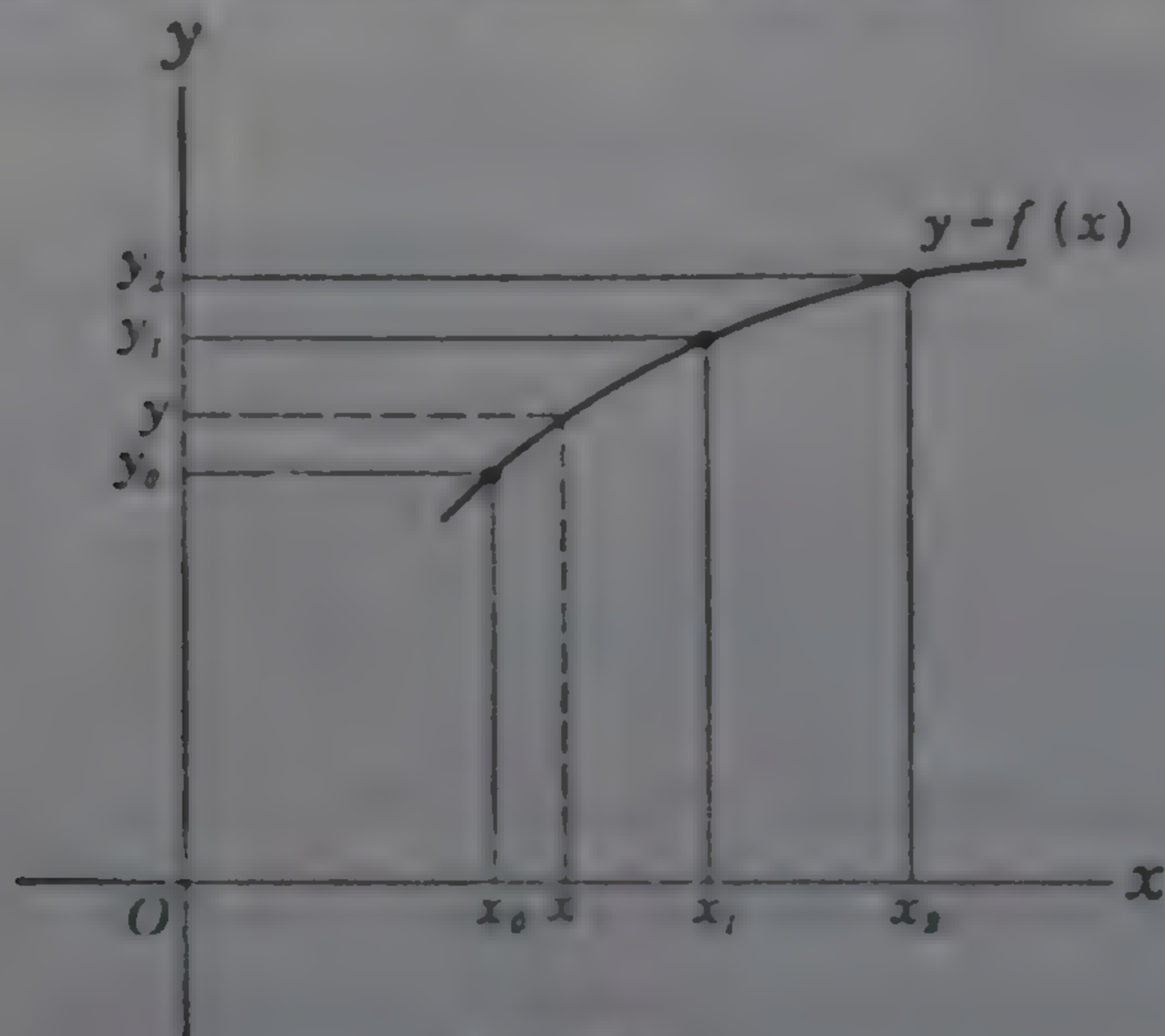


Fig. 65.

The method commonly used in interpolating in tables is called the method of proportional parts or linear interpolation. It is illustrated in the following example. (See section 159 for further illustrations.)

*Example.* Suppose we have the following table:

|     |   |   |   |     |
|-----|---|---|---|-----|
| $x$ | 1 | 2 | 3 | 4   |
| $y$ | 4 | 6 | 2 | -14 |



To find the value of  $y$  corresponding to  $x = 2.75$ , we say that 2.75 is  $3/4$  of the way from 2 to 3, so that the desired value of  $y$  will be approximately  $3/4$  of the way between 6 and 2, or  $y = 3$ . The work may be arranged as follows:

$$0.75 \left[ \begin{array}{c|c} x & y \\ \hline 2 & 6 \\ 1 & 2.75 \\ \hline 3 & 2 \end{array} \right] - 4$$

$$\frac{d}{-4} = \frac{0.75}{1}, d = -3, y = 6 - 3 = 3.$$

This method amounts to replacing the actual curve, whatever it is, by the straight line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$

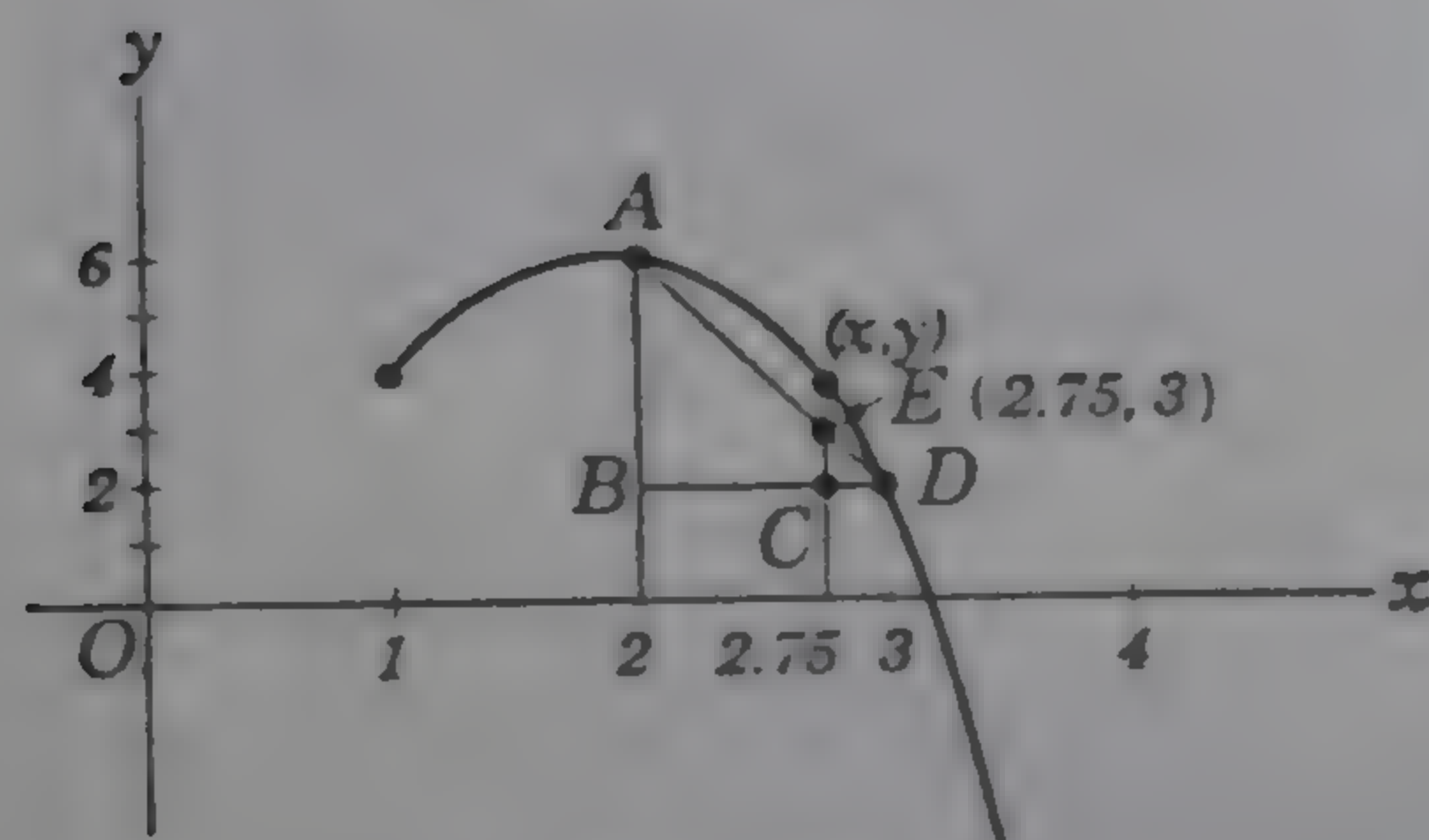


Fig. 66.

between which we are interpolating. In the example above, the straight line joining the points  $(2, 6)$  and  $(3, 2)$  is  $y = -4x + 14$ . When  $x = 2.75$ , the value of  $y$  determined from this line is  $y = 3$ .

The table of the example above was actually calculated from the function  $y = 2 + 3x^2 - x^3$ . The actual value of  $y$  corre-

sponding to  $x = 2.75$  therefore is  $y = 3.890625$ . The connection between the idea of proportional parts and the idea of approximating the curve by the straight line should be clear from the similar triangles  $ABD$  and  $ECD$  (Fig. 66), since  $\frac{CE}{BA} = \frac{CD}{BD} = \frac{1}{4}$ .

The accuracy of this method clearly depends on how closely the chord joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  approximates the actual curve. In practice this often depends on how close together are the values of  $x$  between which we are interpolating;\* interpolating between values of  $x$  which are closer together is apt to be more reliable. If the values  $y_1$  and  $y_2$  are taken from a table, they are themselves not always exact since they may be subject to rounding-off errors.

\* That this is not always the case may be seen from the example  $y = 1/10x$ , by interpolating by proportional parts between the points  $(-0.1, -1)$  and  $(0.5, 0.2)$ , obtaining the point  $(0.2, -0.4)$ , which is far from the true point  $(0.2, 0.2)$ . See M. Richardson, *Fundamentals of Mathematics*, Macmillan, 1941, p. 277.



## EXERCISES

Using the table of the example above, calculate, by means of interpolation by proportional parts, the value of  $y$  corresponding to:

1.  $x = 1.5$ .      2.  $x = 2.25$ .      3.  $x = 3\frac{1}{4}$ .      4.  $x = 1.4$ .

5.  $x = 2.2$ .      6.  $x = 2.7$ .

7. Given that  $\sqrt[3]{2.50} = 1.357$  and  $\sqrt[3]{2.60} = 1.375$ , find by linear interpolation the value of  $\sqrt[3]{2.54}$  to three decimal places.

8. Given that  $\sqrt[3]{4.60} = 1.663$  and  $\sqrt[3]{4.70} = 1.675$ , find by linear interpolation the value of  $\sqrt[3]{4.66}$  to three decimal places.

9. From table I, find by linear interpolation the value of  $\sqrt[3]{3.925}$  to four decimal places.

10. From table I, find by linear interpolation the value of  $\sqrt[3]{66.25}$  to four decimal places.

**186. Polynomial interpolation.** It is obviously plausible that replacing the actual (unknown) curve by a polynomial of degree two is likely to give a better approximation than the linear polynomial used in the preceding section, and that in general a polynomial of higher degree is likely to be better than one of lower degree.

**THEOREM.** If  $x_0, x_1, x_2, \dots, x_r$  are  $r + 1$  different values of  $x$ , and  $y_0, y_1, y_2, \dots, y_r$  are any  $r + 1$  values of  $y$ , then there exists one and only one polynomial  $f(x)$  of degree less than or equal to  $r$  such that the curve  $y = f(x)$  passes through the  $r + 1$  points  $(x_0, y_0), (x_1, y_1), \dots, (x_r, y_r)$ .

*Proof.* This follows at once from (2) of section 187, and corollary 2, section 104, chapter XV.

This unique polynomial may be determined by the method of undetermined coefficients, as in the following example.

**Example 1.** Find the unique polynomial of degree 3 or less whose graph passes through the four points  $(1, 4), (2, 6), (3, 2), (4, -14)$ .

**Solution.** Let  $y = ax^3 + bx^2 + cx + d$ . Since  $x = 1, y = 4$  must satisfy this equation, we have

$$(1) \quad 4 = a + b + c + d,$$

a linear equation in the coefficients  $a, b, c$ , and  $d$ . Each of the other three given points similarly yields a linear equation in  $a, b, c$ , and  $d$ :

$$(2) \quad 6 = 8a + 4b + 2c + d,$$

$$(3) \quad 2 = 27a + 9b + 3c + d,$$

$$(4) \quad -14 = 64a + 16b + 4c + d.$$



Solving this system of equations (1), (2), (3), and (4), we find  $a = -1$ ,  $b = 3$ ,  $c = 0$ ,  $d = 2$ . Hence the desired polynomial is  $y = -x^3 + 3x^2 + 2$ .

## EXERCISES

Use the method of undetermined coefficients in each of the following exercises:

1. (a) Determine the unique polynomial of degree  $\leq 3$  whose graph passes through the points given in the following table:

|     |    |    |    |    |
|-----|----|----|----|----|
| $x$ | 1  | 3  | 4  | 5  |
| $y$ | -2 | 14 | 40 | 86 |

(b) Use the polynomial obtained in (a) to interpolate for the value of  $y$  corresponding to  $x = 2$ .

2. Same questions as exercise 1 for the table

|     |    |    |   |   |
|-----|----|----|---|---|
| $x$ | 1  | 3  | 4 | 5 |
| $y$ | 14 | 12 | 8 | 2 |

3. Same questions as exercise 1 for the table

|     |    |   |   |   |
|-----|----|---|---|---|
| $x$ | 1  | 3 | 4 | 5 |
| $y$ | -1 | 3 | 5 | 7 |

4. (a) Find the unique polynomial of degree  $\leq 2$  whose graph passes through the points (1,3), (2,3), and (4,9).

(b) Use the polynomial obtained in (a) to find the values of  $y$  corresponding to  $x = 3$  and to  $x = 1.5$  respectively.

5. (a) Find the unique polynomial  $f(x)$  of degree  $\leq 3$  such that  $f(1) = 0$ ,  $f(2) = -4$ ,  $f(3) = -6$ , and  $f(5) = -4$ .

(b) Find the values of  $f(4)$  and  $f(2.25)$ .

6. (a) Find the unique polynomial  $f(x)$  of degree  $\leq 3$  whose graph passes through the points given in the following table:

|     |    |   |    |    |
|-----|----|---|----|----|
| $x$ | 1  | 3 | 4  | 5  |
| $y$ | -8 | 2 | 25 | 68 |

(b) Use the polynomial obtained in (a) to find the values of  $f(2)$  and  $f(3.3)$ .

7. (a) Find the unique polynomial of the smallest possible degree whose graph passes through the points given in the following table:

|     |   |    |    |     |     |
|-----|---|----|----|-----|-----|
| $x$ | 1 | 2  | 3  | 4   | 5   |
| $y$ | 9 | 18 | 73 | 234 | 585 |

(b) Use the polynomial obtained in (a) to find the value of  $y$  corresponding to  $x = 2.2$ .



8. (a) Find the unique polynomial of the smallest possible degree whose graph passes through the points given in the following table:

$$\begin{array}{c|c|c|c} x & 2 & 4 & 8 \\ \hline y & 18 & 18 & -8 \end{array}.$$

(b) Using the polynomial obtained in (a), find the values of  $y$  corresponding to  $x = 3$ ,  $x = 5$ ,  $x = 6$ , and  $x = 9$ , respectively.

**\*187. Lagrange's interpolation formula.** The unique polynomial of degree 2 or less passing through the points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , where  $x_0, x_1, x_2$  are all distinct, is easily seen by direct substitution to be

$$(1) \quad y = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

Similarly the unique polynomial of degree  $r$  or less passing through the  $r + 1$  points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $\dots$ ,  $(x_r, y_r)$ , where  $x_0, x_1, \dots, x_r$  are all distinct, is given by

$$(2) \quad y = y_0 \frac{(x - x_1)(x - x_2) \cdots (x - x_r)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_r)} + y_1 \frac{(x - x_0)(x - x_2) \cdots (x - x_r)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_r)} + \cdots + y_r \frac{(x - x_0)(x - x_1) \cdots (x - x_{r-1})}{(x_r - x_0)(x_r - x_1) \cdots (x_r - x_{r-1})}.$$

*Example.* The unique polynomial of degree 3 or less passing through the points  $(1, 4)$ ,  $(2, 6)$ ,  $(3, 2)$ ,  $(4, -14)$  is

$$y = 4 \frac{(x - 2)(x - 3)(x - 4)}{(1 - 2)(1 - 3)(1 - 4)} + 6 \frac{(x - 1)(x - 3)(x - 4)}{(2 - 1)(2 - 3)(2 - 4)} + 2 \frac{(x - 1)(x - 2)(x - 4)}{(3 - 1)(3 - 2)(3 - 4)} - 14 \frac{(x - 1)(x - 2)(x - 3)}{(4 - 1)(4 - 2)(4 - 3)}$$

which reduces to  $y = -x^3 + 3x^2 + 2$ .

*Note.* Formula (2) is credited to J. L. Lagrange (French, 1736-1813), who made many remarkable contributions to the progress of mathematics, both pure and applied.

## EXERCISES

1-8. Work exercises 1-8 of section 186 by means of Lagrange's interpolation formula.

\* This section may be omitted without disturbing the continuity of the chapter.



**188. The Gregory-Newton interpolation formula.** Another method for determining the unique polynomial  $y = f(x)$  of degree  $r$  or less passing through the  $r + 1$  points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $\dots$ ,  $(x_r, y_r)$ , where  $x_0, x_1, \dots, x_r$  are all distinct, is often convenient.

*The student should review sections 150 and 151, chapter XXI, at this point.* From formula (6) of section 150, we have, for  $k = 0, 1, 2, \dots, r$ ,

$$(1) \quad y_k = y_0 + k\Delta y_0 + \frac{k(k-1)}{2!} \Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!} \Delta^3 y_0 \\ + \dots + \frac{k(k-1)(k-2) \dots (k-r+1)}{r!} \Delta^r y_0.$$

By theorem 2, section 151, all differences of order higher than the  $r$ th are equal to zero since  $f(x)$  is a polynomial of degree  $r$  or less.

For simplicity, we now restrict ourselves to the case in which the values  $x_0, x_1, \dots, x_r$  are evenly spaced. Let  $h$  be the constant difference between the successive values  $x_0, x_1, \dots, x_r$ . Thus  $x_1 = x_0 + h$ ,  $x_2 = x_0 + 2h$ ,  $\dots$ ,  $x_r = x_0 + rh$ . That is,

$$(2) \quad x_k = x_0 + kh \quad (k = 0, 1, \dots, r)$$

or

$$(3) \quad k = \frac{x_k - x_0}{h}.$$

Hence

$$(4) \quad \begin{cases} k - 1 = \frac{x_k - x_0 - h}{h} \\ k - 2 = \frac{x_k - x_0 - 2h}{h} \\ \vdots \\ k - r + 1 = \frac{x_k - x_0 - [r - 1]h}{h} \end{cases}$$

Substituting (3) and (4) in (1), we get

$$(5) \quad y_k = y_0 + \frac{x_k - x_0}{h} \Delta y_0 + \frac{\left(\frac{x_k - x_0}{h}\right) \left(\frac{x_k - x_0 - h}{h}\right)}{2!} \Delta^2 y_0 \\ + \frac{\left(\frac{x_k - x_0}{h}\right) \left(\frac{x_k - x_0 - h}{h}\right) \left(\frac{x_k - x_0 - 2h}{h}\right)}{3!} \Delta^3 y_0 + \dots \\ + \frac{\left(\frac{x_k - x_0}{h}\right) \left(\frac{x_k - x_0 - h}{h}\right) \left(\frac{x_k - x_0 - 2h}{h}\right) \dots \left(\frac{x_k - x_0 - [r - 1]h}{h}\right)}{r!} \Delta^r y_0.$$



Replacing  $x_k$  by  $x$  and simplifying, we find that the polynomial

$$(6) \quad y = f(x) = y_0 + \frac{x - x_0}{h} \Delta y_0 + \frac{(x - x_0)(x - x_0 - h)}{2!h^2} \Delta^2 y_0 \\ + \frac{(x - x_0)(x - x_0 - h)(x - x_0 - 2h)}{3!h^3} \Delta^3 y_0 + \cdots \\ + \frac{(x - x_0)(x - x_0 - h)(x - x_0 - 2h) \cdots (x - x_0 - [r - 1]h)}{r!h^r} \Delta^r y_0$$

satisfies the requirements of the problem. It is of degree  $r$  or less, and for  $k = 0, 1, \cdots, r$  we have  $y = f(x_k) = y_k$ .

*Example 1.* Find the unique polynomial of degree 3 or less passing through the points  $(1, 4)$ ,  $(2, 6)$ ,  $(3, 2)$ ,  $(4, -14)$ .

*Solution.* We construct the successive differences of  $y_0, y_1, y_2, y_3$ :

$$\begin{array}{rcl} y_k: & 4, & 6, \quad 2, \quad -14 \\ \Delta y_k: & 2, & -4, \quad -16 \\ \Delta^2 y_k: & -6, & -12 \\ \Delta^3 y_k: & -6. \end{array}$$

Here  $y_0 = 4$ ,  $\Delta y_0 = 2$ ,  $\Delta^2 y_0 = -6$ ,  $\Delta^3 y_0 = -6$ ,  $x_0 = 1$ , and  $h = 1$ . Substituting in (6) we get

$$y = 4 + \frac{x - 1}{1} \cdot 2 + \frac{(x - 1)(x - 2)}{2!1^2} (-6) \\ + \frac{(x - 1)(x - 2)(x - 3)}{3!1^3} (-6),$$

which reduces to  $y = -x^3 + 3x^2 + 2$ .

The formula (6) is often called Newton's formula, after Isaac Newton (1642–1727) although it was discovered by James Gregory (1638–1675).

*Example 2.* By the Gregory-Newton interpolation formula find an approximate value of  $\sqrt[3]{28.8}$  given that  $\sqrt[3]{27.0} = 3.000$ ,  $\sqrt[3]{28.0} = 3.037$ ,  $\sqrt[3]{29.0} = 3.072$ ,  $\sqrt[3]{30.0} = 3.107$ , approximately.

*Solution.* We form the table of differences:

$$\begin{array}{rcll} y_k: & 3.000 & 3.037 & 3.072 & 3.107 \\ \Delta y_k: & 0.037 & 0.035 & 0.035 & \\ \Delta^2 y_k: & -0.002 & 0.000 & & \\ \Delta^3 y_k: & 0.002. & & & \end{array}$$

Here  $h = 1$ ,  $x_0 = 27$ ,  $y_0 = 3$ ,  $\Delta y_0 = 0.037$ ,  $\Delta^2 y_0 = -0.002$ ,  $\Delta^3 y_0 = 0.002$ , and  $x = 28.8$ . From formula (6) we have



$$\begin{aligned}
 (1) \quad y &= 3 + \frac{1.8}{1} (0.037) + \frac{(1.8)(0.8)}{2!1^2} (-0.002) \\
 &\quad + \frac{(1.8)(0.8)(-0.2)}{3!1^3} (0.002) \\
 &= 3 + 0.0666 - 0.00144 - 0.000096 = 3.065064
 \end{aligned}$$

approximately. This answer is correct to the nearest thousandth. We have essentially approximated part of the curve  $y = \sqrt[3]{x}$  by a cubic polynomial. If we had used linear interpolation between the values 3.037 and 3.072 we would have obtained 3.065. Thus in this example we have not gained in accuracy by increasing the degree of the approximating curve. This might have been foreseen from the fact that the first differences were nearly equal, suggesting that a first-degree polynomial would be almost a perfect fit. In general, however, one may expect better accuracy from higher degree curves. The problem of estimating the accuracy to be expected lies beyond the scope of this book.

## EXERCISES

Use the Gregory-Newton interpolation formula in each of the following exercises:

1. (a) Find the unique polynomial of lowest degree whose graph passes through the points given in the following table:

|     |    |   |    |    |    |
|-----|----|---|----|----|----|
| $x$ | 2  | 3 | 4  | 5  | 6  |
| $y$ | -3 | 4 | 15 | 30 | 49 |

(b) Use the polynomial obtained in (a) to find the values of  $y$  corresponding to  $x = 4.2$ , and to  $x = 7$ .

2. (a) Find the unique polynomial of lowest degree whose graph passes through the points given in the following table:

|     |     |      |      |      |
|-----|-----|------|------|------|
| $x$ | 2.0 | 2.4  | 2.8  | 3.2  |
| $y$ | 8.0 | 10.4 | 12.8 | 15.2 |

(b) Use the polynomial obtained in (a) to find the values of  $y$  corresponding to  $x = 2.7$  and  $x = 4.0$ .

3. (a) Find the unique polynomial of lowest degree whose graph passes through the points given in the table:

|     |   |   |    |     |     |      |
|-----|---|---|----|-----|-----|------|
| $x$ | 1 | 2 | 3  | 4   | 5   | 6    |
| $y$ | 2 | 7 | 56 | 209 | 550 | 1187 |

(b) Use the polynomial obtained in (a) to find the value of  $y$  corresponding to  $x = 2.2$ .



4. (a) Find the unique polynomial of lowest degree whose graph passes through the points given in the table:

|     |   |    |     |      |
|-----|---|----|-----|------|
| $x$ | 1 | 3  | 5   | 7    |
| $y$ | 6 | -4 | -70 | -240 |

(b) Use the polynomial obtained in (a) to find the values of  $y$  corresponding to  $x = 4$  and  $x = 0.5$ .

5. (a) Find the unique polynomial of lowest degree whose graph passes through the points given in the table:

|     |   |    |    |     |     |
|-----|---|----|----|-----|-----|
| $x$ | 1 | 3  | 5  | 7   | 9   |
| $y$ | 6 | -4 | 18 | 120 | 350 |

(b) Use the polynomial obtained in (a) to find the values of  $y$  corresponding to  $x = 4$ , and  $x = 10$ .

6. Find the unique polynomial  $f(x)$  of lowest degree such that  $f(1) = 9$ ,  $f(2.5) = 10.5$ ,  $f(4) = 3$ ,  $f(5.5) = -13.5$ .

7. Find the unique polynomial  $f(x)$  of lowest degree such that  $f(0.8) = 0.788$ ,  $f(1.2) = 0.508$ ,  $f(1.6) = 0.612$ ,  $f(2) = 1.1$ , and use it to find  $f(1)$  and  $f(1.4)$ .

8. Work exercise 7 of section 186 by means of the Gregory-Newton formula.

9. Given that  $22^3 = 10648.0$ ,  $23^3 = 12167.0$ ,  $24^3 = 13824.0$ , approximate  $23.3^3$  by the method of example 2, above.

10. Given that  $\sqrt[3]{670} = 8.750$ ,  $\sqrt[3]{680} = 8.794$ ,  $\sqrt[3]{690} = 8.837$ , approximate  $\sqrt[3]{686}$  by the method of example 2, above.

**189. Curve fitting and empirical equations.** Suppose numerous observations are tabulated which relate two variables  $x$  and  $y$ :

|     |       |       |       |         |       |
|-----|-------|-------|-------|---------|-------|
| $x$ | $x_0$ | $x_1$ | $x_2$ | $\dots$ | $x_m$ |
| $y$ | $y_0$ | $y_1$ | $y_2$ | $\dots$ | $y_m$ |

Plotting the  $m + 1$  points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $\dots$ ,  $(x_m, y_m)$ , we obtain a so-called scatter diagram (Fig. 67). To be sure, if the values  $x_0, x_1, \dots, x_m$  are all different, we can pass a curve  $y = f(x)$ , where  $f(x)$  is a polynomial of degree not greater than  $m$ , through the  $m + 1$  points *exactly*. But if our purpose is practical, we desire the simplest possible equation  $y = f(x)$ . Furthermore, in practice, the coordinates of the  $m + 1$  given points are obtained by some sort of measurement and may therefore be assumed to be somewhat in error.\* Hence we would prefer a simple equation which *nearly* goes through the given points to a complicated equation which goes through them *exactly*.

\* It is actually assumed that any errors in the  $x$ 's are negligible, while the  $y$ 's are subject to error. This is a situation often attained in experimental practice, and the subtleties involved may well be passed over here.



The pattern of points on the scatter diagram may suggest the form of a simple equation, such as (a) a linear function  $y = ax + b$  or straight line, (b) a quadratic function  $y = ax^2 + bx + c$  or parabola, (c) direct variation as some power, as  $y = cx^n$ , (d) a polynomial of some definite degree, or (e) an exponential curve  $y = cb^{ax}$ . Of course, there may be doubt as to the appropriate form (see Fig. 67), and, of course, other forms besides those treated here are possible.\*

If the form of the equation is decided upon, either for theoretical reasons or by intelligent guesswork, the next step is to determine the values of the coefficients in the equation so as to fit the given

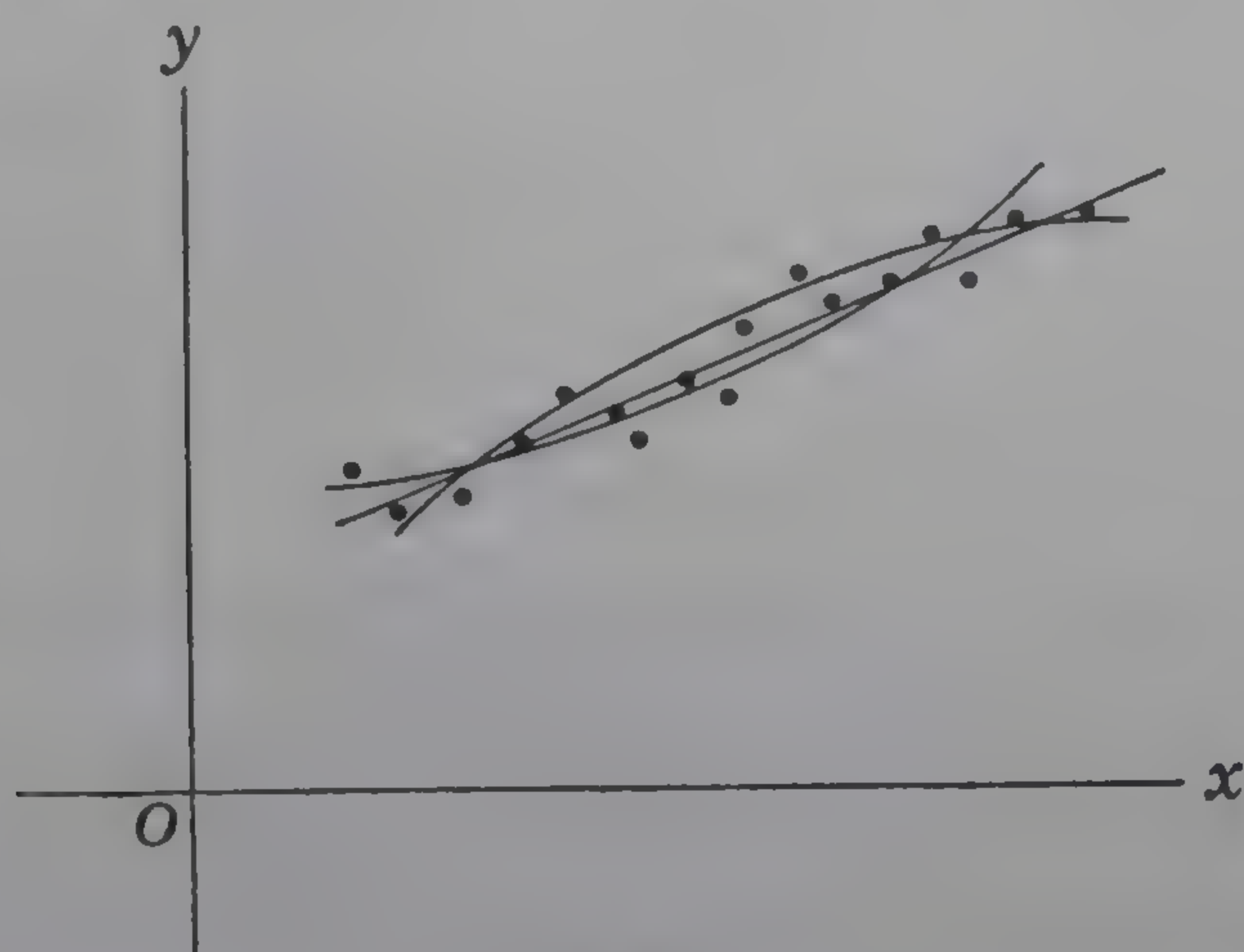


Fig. 67.

points well. Methods for doing this will be taken up later. If a straight line seems appropriate, a crude but quick method is to draw a line with a ruler so that it passes close to most of the points, and find approximate values of  $a$  and  $b$  from the graph itself. The process of forming an equation to fit empirical data approximately is called **curve fitting** and the resulting equation is often called an **empirical equation**. In practice the resulting equation may be used for interpolating, or for suggesting laws connecting the variables, or for prediction of further corresponding values of  $x$  and  $y$ , or for the determination of the values of unknown constants in laws whose form is theoretically known.

**190. The method of differences.** A convenient method of curve fitting uses differences and the Gregory-Newton formula.

\* For further study along these lines, see Whittaker and Robinson, *Calculus of Observations*, Blackie and Sons, 1932; Scarborough, *Numerical Mathematical Analysis*, Johns Hopkins Press, 1930; and textbooks on mathematical statistics.



Suppose the given points  $(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m)$  are such that the values of  $x_0, x_1, \dots, x_m$  are equally spaced.\* If these points lay exactly on the graph of an equation  $y = f(x)$  where  $f(x)$  is a polynomial of degree  $r$ , then the differences of  $r$ th order,  $\Delta^r y_0, \Delta^r y_1, \Delta^r y_2, \dots$ , would be exactly equal, and conversely (see theorems 1 and 2, section 151, chapter XXI). Therefore it is plausible that if the differences of  $r$ th order,  $\Delta^r y_0, \Delta^r y_1, \Delta^r y_2, \dots$ , are nearly equal, then the given points will nearly lie on the graph of an equation  $y = f(x)$  where  $f(x)$  is a polynomial of degree  $r$ . Hence construction of a table of differences will reveal whether the given points may be approximately fitted by a curve  $y = f(x)$  where  $f(x)$  is a polynomial of degree  $r$ . In fact, the Gregory-Newton formula may be used to determine the desired polynomial, as in the following examples.

*Example 1.* Fit a curve to the points given by the table:

|     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $x$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
| $y$ | 2.9 | 3.5 | 4.1 | 4.4 | 5.0 | 5.6 | 5.9 |

*Solution.* We obtain the table of differences:

|                |     |     |     |     |     |      |     |
|----------------|-----|-----|-----|-----|-----|------|-----|
| $y_k$ :        | 2.9 | 3.5 | 4.1 | 4.4 | 5.0 | 5.6  | 5.9 |
| $\Delta y_k$ : | 0.6 | 0.6 | 0.3 | 0.6 | 0.6 | 0.3. |     |

The first differences are nearly equal. This suggests approximating the given points by a straight line or linear function. Plotting the seven given points on a graph bears out this conclusion. Hence we use the first two terms of formula (6), section 188,

$$y = y_0 + \frac{x - x_0}{h} \Delta y_0$$

with  $y_0 = 2.9, x_0 = 1, h = 1, \Delta y_0 = 0.6$ . We obtain

$$y = 2.9 + (x - 1)(0.6)$$

or

$$(1) \quad y = 0.6x + 2.3.$$

The student should plot this line and the given points on a graph.

*Example 2.* A projectile is thrown vertically upward from an initial height of 5 feet. Its height  $y$ , in feet, at the end of  $x$  seconds is observed for  $x = 0, 1, \dots, 5$ , as in the following table:

|     |   |     |     |     |     |     |
|-----|---|-----|-----|-----|-----|-----|
| $x$ | 0 | 1   | 2   | 3   | 4   | 5   |
| $y$ | 5 | 115 | 193 | 240 | 255 | 238 |

\* Modifications of the Gregory-Newton formula which do not require this restriction can be found, for example, in Whittaker and Robinson, *Calculus of Observations*, Blackie and Sons, 1932.



Find an approximate formula expressing  $y$  in terms of  $x$ .

*Solution.* We obtain the difference table:

|               |     |     |     |     |     |     |
|---------------|-----|-----|-----|-----|-----|-----|
| $y:$          | 5   | 115 | 193 | 240 | 255 | 238 |
| $\Delta y:$   | 110 | 78  | 47  | 15  | -17 |     |
| $\Delta^2 y:$ | -32 | -31 | -32 | -32 |     |     |

Since the differences of second order are nearly equal, we expect a quadratic polynomial to be a good fit. Hence we use the first three terms of formula (6), section 188,

$$y = y_0 + \frac{x - x_0}{h} \Delta y_0 + \frac{(x - x_0)(x - x_0 - h)}{2!h^2} \Delta^2 y_0$$

with  $y_0 = 5$ ,  $h = 1$ ,  $x_0 = 0$ ,  $\Delta y_0 = 110$ ,  $\Delta^2 y_0 = -32$ , obtaining

$$y = 5 + 110x + \frac{x(x-1)}{2} (-32), \text{ or}$$

$$(2) \quad y = 5 + 126x - 16x^2.$$

## EXERCISES

Use the method of differences to verify the appropriateness of the suggested degree of a polynomial fitting the tabulated data well, and to find such a polynomial; plot the graph of the calculated polynomial and the given points:

1. Fit a linear polynomial to the following data:

|     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|
| $x$ | 1   | 2   | 3   | 4   | 5   | 6   |
| $y$ | 1.5 | 2.0 | 2.4 | 2.9 | 3.4 | 4.0 |

2. Fit a quadratic polynomial to the following data:

|     |     |     |     |      |      |      |
|-----|-----|-----|-----|------|------|------|
| $x$ | 1   | 2   | 3   | 4    | 5    | 6    |
| $y$ | 1.0 | 2.0 | 5.1 | 10.1 | 17.1 | 26.2 |

3. Fit a quadratic polynomial to the following data:

|     |     |     |      |      |      |       |
|-----|-----|-----|------|------|------|-------|
| $x$ | 1   | 3   | 5    | 7    | 9    | 11    |
| $y$ | 4.0 | 8.1 | 19.8 | 40.2 | 67.7 | 104.0 |

4. Fit a quadratic polynomial to the following data:

|     |      |      |      |      |      |      |
|-----|------|------|------|------|------|------|
| $x$ | 1    | 2    | 3    | 4    | 5    | 6    |
| $y$ | 21.1 | 38.0 | 52.9 | 66.0 | 77.1 | 86.2 |

5. A thin wire is stretched by means of weights. Its length  $y$ , in inches, is observed for various weights  $x$ , in pounds, as in the following table:

|     |       |                |       |                |       |                |       |
|-----|-------|----------------|-------|----------------|-------|----------------|-------|
| $x$ | 1     | $1\frac{1}{2}$ | 2     | $2\frac{1}{2}$ | 3     | $3\frac{1}{2}$ | 4     |
| $y$ | 36.00 | 36.06          | 36.13 | 36.18          | 36.25 | 36.30          | 36.36 |



- (a) Fit a straight line to the above data.  
 (b) What would the predicted length be for  $x = 5$  pounds?  
 (c) Would it be safe to use the formula found in (a) to predict the length of the wire for  $x = 1000$  pounds?

6. The cost  $y$  cents per item of manufacturing a certain article varies with the number  $x$  of hundreds produced as in the following table:

|     |      |      |      |      |
|-----|------|------|------|------|
| $x$ | 1    | 3    | 5    | 7    |
| $y$ | 50.0 | 48.9 | 47.9 | 47.0 |

- (a) Fit a linear polynomial  $y = ax + b$  to the above data.  
 (b) Use the linear polynomial calculated in (a) to predict the value of  $y$  corresponding to  $x = 10$ .  
 (c) Would it be safe to use the polynomial obtained in (a) to predict the value of  $y$  corresponding to  $x = 150$ ?

7. The height  $y$ , in feet, of a projectile fired vertically upward at the end of  $t$  seconds is given by the following table:

|     |    |     |     |     |     |     |
|-----|----|-----|-----|-----|-----|-----|
| $t$ | 0  | 2   | 4   | 6   | 8   | 10  |
| $y$ | 10 | 350 | 560 | 630 | 590 | 410 |

- (a) Fit a quadratic polynomial  $y = a + bt + ct^2$  to the above data.  
 (b) Use the polynomial calculated in (a) to predict the values of  $y$  corresponding to  $t = 3$  and  $t = 11$  respectively.

8. The relation between the number  $x$  of milligrams of maltose and its reducing effect on Fehling's solution as measured by the number  $y$  of milliliters of 0.1N sodium thiosulfate solution is given by the following table:\*

|     |      |      |      |      |      |       |       |       |
|-----|------|------|------|------|------|-------|-------|-------|
| $x$ | 10   | 20   | 30   | 40   | 50   | 60    | 70    | 80    |
| $y$ | 1.89 | 3.63 | 5.33 | 6.95 | 8.90 | 10.48 | 12.25 | 13.80 |

- (a) Fit a quadratic polynomial to the above data.  
 (b) Use the formula found in (a) to predict the value of  $y$  corresponding to  $x = 100$ .

9. The lift coefficient  $C_L$  of a certain airfoil section is related to the number  $\alpha$  of degrees in the angle of attack (or angle of incidence) by the following table:

|          |       |       |       |       |
|----------|-------|-------|-------|-------|
| $\alpha$ | 0     | 2     | 4     | 6     |
| $C_L$    | 0.385 | 0.531 | 0.677 | 0.823 |

- (a) Fit a linear polynomial  $C_L = A\alpha + B$  to the above data.  
 (b) Use the polynomial calculated in (a) to predict the value of  $C_L$  corresponding to  $\alpha = 8^\circ$ .  
 (c) Would it be safe to use the polynomial obtained in (a) to predict the value of  $C_L$  corresponding to  $\alpha = 80^\circ$ ?

\* The author is indebted to Professor L. Sattler of the Department of Chemistry of Brooklyn College for supplying the data for this problem and for exercise 10. section 191.



10. Fit a cubic polynomial to the following data:

|     |     |     |      |      |       |       |
|-----|-----|-----|------|------|-------|-------|
| $x$ | 1   | 2   | 3    | 4    | 5     | 6     |
| $y$ | 1.1 | 7.2 | 24.9 | 60.8 | 121.1 | 211.8 |

191. The method of averages. We shall explain this method of curve fitting by means of examples.

*Example 1.* As in example 1, section 190, we wish to fit a curve to the points

|     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $x$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
| $y$ | 2.9 | 3.5 | 4.1 | 4.4 | 5.0 | 5.6 | 5.9 |

*Solution.* (a) By plotting the points (or else by examining a table of differences), we decide that a linear polynomial will suffice.

(b) Hence we must determine  $a$  and  $b$  so that

$$(1) \quad y = ax + b$$

will fit the given points well. Substituting the given values of  $x$  and  $y$  in (1), we get seven equations of first degree in  $a$  and  $b$ :

$$(2) \quad \begin{cases} 2.9 = a + b \\ 3.5 = 2a + b \\ 4.1 = 3a + b \\ 4.4 = 4a + b \\ 5.0 = 5a + b \\ 5.6 = 6a + b \\ 5.9 = 7a + b. \end{cases}$$

In general, a system of more than two linear equations in two unknowns is inconsistent,\* and we can hope to satisfy it at best only approximately, which is all we expect. Since there are two constants,  $a$  and  $b$ , to be determined, we divide the seven equations into two nearly equal groups, say the first four and the last three. Adding the equations in each group we get

$$(3) \quad \begin{cases} 14.9 = 10a + 4b \\ 16.5 = 18a + 3b, \end{cases}$$

the first equation in (3) being the sum of the first four equations of (2), and the second equation of (3) being the sum of the last three equations of (2). Solving the system (3) for  $a$  and  $b$  and rounding off the solutions to the nearest tenth (since the data are given to the nearest tenth), we find  $a = 0.5$  and  $b = 2.5$ , approximately. Hence

$$(4) \quad y = 0.5x + 2.5$$

\* See section 129.



is an approximate fit. See Fig. 68. This answer differs only slightly from the solution obtained in section 190.

*Example 2.* A projectile is thrown vertically upward from an initial height of 5 feet. Its height  $y$ , in feet, at the end of  $x$  seconds is observed for  $x = 0, 1, \dots, 5$  as in the following table:

|     |   |     |     |     |     |     |
|-----|---|-----|-----|-----|-----|-----|
| $x$ | 0 | 1   | 2   | 3   | 4   | 5   |
| $y$ | 5 | 115 | 193 | 240 | 255 | 238 |

Find an approximate formula for  $y$  in terms of  $x$ .

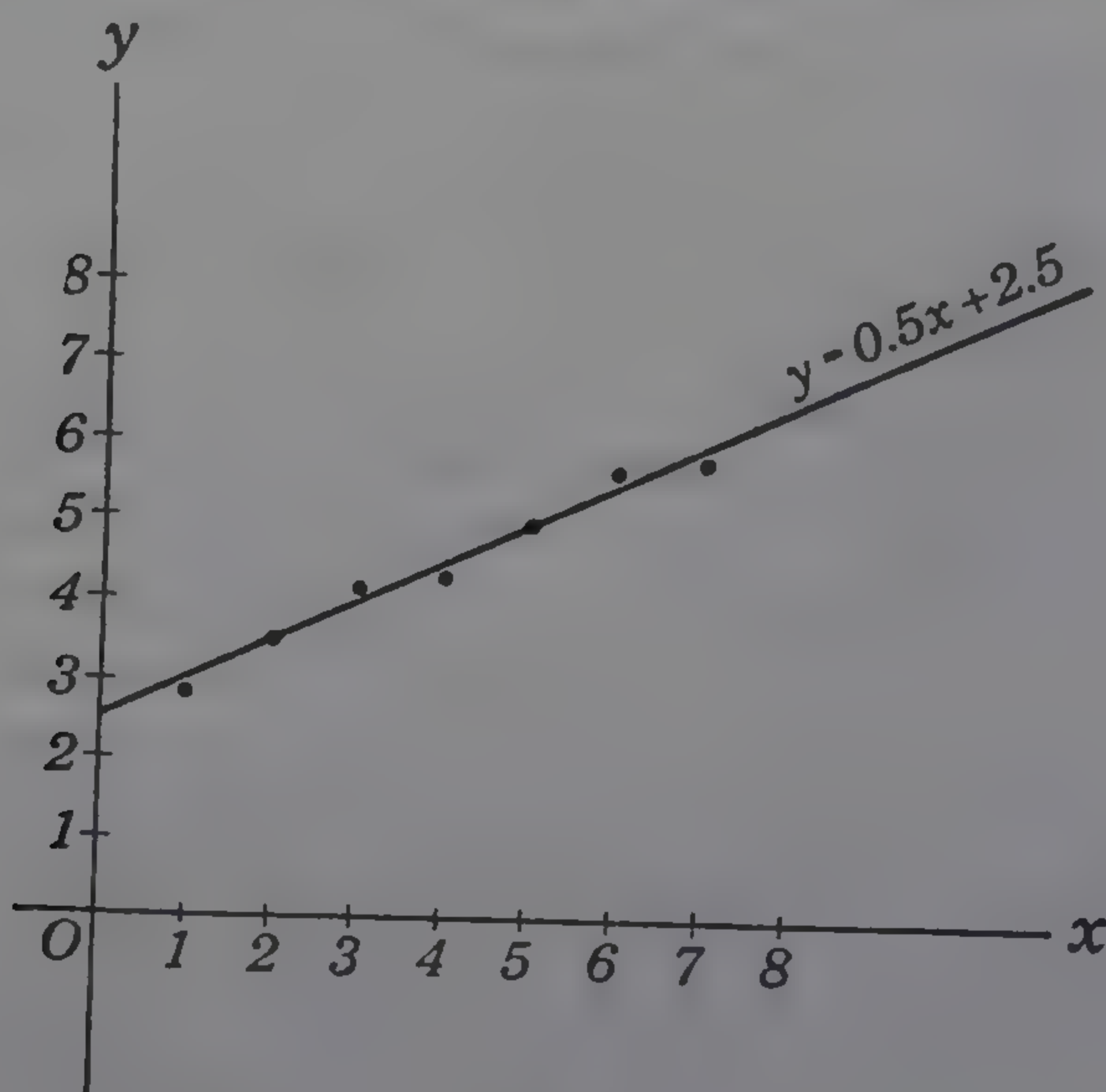


Fig. 68.

*Solution.* (a) By plotting the given points (or else by examining a table of differences) we decide that a quadratic function, or parabola, will fit well.

(b) Hence we must determine  $a$ ,  $b$ , and  $c$  so that

$$(5) \quad y = a + bx + cx^2$$

will fit the given six points approximately. Substituting the given values of  $x$  and  $y$  in (5), we get six equations of first degree in  $a$ ,  $b$ , and  $c$ :

$$(6) \quad \begin{cases} 5 = a \\ 115 = a + b + c \\ 193 = a + 2b + 4c \\ 240 = a + 3b + 9c \\ 255 = a + 4b + 16c \\ 238 = a + 5b + 25c \end{cases}$$



Since the system (6) has more equations than unknowns, it is likely to be inconsistent, but we may expect to satisfy it approximately. Since there are three constants,  $a$ ,  $b$ , and  $c$ , to be determined, we divide the six equations into three groups, say the first two, the middle two, and the last two. Adding the equations in each group, we get

$$(7) \quad \begin{cases} 120 = 2a + b + c \\ 422 = 2a + 2b + 12c \\ 493 = 2a + 9b + 41c \end{cases}$$

Solving the system (7), and rounding off the solutions to the nearest unit (since the data are given to the nearest unit), we have  $a = 5$ ,  $b = 126$ ,  $c = -16$ . Hence

$$(8) \quad y = 5 + 126x - 16x^2.$$

This answer is in accord with the result of example 2, section 190. Such exact agreement between the results of two different methods of curve fitting is not to be expected in every case.

*Note 1.* In this method it is not necessary that the values of  $x$  be evenly spaced as it was in section 190.

*Note 2.* In every case the problem reduces to solving approximately a system of linear equations with more equations than unknowns. Consider the system (2), to be specific. The difference between the right member and the left member of each equation may be called a residual, or error, or deviation. Denoting the residuals of the seven equations of (2) by  $e_1, e_2, \dots, e_7$ , respectively, we have

$$\begin{aligned} e_1 &= 3 + b - 2.9 & e_5 &= 5a + b - 5.0 \\ e_2 &= 2a + b - 3.3 & e_6 &= 6a + b - 5.6 \\ e_3 &= 3a + b - 4.1 & e_7 &= 7a + b - 5.9. \\ e_4 &= 4a + b - 4.4 \end{aligned}$$

To demand that every residual be zero is unreasonable. Since setting  $e_1 = e_2 = \dots = e_7 = 0$  yields a system of equations which is likely to be inconsistent. To demand that the sum (or average) of all the residuals be zero would not serve to determine  $a$  and  $b$ , since it would lead to only one equation containing these two unknowns. Our method consists in demanding merely that

$$e_1 + e_2 + e_3 + e_4 = 0 \text{ and } e_5 + e_6 + e_7 = 0,$$

that is, that the averages

$$\frac{e_1 + e_2 + e_3 + e_4}{4} \text{ and } \frac{e_5 + e_6 + e_7}{3}$$



of the first four residuals and of the last three residuals respectively be zero. Similar remarks may be made for example 2, and hold in general. This accounts for the name "method of averages."

## EXERCISES

By the method of averages, fit a polynomial of the indicated degree to the data; plot the graph of the estimated polynomial and the given points:

1-9. Exercises 1-9 of section 190.

10. A mixture of equal weights of dextrose and levulose when analyzed\* yields the following equivalence:

| $x$ (mg. sugar)                   | 5    | 20   | 35    | 50    | 70    | 90    |
|-----------------------------------|------|------|-------|-------|-------|-------|
| $y$ (ml. 0.1N sodium thiosulfate) | 1.59 | 8.00 | 14.52 | 18.89 | 22.69 | 25.22 |

Fit a quadratic polynomial  $y = a + bx + cx^2$  to these data.

192. The method of least squares. We shall explain this method by means of examples.

*Example 1.* As in example 1, sections 190 and 191, we wish to fit a curve to the points

| $x$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $y$ | 2.9 | 3.5 | 4.1 | 4.4 | 5.0 | 5.6 | 5.9 |

*Solution.* (a) By plotting the points (or else by examining a table of differences), we decide that a linear function will suffice.

(b) Hence we must determine  $a$  and  $b$  so that

$$(1) \quad y = ax + b$$

will fit the given points well. Substituting the given values of  $x$  and  $y$  in (1), we get seven linear equations in  $a$  and  $b$ :

$$(2) \quad \begin{cases} 2.9 = a + b \\ 3.5 = 2a + b \\ 4.1 = 3a + b \\ 4.4 = 4a + b \\ 5.0 = 5a + b \\ 5.6 = 6a + b \\ 5.9 = 7a + b \end{cases}$$

Multiplying each equation of (2) by the coefficient of  $a$  in it and adding the resulting equations, we get what is called the normal equation corresponding to  $a$ ; multiplying each equation of (2) by

\* See footnote, page 419.



the coefficient of  $b$  in it and adding the resulting equations, we get what is called the normal equation corresponding to  $b$ .

Thus,

$$2.9 = a + b$$

$$7.0 = 4a + 2b$$

$$12.3 = 9a + 3b$$

$$17.6 = 16a + 4b$$

$$25.0 = 25a + 5b$$

$$33.6 = 36a + 6b$$

$$41.3 = 49a + 7b$$

$$(3) \quad 139.7 = 140a + 28b \quad (\text{the normal equation corresponding to } a).$$

Similarly, the normal equation corresponding to  $b$  is

$$(4) \quad 31.4 = 28a + 7b.$$

Solving the system consisting of the normal equations (3) and (4), and rounding off the answers to the nearest tenth (since the data are given to the nearest tenth), we get  $a = 0.5$  and  $b = 2.5$ . Hence

$$(5) \quad y = 0.5x + 2.5.$$

This is in accord with the result of example 1, section 191. (See Fig. 68, page 421.) Such exact agreement between the results of two different methods of curve fitting is not to be expected in every case.

*Example 2.* A projectile is thrown vertically upward from an initial height of 5 feet. Its height  $y$ , in feet, at the end of  $x$  seconds is observed for  $x = 0, 1, \dots, 5$  as in the following table:

|     |   |     |     |     |     |     |
|-----|---|-----|-----|-----|-----|-----|
| $x$ | 0 | 1   | 2   | 3   | 4   | 5   |
| $y$ | 5 | 115 | 193 | 240 | 255 | 238 |

Find an approximate formula for  $y$  in terms of  $x$ .

*Solution.* (a) By plotting the given points (or else by examining a table of differences), we decide that a quadratic function, or parabola, will fit well.

(b) Hence we must determine  $a$ ,  $b$ , and  $c$  so that

$$(6) \quad y = a + bx + cx^2$$

will fit the given six points approximately. Substituting the given values of  $x$  and  $y$  in (6), we get six linear equations in  $a$ ,  $b$ , and  $c$ :



$$(7) \quad \begin{cases} 5 = a \\ 115 = a + b + c \\ 193 = a + 2b + 4c \\ 240 = a + 3b + 9c \\ 255 = a + 4b + 16c \\ 238 = a + 5b + 25c. \end{cases}$$

Since the system (7) has more equations than unknowns, it is likely to be inconsistent, but we may expect to satisfy it approximately.

Multiplying each equation of (7) by the coefficient of  $a$  in it, and adding the resulting equations, we get the **normal equation corresponding to  $a$** ; repeating this process with the coefficients of  $b$  and  $c$ , in turn, we get the normal equations corresponding to  $b$  and  $c$ , respectively. Thus the normal equations corresponding to  $a$ ,  $b$ , and  $c$  are respectively

$$(8) \quad \begin{cases} 1046 = 6a + 15b + 55c \\ 3431 = 15a + 55b + 225c \\ 13077 = 55a + 225b + 979c. \end{cases}$$

Solving the system of normal equations (8), and rounding off the answers to the nearest unit (because the data are given to the nearest unit), we get  $a = 5$ ,  $b = 126$ ,  $c = -16$ . Hence

$$(9) \quad y = 5 + 126x - 16x^2.$$

This is in accord with the result of example 2, sections 190 and 191. Such exact agreement between the results of different methods of curve fitting is not to be expected in every case.

*Note 1.* In this method it is not necessary that the values of  $x$  be evenly spaced as it was in section 190.

*Note 2.* The problem of fitting a polynomial (or even certain other types of functions) reduces to solving approximately a system of linear equations with more equations than unknowns. To be specific, consider the system of three equations in two unknowns  $a$  and  $b$ :

$$(10) \quad \begin{aligned} m_1a + n_1b &= p_1 \\ m_2a + n_2b &= p_2 \\ m_3a + n_3b &= p_3. \end{aligned}$$

Let the residuals, or errors, or deviations,  $e_1, e_2, e_3$ , be defined by

$$(11) \quad \begin{aligned} e_1 &= m_1a + n_1b - p_1, \quad e_2 = m_2a + n_2b - p_2, \\ e_3 &= m_3a + n_3b - p_3. \end{aligned}$$

To demand that every residual be zero is unreasonable, since



setting  $e_1 = e_2 = e_3 = 0$  yields the system (10) which is likely to be inconsistent. We shall prove that *our method of solving the normal equations amounts to requiring that the sum of the squares of the residuals  $e_1^2 + e_2^2 + e_3^2$  shall be a minimum.* This accounts for the name "least squares."

*Proof.* From (11) we find that

$$\begin{aligned} e_1^2 &= m_1^2 a^2 + n_1^2 b^2 + p_1^2 + 2m_1 n_1 ab - 2m_1 p_1 a - 2n_1 p_1 b \\ e_2^2 &= m_2^2 a^2 + n_2^2 b^2 + p_2^2 + 2m_2 n_2 ab - 2m_2 p_2 a - 2n_2 p_2 b \\ e_3^2 &= m_3^2 a^2 + n_3^2 b^2 + p_3^2 + 2m_3 n_3 ab - 2m_3 p_3 a - 2n_3 p_3 b. \end{aligned}$$

Adding these three equations, we find that the sum of the squares of the residuals is given by

$$\begin{aligned} (12) \quad e_1^2 + e_2^2 + e_3^2 &= (m_1^2 + m_2^2 + m_3^2)a^2 \\ &+ (n_1^2 + n_2^2 + n_3^2)b^2 + (p_1^2 + p_2^2 + p_3^2) \\ &+ 2(m_1 n_1 + m_2 n_2 + m_3 n_3)ab \\ &- 2(m_1 p_1 + m_2 p_2 + m_3 p_3)a - 2(n_1 p_1 + n_2 p_2 + n_3 p_3)b. \end{aligned}$$

The right member of (12) is a quadratic function of  $a$  and  $b$  which we may denote by  $f(a, b)$ .

We wish to determine  $a$  and  $b$  so that  $f(a, b)$  will be minimum. Consider  $b$  to be a constant for a moment. Then  $f(a, b)$  may be thought of as a quadratic function of  $a$  alone. By exercise 21, section 82, chapter XI, we get the minimum when  $a$  is equal to minus the coefficient of  $a$  divided by twice the coefficient of  $a^2$ . Therefore we must have, from (12),

$$a = - \frac{2(m_1 n_1 + m_2 n_2 + m_3 n_3)b - 2(m_1 p_1 + m_2 p_2 + m_3 p_3)}{2(m_1^2 + m_2^2 + m_3^2)}$$

or

$$\begin{aligned} (13) \quad (m_1^2 + m_2^2 + m_3^2)a + (m_1 n_1 + m_2 n_2 + m_3 n_3)b \\ = m_1 p_1 + m_2 p_2 + m_3 p_3. \end{aligned}$$

But this is exactly the normal equation corresponding to  $a$ , derived from the equations (10).

Similarly, regarding  $a$  as constant and  $f(a, b)$  as a quadratic function of  $b$  alone, we get

$$\begin{aligned} (14) \quad (m_1 n_1 + m_2 n_2 + m_3 n_3)a + (n_1^2 + n_2^2 + n_3^2)b \\ = n_1 p_1 + n_2 p_2 + n_3 p_3 \end{aligned}$$

which is the normal equation corresponding to  $b$ , derived from the equations (10). Hence, in order for (12) to be minimum it is neces-



sary that  $a$  and  $b$  satisfy the normal equations (13) and (14). To investigate whether this is also sufficient lies beyond the scope of this book.

*Note 3.* The term "best-fitting curve" is frequently used for the one obtained by the method of least squares. This usage arises from the fact that the criterion that the sum of the squares of the errors be minimum is closely related to other theoretical considerations occurring in the more advanced theory of probability and statistics.

## EXERCISES

*By the method of least squares, fit a polynomial of the indicated degree to the data; plot the graph of the calculated polynomial and the given points:*

1-9. Exercises 1-9, section 190.

10. Exercise 10, section 191.

11. Fit a curve of the form  $y = ax + \frac{b}{x}$  to the following data:

|     |      |     |      |      |      |
|-----|------|-----|------|------|------|
| $x$ | 1    | 2   | 4    | 5    | 10   |
| $y$ | 12.1 | 9.0 | 10.5 | 12.0 | 20.9 |

193. Exponential and power functions. Many laws of physics, chemistry, biology, etc., are of the exponential form

$$(1) \quad y = ab^x$$

or of the power form

$$(2) \quad y = ax^b.$$

(a) If the scatter diagram or any other information suggests that the exponential function (1) will fit, we proceed as follows.

From (1), we get

$$(3) \quad \log y = \log a + x \log b.$$

Setting

$$(4) \quad Y = \log y, A = \log a, B = \log b,$$

we get

$$(5) \quad Y = A + Bx.$$

Hence we may fit a straight line to the points  $(x_1, Y_1 = \log y_1)$ ,  $(x_2, Y_2 = \log y_2)$ ,  $\dots$ ,  $(x_m, Y_m = \log y_m)$  by any of the preceding methods. From the values of  $A$  and  $B$  thus found we calculate  $a$  and  $b$  from (4).

*Note 1.* Semi-logarithmic graph paper is helpful. Its use will remove the necessity of consulting a table of logarithms in order



to plot equation (3) or (5). Furthermore, if the data can be plotted on it, and if the points  $(x_1, Y_1)$ ,  $(x_2, Y_2)$ , . . . ,  $(x_m, Y_m)$  lie nearly in a straight line, the exponential form (1) is desirable.

*Example 1.* Fit a curve of the exponential form  $y = ab^x$  to the following data:

|     |     |     |     |     |     |      |
|-----|-----|-----|-----|-----|-----|------|
| $x$ | 0   | 1   | 2   | 3   | 4   | 5    |
| $y$ | 100 | 175 | 305 | 535 | 940 | 1640 |

*Solution.* Let  $Y = \log y$ . Then, we have

|     |   |        |        |        |        |        |
|-----|---|--------|--------|--------|--------|--------|
| $x$ | 0 | 1      | 2      | 3      | 4      | 5      |
| $Y$ | 2 | 2.2430 | 2.4843 | 2.7284 | 2.9731 | 3.2148 |

By means of a graph (Fig. 69) of  $Y$  against  $x$  (or by calculating the first differences of the  $Y$ 's), we see that a straight line will fit.

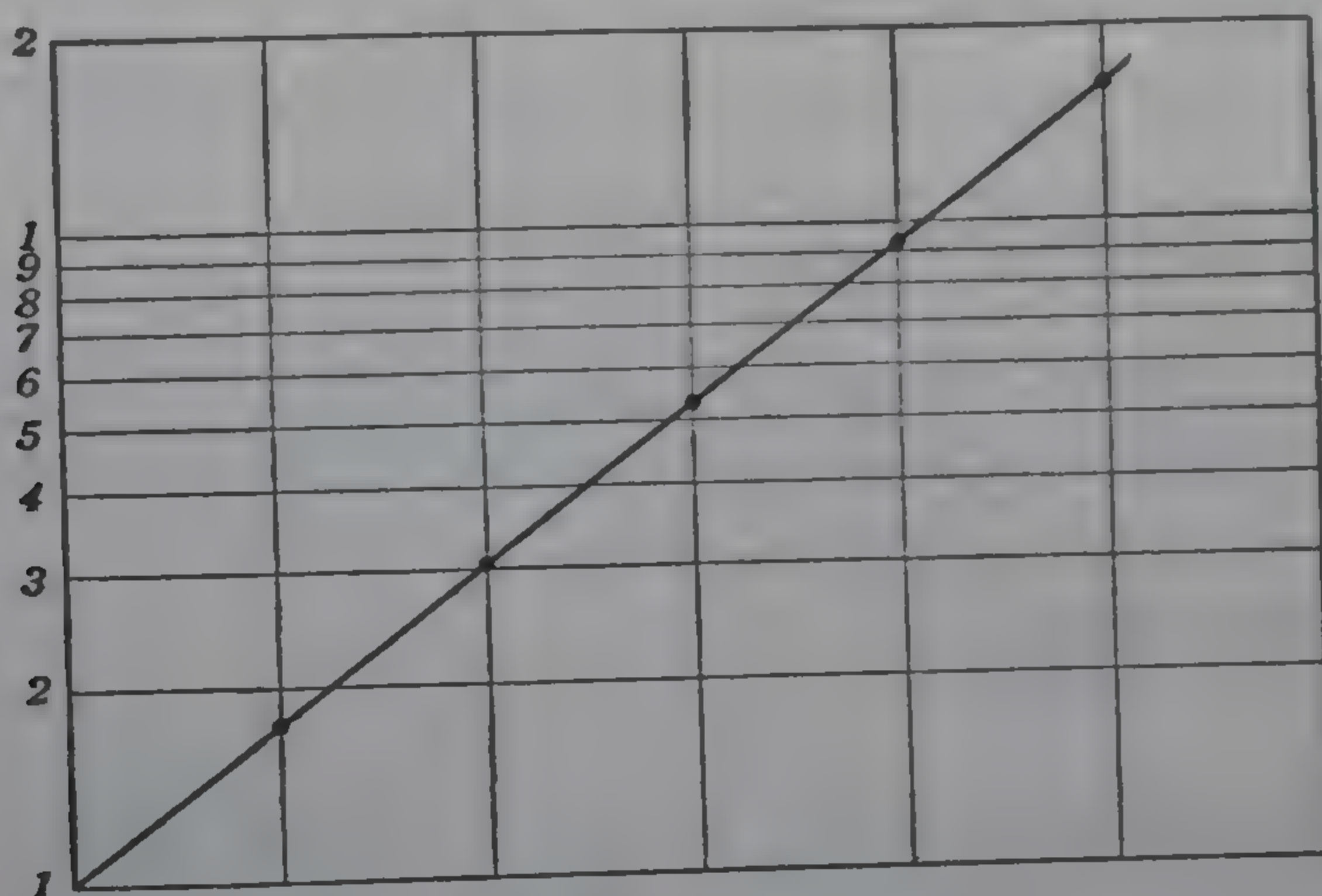


Fig. 69.

By the method of differences, we get  $Y = 2 + 0.2430x$ . From (4) we get  $a = 100$ ,  $b = 1.75$ . Hence  $y = 100(1.75)^x$ .

(b) If the scatter diagram, or other information, suggests that the power function (2) will fit, we proceed as follows. From (2), we get

$$(6) \quad \log y = \log a + b \log x.$$

Let

$$(7) \quad Y = \log y, A = \log a, X = \log x.$$

Then

$$(8) \quad Y = A + bX.$$



A straight line may be fitted to the points  $(X_1 = \log x_1, Y_1 = \log y_1), \dots, (X_m = \log x_m, Y_m = \log y_m)$  by the methods of the preceding sections. The value of  $a$  can be determined from the resulting value of  $A$  by (7).

*Note 2.* Logarithmic (log-log) graph paper is helpful. Its use will remove the necessity of consulting a table of logarithms in order to plot equation (6) or (8). Furthermore, if the data are plotted on it and if the points  $(X_1, Y_1), \dots, (X_m, Y_m)$  lie nearly on a straight line, the power function (2) is desirable.

*Example 2.* Fit a curve of the power form  $y = ax^b$  to the following data:

|     |     |      |         |           |
|-----|-----|------|---------|-----------|
| $x$ | 1   | 10   | 100     | 1000      |
| $y$ | 100 | 3160 | 100,000 | 3,160,000 |

*Solution.* Setting  $Y = \log y$  and  $X = \log x$ , we get

|     |   |        |        |        |
|-----|---|--------|--------|--------|
| $X$ | 0 | 1      | 2      | 3      |
| $Y$ | 2 | 3.4997 | 5.0000 | 6.4997 |

By means of a graph of  $Y$  against  $X$  (or by calculating the first differences of the  $Y$ 's), we see that a straight line will fit. By the method of differences, we get  $Y = 2 + 1.4997X$ . From (7), we get  $y = 100x^{1.4997}$ .

## EXERCISES

1. The number  $y$  of bacteria in a culture at the end of  $x$  hours is given by:

|     |     |     |     |     |
|-----|-----|-----|-----|-----|
| $x$ | 0   | 1   | 2   | 3   |
| $y$ | 100 | 112 | 125 | 140 |

(a) Fit an exponential curve  $y = ab^x$  to the above data.

(b) Use the formula calculated in (a) to predict the number of bacteria present at  $x = 4$ .

2. The number  $y$  of millions of population of a certain country at the end of  $x$  years is given by

|     |      |      |      |      |
|-----|------|------|------|------|
| $x$ | 0    | 5    | 10   | 15   |
| $y$ | 4.50 | 5.47 | 6.66 | 8.10 |

(a) Fit an exponential curve  $y = ab^x$  to the above data.

(b) Use the formula calculated in (a) to predict the population at  $x = 20$ .

3. Suppose that a body moves through a resisting medium so that its velocity  $y$  feet per minute at the end of  $x$  minutes is given by

|     |     |    |    |    |
|-----|-----|----|----|----|
| $x$ | 0   | 1  | 2  | 3  |
| $y$ | 100 | 76 | 57 | 43 |



(a) Fit an exponential curve of the form  $y = ab^x$  to the above data.

(b) Use the formula calculated in (a) to predict the velocity at  $x = 4$ .

(c) Would it be safe to use the formula of (a) to predict the velocity at  $x = 120$ ?

4. The period,  $y$  years, of revolution about the sun for each planet, and the radius,  $x$  millions of miles, of its orbit are given in the following table:

|     | Mercury | Venus | Earth | Mars | Jupiter | Saturn | Uranus | Neptune |
|-----|---------|-------|-------|------|---------|--------|--------|---------|
| $x$ | 36.0    | 67.2  | 92.9  | 142  | 483     | 886    | 1780   | 2790    |
| $y$ | 0.24    | 0.62  | 1.00  | 1.88 | 11.9    | 29.5   | 84.0   | 165     |

Fit a power function  $y = ax^b$  to the above data.

5. The pressure,  $p$  inches of mercury, of the atmosphere, under certain conditions, is related to the density ratio  $\sigma$  by the following table:

| $\sigma$ | 1    | 0.862 | 0.738 | 0.630 |
|----------|------|-------|-------|-------|
| $p$      | 29.9 | 24.3  | 19.5  | 15.7  |

Fit a power function  $p = a\sigma^b$  to the above table.

6. Prove that if the sequence  $x_1, x_2, x_3, \dots, x_n, \dots$  is a geometric progression, then the sequence  $\log x_1, \log x_2, \log x_3, \dots, \log x_n, \dots$  is an arithmetic progression.

7. Prove that if the sequence  $x_1, x_2, \dots, x_n, \dots$  is a geometric progression, then the sequence of differences of first order  $\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_n, \dots$  is a geometric progression.

8. Prove that if the values  $x_1, x_2, \dots, x_n, \dots$  are in arithmetic progression with common difference  $h$ , and the corresponding values  $y_1, y_2, \dots, y_n$  are in geometric progression with common ratio  $r$ , then  $y_n = ab^{x_n}$  where  $a = y_1 r^{-x_1/h}$  and  $b = r^{1/h}$ .

9. Prove that if the values  $x_1, x_2, \dots, x_n, \dots$  are in geometric progression with common ratio  $r$ , and the corresponding values  $y_1, y_2, \dots, y_n, \dots$  are in geometric progression with common ratio  $s$ , then  $y_n = ax_n^b$  where  $b = \log s / \log r$  and  $a = y_1 / x_1^b$ .

**194. Extrapolation.** The equations or curves arrived at by the various methods explained in this chapter are, in practice, used not merely for interpolation, or reading between the known values, but also for **extrapolation**, or reading beyond the range of known values. Needless to say, such predictions must be accompanied by an appropriate amount of scientific caution and skepticism. Thus, if one fitted a curve to the graph of some stock market index over a period of weeks and then used the curve to predict what would happen next week, he would be most unwise. If he had done that just before the crash of 1929, he might well have been ruined. On the other hand, if an astronomer thus predicts the orbit of a



comet for the next few centuries from a very few observations, he is likely to be astonishingly accurate. But this is due to the overwhelming amount of corroborative evidence, both theoretical and empirical, which makes him sure of the form of the functions involved. Even in the physical sciences, a curve which gives good predictions for some limited range of values may be useless too far beyond that range. However, when properly used, methods of extrapolation are extremely helpful in the investigation of natural laws.







## Answers to Odd-Numbered Exercises

### Section 1. Pages 3–5

1. (a) Must be true. (b) May be true or false.  
(c) May be true or false. (d) Must be false.  
3. (d), (e). 5. Sufficient. 7. No.  
9. (a) Invalid. (b) Invalid. 11. All invalid. 13. Invalid.  
15. Invalid. 17. Valid.

### Section 4. Page 11

1. Commutative law for addition.  
3. Associative law for addition.  
5. Associative law for multiplication.  
7. Distributive law. 9. Distributive law.  
11. 23. 13. 29. 15. 77. 17. 207. 19. 183.

### Section 5. Pages 13–14

1.  $2 \cdot 4 = 8$  3. (b), (c), (d), (e).  
5. (b)  $a > b$ . (d)  $a$  is a multiple of  $b$ .

### Section 6. Pages 16–17

3.  $3 \cdot 5$ . 5.  $2 \cdot 43$ . 7.  $3 \cdot 11 \cdot 11$ . 9.  $3 \cdot 5 \cdot 11 \cdot 19$ .  
11. (a) 20. (b) 240. 13. (a) 12. (b) 72. 15. (a) 1. (b) 6450.  
17. (a) 12. (b) 360. 19. (a) 20. (b) 240. 23. Yes.  
25. Yes. 27. Yes.

### Section 7. Page 21

1.  $2/3$ . 3.  $3/2$ . 5.  $2/3$ . 7.  $2a/3d$ . 9.  $\frac{m+3n}{mn}$ .  
11.  $5/9$ . 13.  $5/16$ . 15.  $a/c$ . 17.  $2x$ . 19.  $24/40$ .

### Section 8. Pages 23–24

1.  $7/12$ . 3.  $5/8$ . 5.  $7/6$ . 7.  $1/20$ . 9.  $5/8$ .  
11.  $4/3$ . 13.  $1/6$ . 15.  $\frac{2+x}{2x}$ . 17.  $1/2$ . 19. Less than.  
21.  $21/23$ ;  $34/667$ . 23. (b),  $x > y$ . 25.  $2/3$ .

### Section 9. Page 28

1. 3. 3. -8. 5. -3. 7. 0. 9. -10. 11. 0.  
13. -90. 15. -1. 17. -10. 19. 3.



## Section 10. Pages 31–32

1.  $-8$ .    3.  $3$ .    5.  $-8$ .    7.  $-3$ .    9.  $-9$ .    11.  $1$ .  
 13.  $4$ .    15.  $1$ .    17.  $40$ .    19.  $-1/10$ .    21.  $13/2$ .    23.  $20$ .

## Section 11. Page 34

1. Yes.    3. Yes.    5. Yes.    7. Yes.

## Section 12. Page 37

1.  $3$ .    3.  $-5$ .    5.  $2$ .    7.  $2$ .    9.  $3$ .  
 11. (a) No. (b)  $a^2 + 2ab + b^2$ .    13. No.    15.  $5$ .    17.  $a$ .  
 19.  $x$ .    21.  $1/2$ .    23.  $144$ .    25.  $80$ .    27.  $196$ .  
 29.  $100$ .

## Section 14. Page 45

1. (a)  $1.732$ . (b)  $1.73$ .    3. (a)  $1.259$ . (b)  $1.26$ .  
 5. (a)  $1.710$ . (b)  $1.71$ .    13. False.

## Section 15. Page 47

1.  $25$ .    3.  $27$ .    5.  $487$ .    7.  $706$ .    9.  $8.74$ .  
 11.  $2.17$ .    13.  $19.131$ .    15.  $7.517$ .    17.  $8.883$ .    19.  $5.404$ .

## Section 16. Pages 50–51

1.  $3i$ .    3.  $i\sqrt{5}$ .    5.  $ci$ .    7.  $15i$ .    9.  $3 + 2i$ .  
 11.  $4 + 6i$ .    13.  $9 + 4i$ .    15.  $3 + 3i$ .    17.  $-3 - 5i$ .  
 19.  $8 - 2i$ .    21.  $-12 + 8i$ .    23.  $7 + 17i$ .    25.  $14 - 8i$ .  
 27.  $4 - 19i$ .    29.  $9 - 7i$ .    31.  $24 + 23i$ .    33. Complex, imaginary.  
 35. Complex, imaginary, pure imaginary.  
 37. Complex, real, irrational, positive.  
 39. Complex, real, rational, negative.  
 41. Complex, real, irrational, negative.  
 43. Complex, real, rational, integral, zero.

## Section 23. Pages 58–59

31.  $a^8$ .    33.  $1/a^6$ .    35.  $18a^5b^7$ .    37.  $\frac{1}{4a^5}$ .    39.  $16x^8y^{12}$ .  
 41.  $2x^2y^5$ .    43.  $\frac{2a^8b^3}{27}$ .    45.  $-72$ .    47.  $-x^6y^9$ .  
 49.  $2$ .    51. False;  $3^5$ .    53. False;  $3^5$ .    55. False;  $36$ .  
 57. False;  $2 \cdot 5^3 = 250$ .    59. False;  $3^6$ .    61. True.  
 63.  $3x^4y^5$ .    65.  $2x^4y^2$ .

## Section 24. Page 62

1. (a)  $A$ ; (b)  $4$ .    3. (a)  $B, C$ ; (b)  $7$ .    5. (a)  $B$ ; (b)  $2$ .  
 7. (a)  $B, C$ ; (b)  $12 + \sqrt{2}$ .    9. (a)  $B, C$ ; (b)  $3$ .  
 11. (a)  $-2$ ; (b)  $3$ .    13. (a)  $-1$ ; (b)  $5$ .    15. (a)  $-1/2$ ; (b)  $5$ .



## Section 25. Pages 63–64

1.  $x + 7y$ . 3.  $-2x^2 + 3xy + 2y^2$ . 5.  $5x - 4y$ . 7.  $-x + 3y + z$ .  
 9.  $3x^2 + x + 7$ . 11.  $-2x + 5y$ . 13.  $3x^2 - 2x + 2$ . 15.  $y$ .  
 17.  $-5x + 5y + 4$ .  
 19. (a)  $(-x + y) + (2a - 3b)$ ; (b)  $-(x - y) - (-2a + 3b)$ .

## Section 26. Page 66

1.  $3x^2 - 3x + 4$ ; 2. 3.  $4x + 4$ ; 1. 5.  $x^4y - x^3y + x^2y^2 - 2x + 3$ ; 5.  
 7.  $3x + 1$ ; 1. 9. An integer. 11. No.

## Section 28. Page 69

5. Identity. 7. Conditional. 9. Identity.

## Section 29. Page 71

1. 5. 3. No root. 5. No root.

## Section 30. Page 72

1.  $3x^3 - x^2 - 2x + 4 = 0$ ; 3. 3.  $2x^2 + x + 7 = 0$ ; 2.  
 5.  $2x^3 - 16 = 0$ ; 3.

## Section 31. Pages 74–75

1.  $y = 15$ . 3.  $y = 2$ . 5.  $z = 2$ . 7. 2. 9. 2, 3.

## Section 32. Pages 75–76

1. 10. 3. 4. 5. 0. 7.  $2n + 6$ . 9.  $-2r + 6$ . 11.  $-5$ .  
 13. 0. 15.  $r^2 - 9$ . 17. 17. 19.  $3/4$ . 21. 2.92.  
 23. 7. 25.  $8/9$ . 27. Not defined. 29.  $3/2$ .

## Section 33. Page 77

1. (a) 144 ft.; (b) 256 ft. 3. 1 sq. ft. 5. \$14.88.  
 7.  $104^\circ\text{F}$ . 9. 113.0 sq. ft.

## Section 34. Page 79

3. 0.

## Section 35. Page 81

1. 3. 3.  $-2$ . 5.  $\pm 2$ . 7. None. 9. 3,  $-2$ . 11. 1, 3,  $-2$ .

## Section 37. Page 84

1. 4. 3.  $-2$ . 5.  $\pm 2$ . 7. 1, 3. 9. 3,  $-1$ .  
 11. 2, 3,  $-1$ . 13.  $4/3$ ,  $-1/2$ . 15.  $x = \pm 1.4$ , approx.

## Section 38. Page 85

1.  $x = 0$ . 3.  $x = 3$ . 5.  $x = -2/3$ . 7.  $x = \pm 2$ .  
 9.  $x = 1$  and  $x = 3$ . 11.  $x = 2$ . 13. None.



## Section 40. Page 89

1.  $5x^2 - 2x + 3$ .      3.  $6x^3 - 3x^2y + 2xy^2 + 4y^2$ .      5.  $2x^2 - 3x - 3$ .  
 7.  $2x^2 + xy - 5y^2$ .      9.  $4x^2 - 5$ .      11.  $-3x^3 + x^2y - 3y^3$ .

## Section 41. Page 90

1.  $10xy + 6xz$ .      3.  $-6xy + 10yz$ .      5.  $2x^3 - 6x^2 + 8x$ .  
 7.  $6x^3 + 3x^2y - 6xy^2$ .      9.  $-15x^5 + 6x^2y$ .      11.  $x^2 - 7x + 10$ .  
 13.  $x^2 + x - 12$ .      15.  $6x^2 - 7x + 2$ .      17.  $x^3 - 2x^2 + 2x - 15$ .  
 19.  $6x^3 - 13x^2 + 14x - 12$ .      21.  $x^3 + x^2y - xy^2 + 2y^3$ .      23.  $x^3 + y^3$ .  
 25.  $x^2 + 2xy + y^2$ .      27.  $a^2x^{2c} - b^2$ .      29.  $a^2x^{2c} - 2abx^c + b^2$ .  
 31.  $6x^4 - 4x^3 + 3x^2 + 7x - 6$ .

## Section 42. Page 91

1.  $5x^4$ .      3.  $2x^4y^8$ .      5.  $7x^2 + 6x^3y^3$ .      7.  $-3x^6 + 2x^4 - 1$ .  
 9.  $-5x^2y^2 + 4x + 3$ .

## Section 43. Page 95

1.  $x - 5$ .      3.  $2x - 1$ .      5.  $x^2 - x + 2$ .      7.  $x^2 + 3x - 2$ .  
 9.  $3x - 1$ .      11.  $2x^2 + 4$ .      13.  $x^2 - 2xy + 4y^2$ .  
 15.  $x^2 - 2xy + 3y^2$ .      17. Quotient  $x - 3$ , remainder 4.  
 19. Quotient  $2x^2 - 3x + 1$ , remainder  $-5$ .  
 21. Quotient  $2x^2 + 3x - 1$ , remainder  $-6$ .  
 23. Quotient  $x^2 - 3$ , remainder  $2x - 1$ .  
 25. Quotient  $2x - 1$ , remainder  $3x + 5$ .  
 27. Quotient  $x^2 + 3$ , remainder 2.      29.  $x^2 - xy + y^2$ .  
 31.  $x^2 + xy + y^2$ .      33.  $x^3 - x^2y + xy^2 - y^3$ .      35.  $x^2 - 4$ .

## Section 45. Page 98

1.  $6ax + 15ay$ .      3.  $-6a^5b + 2a^3b^2$ .      5.  $x^2 - 9y^2$ .      7.  $1 - 9x^4$ .  
 9.  $9x^2 + 12x + 4$ .      11.  $25x^2 - 30xy + 9y^2$ .  
 13.  $9c^2x^2 - 6acxy + a^2y^2$ .      15.  $16x^4 - 24x^2y^2 + 9y^2$ .  
 17.  $x^2 + 7x + 10$ .      19.  $x^2 + 5x - 14$ .      21.  $u^2 - 7u + 12$ .  
 23.  $2x^2 + 7x - 4$ .      25.  $12x^2 + 16xy - 3y^2$ .      27.  $6x^2 + 13xy - 5y^2$ .  
 29.  $6x^4 - 11x^2 - 10$ .      31.  $4x^4 + 5x^2y^2 - 6y^4$ .  
 33.  $x^2 + 2xy + y^2 - 4z^2$ .      35.  $x^2 + 6x + 9 + 4xy + 12y + 4y^2$ .  
 37.  $x^2 + 2xy + y^2 - z^2$ .      39.  $x^2 - 2xy + y^2 - 6x + 6y + 9$ .  
 41.  $x^2 + 2xy + y^2 - u^2 + 2uv - v^2$ .      43.  $x^{2a} + 2x^ay^b + y^{2b}$ .  
 45.  $x^{2a} - y^{2b}$ .

## Section 46. Pages 99-100

1.  $2a(x - 3a)$ .      3.  $2xy(2x^2 - 3y + 4y^2)$ .      5.  $(4x + 3y)(4x - 3y)$ .  
 7.  $(3 - 8xy)(3 + 8xy)$ .      9.  $(12u - 5v)(12u + 5v)$ .      11.  $(2x + 3y)^2$ .  
 13.  $(4x - 3y)^2$ .      15.  $(8xy - 1)^2$ .      17.  $(2ax + b)^2$ .  
 19.  $\left(x + \frac{b}{2a}\right)^2$ .      21.  $x(3x + 2y)(3x - 2y)$ .      23.  $2x^2(4x + 3y)(4x - 3y)$ .  
 25.  $(x^2 + 4)(x + 2)(x - 2)$ .      27.  $(3x + 2y)(9x^2 - 6xy + 4y^2)$ .  
 29.  $(4x - y)(16x^2 + 4xy + y^2)$ .  
 31.  $(x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2)$ .  
 33.  $(2a + b)(x + y)$ .      35.  $(x + y + 3)(x + y - 3)$ .



37.  $(5x + y - z)(5x - y + z)$ . 39.  $(x - 2y - 3)^2$ .  
 41.  $(4x - y)(-2x - 3y)$ . 43.  $2x^2y(y - 2x)(y^2 + 2xy + 4x^2)$ .  
 45.  $(x + y + 2)(x - y + 4)$ . 47.  $(x + 4y)(x^2 + 2xy + 4y^2)$ .  
 49.  $(x + 2)(3x - 4y)(-x - 2y)$ . 51.  $5xy(x^2 + 4y^2)(x + 2y)(x - 2y)$ .  
 53.  $3x^2y(x + 2y)(x^2 - 2xy + 4y^2)$ .

## Section 47. Page 101

1.  $(a + 3)(x - y)$ . 3.  $(x - y)(x - 3)$ . 5.  $(x^2 - 2)(2x + 3)$ .  
 7.  $(x - 2y)(x + 2y + 1)$ . 9.  $(x + y + 3)(x - y + 3)$ .  
 11.  $(3x + y)(1 - 2x)$ . 13.  $(x + 2y + u + 1)(x + 2y - u - 1)$ .  
 15.  $(x^2 - y^2 + xy)(x^2 - y^2 - xy)$ .  
 17.  $(x^2 - y^2 - 2xy)(x^2 - y^2 + 2xy)$ . 19.  $(x^2 + 2x + 3)(x^2 - 2x + 3)$ .  
 21.  $(x^2 + 2x + 2)(x^2 - 2x + 2)$ . 23.  $(x^2 + xy - 3y^2)(x^2 - xy - 3y^2)$ .  
 25.  $(a - b)(x - y)$ .

## Section 48. Page 102

1.  $(x - 3)(x - 4)$ . 3.  $(x - 4)(x + 3)$ . 5.  $(x - 2)(x + 9)$ .  
 7.  $(x + 12)(x - 3)$ . 9.  $(m - 12)(m + 2)$ . 11.  $(x - 24y)(x + 2y)$ .  
 13.  $(x - 12y)(x + 4y)$ . 15.  $(2x - 1)(3x + 5)$ . 17.  $(6x - 1)(x - 3)$ .  
 19.  $(2x - 3)(4x - 5)$ . 21.  $(7x + 4y)(3x - y)$ .  
 23.  $(2x - 3y)(6x + 5y)$ . 25.  $(x + 4)(x + 5)$ .  
 27.  $(x + y - 2)(x + y + 9)$ . 29.  $(2x - y - 12)(2x - y + 2)$ .  
 31.  $(x - 5)(x - 6)$ . 33.  $2xy^2(2x - y)(3x + 2y)$ .  
 35.  $2x^2y(x - 6y)(x + y)$ . 37.  $(x^2 + 3)(x^2 + 4)$ .  
 39.  $(x + 2)(x^2 - 2x + 4)(x - 1)(x^2 + x + 1)$ .  
 41.  $(x^2 + y^2)(x + 3y)(x - 3y)$ .

## Section 49. Page 103

1.  $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$ .  
 3.  $(x - 2y)(x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4)$ .  
 5.  $(4x^2 + y^2)(2x + y)(2x - y)$ .  
 7.  $(xy + 3)(x^4y^4 - 3x^3y^3 + 9x^2y^2 - 27xy + 81)$ .  
 9.  $(x^2 + 4)(x^4 - 4x^2 + 16)$ .  
 11.  $(x + 2y)(x - 2y)(x^2 - 2xy + 4y^2)(x^2 + 2xy + 4y^2)$ .  
 13.  $(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$ .

## Section 50. Page 104

1.  $2xy; 12x^2y^2$ . 3.  $xy; 18x^3y^3$ . 5.  $(x - 3); 6(x - 3)$ .  
 7.  $x - 2; 2(x - 2)(x + 2)$ . 9.  $(x + y); (x - y)(x + y)^2$ .  
 11.  $(x - 2); (x - 2)(x + 2)(x - 3)$ .  
 13.  $(x - 3); (x - 3)(x + 3)(x + 2)(x^2 + 3x + 9)$ .  
 15.  $2(x - 3); 4(x - 3)^2(x + 3)(x + 5)$ .  
 17.  $2x; 2x(2x + y)(2x - y)(3x + 2y)$ .  
 19.  $(x - 2); 3(x - 2)^2(x + 2)$ .  
 21.  $(x - y); (x - y)^2(x + y)(x^2 + xy + y^2)$ .  
 23.  $1; (x^2 + y^2)(x + y)(x - y)(x + 2y)$ .  
 25.  $(2x - 3); 3x(2x - 3)(2x + 3)(x + 2)$ .



## Section 51. Page 106

1.  $4/5$ .    3.  $5x/6y$ .    5.  $7xy^2/8a$ .    7.  $7/8a$ .    9.  $4x/3$ .  
 11.  $3x/4$ .    13.  $\frac{x-y}{3}$ .    15.  $\frac{x-3}{x+3}$ .    17.  $\frac{4x}{x+2}$ .    19.  $\frac{x-3}{x-6}$ .  
 21.  $\frac{x+y}{x^2+xy+y^2}$ .    23.  $\frac{-2}{x+y}$ .    25.  $\frac{y-x}{x}$ .    27.  $-\frac{9x^2+3x+1}{3x+1}$ .  
 29.  $\frac{2x+5}{5x-2}$ .    31.  $\frac{-2(x+2y)}{3(x^2+2xy+4y^2)}$ .

## Section 52. Pages 107-108

1.  $3/4$ .    3. 6.    5.  $3z/2xy^2$ .    7.  $2y^2/3$ .    9.  $\frac{x-2}{2x}$ .    11. 1.  
 13.  $9ax/bcz$ .    15.  $\frac{x-2y}{x-y}$ .    17. 1.    19.  $\frac{x^2}{x-5y}$ .    21.  $1/2$ .  
 23.  $\frac{1}{3x+2y}$ .    25. 1.    27. (a)  $(2x+1)(x-5)$ ; (b)  $\frac{2x+1}{(x+2)^2(x-5)}$ .  
 29.  $\frac{(3x+2y)(x+2y)}{(4x+y)(2x+y)}$ .

## Section 53. Pages 109-110

1.  $1/5$ .    3.  $7/8$ .    5.  $x/3$ .    7.  $\frac{a^2+5}{3a}$ .    9.  $\frac{bc-ac+ab}{abc}$ .  
 11.  $\frac{7x+9}{6x^2}$ .    13.  $\frac{5x^2-6x+6}{4x}$ .    15.  $\frac{x-3}{x-5}$ .    17.  $\frac{3x^2-7x+9}{(x-3)(2x-1)}$ .  
 19.  $\frac{-4xy}{(x+y)(x-y)}$ .    21.  $\frac{2x}{(x+a)^2(x-a)}$ .    23.  $\frac{1}{x-2}$ .  
 25.  $\frac{2}{(x-2)(x-1)}$ .    27.  $\frac{5}{(x+6)(x-9)}$ .    29.  $\frac{1}{(x-y)(x-3y)}$ .  
 31. 0.    33.  $\frac{x^2-2x+8}{(x-6)(x+4)(x+2)}$ .

## Section 54. Pages 112-113

1. 5.    3.  $\frac{b+a}{b-a}$ .    5.  $4/3$ .    7. 1.    9.  $x$ .    11.  $x/y$ .  
 13.  $\frac{2y+2x+1}{x}$ .    15.  $\frac{2x^2-y^2}{y^2}$ .    17. 2.    19.  $\frac{x+2}{x-2}$ .    21.  $\frac{x+2}{x-2}$ .  
 23.  $\frac{1}{z^2}$ .    25.  $x-y$ .    27.  $\frac{2a^2-b^2}{b^2}$ .    29.  $\frac{a^3+2a^2+7a+8}{a^3+7a}$ .

## Section 55. Pages 115-116

1. 4.    3. 4.    5. 2.    7. 9.    9. 3.    11. -2.    13. -5.  
 15.  $5/2$ .    17.  $1/6$ .    19.  $-18/5$ .    21.  $8/3$ .    23.  $25/18$ .  
 25. 5.    27.  $11/8$ .    29. 0.05.



## Section 56. Pages 116–117

1. 3.      3. 7.      5. 0.      7. No solution.      9. 12.      11. 5.  
 13. 3.      15.  $5/2$ .      17.  $-1/12$ .      19. No solution.  
 21. No solution.      23.  $19/5$ .      25.  $-16/5$ .      27. No solution.

## Section 57. Pages 118–119

1.  $5b$ .      3.  $a + 2b$ .      5.  $\frac{3b}{a-2}$  if  $a \neq 2$ .      7.  $\frac{b}{2a-b}$  if  $a \neq \frac{b}{2}$ .  
 9.  $3a^2 + 2a - 5$ .      11.  $2a + b$ .      13.  $a + b$  if  $a \neq b$ .  
 15.  $\frac{2ab}{a+b}$  if  $a \neq -b$ .      17.  $\frac{a+b}{2}$  if  $a \neq b$ .      19.  $\frac{4-2x}{3}$ .  
 21.  $\frac{x+7}{7}$ .      23.  $\frac{C-Ax}{B}$  if  $B \neq 0$ .      25.  $p = \frac{c}{v}$ .      27.  $h = \frac{2A}{b}$ .  
 29.  $w = \frac{P-2l}{2}$ .      31.  $F = \frac{9}{5}C + 32$ .      33.  $h = \frac{3V}{\pi r^2}$ .  
 35.  $m_1 = \frac{Fr^2}{Gm_2}$ .      37.  $d = \frac{2(S-an)}{n(n-1)}$ .      39.  $r = \frac{nE-IR}{In}$ .  
 41.  $w_2 = \frac{d_1w_1}{d_2}$ .      43.  $t = \frac{P-p}{pc}$ .      45.  $h = h_1 + \frac{1}{c}(p_1 - p)$ .  
 47.  $C = \frac{C_1C_2}{C_1 + C_2}$ .

## Section 58. Pages 122–123

13. 3.      15.  $-6$ .      17.  $5/2$ .      19. 6.      21.  $b = -2$ .      23. 3.  
 25.  $y = 4x - 10$ .      27.  $y = \frac{1}{2}x - \frac{5}{2}$ .      29.  $y = 3x - 1$ .  
 31.  $y = 2x + 4$ .      33.  $y = -\frac{1}{3}x - \frac{2}{3}$ .      35.  $y = 2x - 6$ .

## Section 59. Pages 125–126

1.  $(x = 5, y = -2)$ .      3.  $(x = -3, y = 2)$ .      5.  $(x = 1, y = 1/2)$ .  
 7.  $(x = 2, y = -3)$ .      9.  $(x = -5, y = 1)$ .      11.  $(x = -1/5, y = 3/5)$ .  
 13.  $(x = -1, y = -3)$ .      15.  $(x = -2, y = 3)$ .      17.  $(x = 3, y = 0)$ .  
 19.  $(x = 6, y = 12)$ .      21.  $(x = -6, y = 12)$ .      23.  $(x = 3a, y = 2a)$ .  
 25.  $(x = a + 1, y = 2a - 3)$ .      27.  $(x = 3a + b, y = a - 2b)$ .  
 29.  $\left(x = \frac{4b-3a}{11}, y = \frac{5a-3b}{11}\right)$ .      31.  $(x = a + b, y = b - a)$ .  
 33.  $\left(x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1}, y = \frac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1}\right)$ .

## Section 60. Page 128

1. Inconsistent.      3. Dependent.      5. Simultaneous,  $(x = 2, y = 0)$ .  
 7. Simultaneous,  $(x = 2, y = -1)$ .      9. Dependent.  
 11. Inconsistent.

## Section 61. Page 130

1.  $(x = 1, y = 2, z = 3)$ .      3.  $(x = -1, y = 2, z = -3)$ .  
 5.  $(u = 3, v = -2, w = -1)$ .      7.  $(A = 1, B = -1, C = 2)$ .  
 9.  $(x = 2a, y = 3a, z = 4a)$ .      11.  $(x = 1, y = -1, z = 2, t = 3)$ .  
 13.  $(x = 1, y = 0, z = -1, t = 2, u = 3)$ .



## Section 62. Pages 132-133

1.  $(x = 1, y = -2)$ . 3.  $(x = 4, y = 5)$ . 5.  $(x = -1, y = 2)$ .  
 7.  $(x = 2, y = 3)$ . 9.  $(x = -2, y = -1)$ . 11.  $(x = \frac{1}{2}, y = -\frac{1}{3})$ .  
 13.  $(x = \frac{5}{8}, y = 5)$ . 15.  $(x = \frac{1}{2}, y = -1, z = \frac{1}{3})$ .  
 17. Dependent. 19.  $(x = 1/2, y = 1/3, z = -1/2)$ .  
 21.  $(x = 6/5, y = 1/2)$ . 23.  $(x = 0, y = -4/3)$ .

## Section 63. Pages 136-139

1. 12, 18. 3. 30 years. 5. 11 nickels, 9 quarters.  
 7. 40 ft., 10 ft. 9. \$1000 at 4%, \$4000 at 6%. 11. 86.  
 13. 27, 28, 29. 15. 37, 39, 41. 17. 2.4 days. 19. 3 days.  
 21.  $5\frac{5}{8}$  gal. 23. 2 hr., 100 mi. 25. 40 lb. 27. 80 lb.  
 29. 20 mi. per hr., 4 mi. per hr. 31.  $16\frac{4}{11}$  minutes past the hour.  
 33. 11 by 7 feet. 35. 24 mi., 10 mi. per hr.  
 37. 14 lb. at 70¢, 28 lb. at 40¢. 39. 45.

## Section 64. Pages 141-142

1.  $2/x^3$ . 3. 3. 5.  $2y^3/x^2$ . 7.  $b/a$ . 9.  $\frac{b+a}{b-a}$ .  
 11.  $\frac{2b}{2b-a}$ . 13.  $1/8$ . 15. 8. 17. 1. 19.  $1/6$ . 21. 1.  
 23.  $x^6$ . 25.  $x/y^4$ . 27.  $\frac{1}{a^4b^4}$ . 29. 10. 31.  $8y^2$ .  
 33. 720. 35.  $1/xy$ . 37.  $\frac{x^2y^2}{y^2-x^2}$ .

## Section 65. Pages 143-144

1.  $3 \times 10^5$ . 3.  $6.57 \times 10^7$ . 5.  $3.41 \times 10^{-1}$ . 7.  $3.52 \times 10^0$ .  
 9.  $5.69 \times 10^{-2}$ . 11.  $5.61 \times 10^{-7}$ . 13. 1,000,000. 15. 0.0001.  
 17. 65,400,000. 19. 0.0000342. 21. 0.00000753. 23. 120.  
 25.  $5 \times 10^{14}$  per sec. 27. 220. 29.  $1.316385 \times 10^{25}$  lb.  
 31.  $1.6 \times 10^{-19}$ .

## Section 66. Pages 145-146

1. 4. 3.  $1/2$ . 5. 2. 7. 4. 9. 10. 11. 1000. 13. 81.  
 15. 100. 17. 1000. 19. 100. 21.  $1/125$ . 23.  $4a^2$ . 25.  $x^{-9}$ .  
 27.  $a^{15/2}b$ . 29.  $\frac{3}{2}x^2y^2$ . 31.  $(\sqrt[12]{x})^{17}$ . 33.  $x$ .

## Section 68. Pages 149-150

1. 2. 3. 4. 5. 6. 7. 2. 9. 2. 11. 6. 13.  $2\sqrt{3}$ .  
 15.  $5\sqrt{2}$ . 17.  $6\sqrt{3}$ . 19.  $2\sqrt[3]{4}$ . 21.  $\sqrt{2}/2$ . 23.  $\sqrt{6}/4$ . 25.  $\sqrt[12]{2}$ .  
 27.  $\sqrt{5}$ . 29.  $\sqrt{2}$ . 31.  $1/2$ . 33.  $1/2$ . 35. -2. 37. -2.  
 39.  $3x^3y^4$ . 41.  $2x^{18}y^4\sqrt{y}$ . 43.  $-3x^2y^2\sqrt[3]{x^2}$ . 45.  $3x^3y^2\sqrt[3]{2xy^2}$ .  
 47.  $\frac{a^3}{3b^4}$ . 49.  $\frac{5}{2x^3y^4}\sqrt{y}$ . 51.  $\frac{4x^3}{3y^3}\sqrt{2xy}$ . 53.  $\frac{2x^2}{y^2}\sqrt[5]{x^2}$ .



55.  $a\sqrt{b^2 + c^2}$ . 57.  $3\sqrt{x^2 + 1}$ . 59.  $\frac{1 + 3a}{a}\sqrt{a}$ . 61.  $\frac{3 + 2a}{2a}\sqrt{2a}$ .  
 63. 0.632. 65. 1.538. 67. 1.472. 69. 1.186.

## Section 69. Pages 150–151

1.  $6\sqrt{3}$ . 3.  $5\sqrt[3]{5}$ . 5.  $5\sqrt{2}$ . 7.  $3\sqrt{3}$ . 9.  $\frac{11\sqrt{2}}{2}$ . 11.  $\frac{1}{3}\sqrt{3}$ .  
 13.  $-2\sqrt{a}$ . 15.  $(4a - 2b)\sqrt{2x}$ . 17.  $(2a + 3b)\sqrt[3]{2x}$ .  
 19.  $\left(1 - \frac{a}{2}\right)\sqrt{ab}$ . 21.  $\frac{2b}{a^2 - b^2}\sqrt{a^2 - b^2}$ . 23.  $4\sqrt{3} + 4\sqrt{2}$ .  
 25.  $-\frac{b}{a}$ .

## Section 70. Pages 151–152

1.  $\sqrt{10}$ . 3.  $4\sqrt{3}$ . 5.  $10\sqrt{6}$ . 7.  $24\sqrt{3}$ . 9.  $4\sqrt[3]{3}$ . 11.  $6x\sqrt{y}$ .  
 13.  $54x$ . 15.  $48x^2y$ . 17.  $9(a^2 + b^2)$ . 19.  $(x + y)^3$ . 21.  $24x$ .  
 23.  $\sqrt{6} + \sqrt{10}$ . 25.  $2\sqrt{3} - 3\sqrt{2}$ . 27. 1. 29.  $30 - 12\sqrt{6}$ .  
 31.  $9 + 7\sqrt{5}$ . 33.  $-5\sqrt{6}$ . 35.  $6\sqrt{2}$ . 37.  $21 + 8\sqrt{2}$ . 39. 0.  
 41.  $43 + 12\sqrt{2}$ . 43.  $c/a$ .

## Section 71. Page 153

1.  $\frac{1}{2}\sqrt{10}$ . 3.  $\sqrt{5}$ . 5.  $x\sqrt{6}$ . 7.  $\sqrt{15y}/6x^2$ . 9.  $\sqrt[3]{14}/2$ .  
 11.  $\frac{3 + \sqrt{2}}{7}$ . 13.  $3(\sqrt{3} - \sqrt{2})$ . 15.  $\frac{\sqrt{10} + \sqrt{6}}{2}$ . 17.  $\frac{\sqrt{14} + 4}{2}$ .  
 19.  $\sqrt{6} - \sqrt{3} - \sqrt{2} + 1$ . 21.  $\frac{34 - 7\sqrt{10}}{37}$ . 23.  $\frac{3\sqrt{15} + 11}{7}$ .  
 25.  $\frac{\sqrt{x-1} + x - 1}{2 - x}$ . 27. (a) 1.581; (c) 2.236.

## Section 72. Page 154

1.  $\sqrt[6]{x^6}$ . 3.  $x\sqrt[12]{x^5}$ . 5.  $\sqrt[6]{250}$ . 7.  $\sqrt[6]{32}$ . 9.  $\sqrt[6]{5}$ .  
 11.  $\sqrt[12]{2}$ . 13.  $y\sqrt[3]{xy^2}$ .

## Section 73. Page 156

1. 3, -2. 3. 4, -3. 5.  $\pm 4$ . 7.  $\pm 3/2$ . 9. 3, -12.  
 11. 18, -2. 13. -2, -12. 15.  $1/2, -5/3$ . 17.  $1/2, -4/3$ .  
 19. -3,  $-1/2$ . 21.  $\pm\sqrt{3}$ . 23.  $\pm\sqrt{10}/3$ . 25.  $k, k$ .  
 27.  $k/3, -4k/7$ . 29.  $\pm 2, \pm 3$ . 31.  $\pm 1, \pm\sqrt{3}$ . 33.  $\pm 2i$ .  
 35.  $\pm \frac{2}{3}i$ . 37. 18, -2. 39. 4, 5. 41. 5, 0. 43.  $-7/2, 0$ .  
 45. 8, 8. 47.  $-5/2, -5/2$ . 49.  $t = \pm \frac{1}{g}\sqrt{2sg}$ . 51.  $r = \pm \frac{1}{\pi}\sqrt{A\pi}$ .



## Section 74. Page 158

1. 18, -2.      3. -2, -12.      5.  $1/2$ ,  $-5/3$ .      7.  $1/2$ ,  $-4/3$ .  
 9. -3,  $-1/2$ .      11. (a)  $2 \pm \sqrt{3}$ ; (b) 3.732, 0.268.  
 13. (a)  $\frac{5 \pm \sqrt{3}}{2}$ ; (b) 3.366, 1.634.  
 15. (a)  $-1 \pm \sqrt{5}$ ; (b) 1.236, -3.236.      17.  $1 \pm 2i$ .      19.  $-2 \pm 3i$ .  
 21. (a)  $\frac{5 \pm \sqrt{21}}{2}$ ; (b) 4.791, 0.209.      23. 5,  $-4/3$ .  
 25. (a)  $-3 \pm \sqrt{17}$ ; (b) 1.123, -7.123.      27.  $\frac{-p \pm \sqrt{p^2 - 4q}}{2}$ .

## Section 75. Page 160

- 1-26. See §74.      27. -3, 4.      29.  $2/3$ ,  $2/3$ .      31.  $\frac{\sqrt{3} \pm i\sqrt{5}}{4}$ .  
 33.  $\frac{\sqrt{2} \pm i\sqrt{30}}{4}$ .      35.  $\frac{\sqrt{3} \pm \sqrt{21}}{3}$ .      37.  $\frac{h \pm \sqrt{h^2 - k}}{2}$ .  
 39.  $\frac{-d \pm \sqrt{d^2 - ac}}{a}$ .      41.  $a \pm \sqrt{b}$ .      43.  $-i \pm \sqrt{2}$ .

## Section 76. Page 162

1. 64; real, rational, unequal.      3. 169; real, rational, unequal.  
 5. 41; real, irrational, unequal.      7. -11; imaginary, unequal.  
 9. 0; real, rational, equal.      11. -36; imaginary, unequal.  
 13. 25; real, rational, unequal.      15. -5; imaginary, unequal.  
 17. 25; real, unequal.      19. 16; imaginary, unequal.  
 21. 0; imaginary, equal.      23. 9.      25.  $\pm 24$ .      27. 2.  
 29. 0, 4.      31.  $\pm 5$ .      33. 4.      35. (a)  $k < 4$ ; (b)  $k > 4$ .  
 37. (a)  $k > 6$ ,  $k < -6$ ; (b)  $-6 < k < 6$ .

## Section 77. Pages 163-164

11. 2, -1.      13. 1, -3.      15.  $\frac{1}{2}$ ,  $\frac{1}{2}$ .      17.  $\pm 4/3$ .      19. None.  
 21. 3, 0.      29. Tangent.      31. Two points.      33. Not at all.  
 35. Two points.      37. (a)  $k = 1$ ; (b)  $k < 1$ ; (c)  $k > 1$ .  
 39. (a)  $\pm 4$ ; (b)  $k > 4$ ,  $k < -4$ ; (c)  $-4 < k < 4$ .

## Section 78. Pages 165-166

1. 5, 6.      3.  $-5/2$ , -4.      5.  $-1/2$ ,  $-3/2$ .      7. 0,  $-9/2$ .  
 9.  $-5/3$ , 0.      11.  $r_2 = -1$ ,  $h = -3$ .      13.  $r_2 = 3/2$ ,  $k = 1$ .  
 15.  $h = 2$ ; roots  $5/2$ ,  $7/2$ .      17. Roots  $1/2$ ,  $5/2$ ;  $h = 5$ .  
 19.  $h = 2$ ;  $\pm 3/2$ .      21.  $x^2 - 3x - 10 = 0$ .      23.  $6x^2 - x - 1 = 0$ .  
 25.  $6x^2 - 5x - 6 = 0$ .      27.  $x^2 - 18 = 0$ .      29.  $x^2 + 25 = 0$ .  
 31.  $16x^2 - 16x + 1 = 0$ .      33.  $x^2 - 6x + 13 = 0$ .  
 35.  $x^2 - 2ax + a^2 - b = 0$ .



## Section 79. Pages 167-168

1. 3, -1.    3. -2.    5. 0.    7.  $1/2$ , -3.    9. 1,  $-4/3$ .  
 11.  $\pm 3$ ,  $\pm 2$ .    13.  $\pm 3$ ,  $\pm 4/3$ .    15. 7, 1,  $4 \pm \sqrt{2}$ .    17. 4, -2, 1, 1.  
 19. -2,  $\pm i$ .    21. -2,  $1 \pm i\sqrt{3}$ ,  $1, \frac{-1 \pm i\sqrt{3}}{2}$ .

## Section 80. Page 169

1. 7.    3. 4.    5. 4, 20.    7. No root.    9. -2.    11.  $9/8$ .  
 13. No root.    15.  $65/32$ .    17. 1.    19. -2.

## Section 81. Pages 170-172

1. 8 yd., 7 yd.    3. 5 ft., 12 ft.    5. 23, 24.    7. 27, 29.  
 9. 4 inches.    11. 20 miles per hour, 30 miles per hour.  
 13. 1.5 seconds.    15. 23.    17. A, 6 days; B, 12 days.    19. 2 yd.  
 21. 73.    23. 28 ft., 22 ft.    25.  $\frac{1}{2L}(-R \pm \sqrt{R^2 - (4L/C)})$ .  
 27. 16 miles per hour; 1.5 and 2.5 hours.    29. 37.    31. 3%.

## Section 82. Pages 174-175

1. Min. (3, -7).    3. Max. (2, 7).    5. Min. (3, -11).  
 7. Min. ( $-\frac{1}{2}$ , 2).    9. Max. ( $-1/2$ , 16.5).    11. 10, 10.    13. 10, 10.  
 15. 3 sec., 144 ft.    17. 65 cents.    19. 50 yd., 100 yd.; 5,000 sq. yd.

## Section 83. Page 178

1. Circle.    3. Parabola.    5. Hyperbola.    7. Hyperbola.    9. Ellipse.  
 11. One point.    13. Non-existent.    15. Two intersecting straight lines.  
 17. Two coincident straight lines.

## Section 84. Page 179

1.  $x = 4, y = 3; x = -4, y = -3$ .    3.  $x = 5, y = 0; x = 3, y = 4$ .  
 5.  $x = 3, y = 1; x = 0, y = 2$ .    7.  $x = 3, y = 4; x = 4, y = 3$ .  
 9.  $x = 4, y = -4; x = 1, y = 2$ .  
 11.  $x = 3, y = 2; x = 21/4, y = -19/4$ .  
 13.  $x = 0, y = 0; x = 2, y = 2$ .    15.  $x = 4, y = \pm 3$ .  
 17.  $x = 3, y = \pm 1; x = -3, y = \pm 1$ .  
 19.  $x = 3, y = \pm 2; x = -3, y = \pm 2$ .  
 21.  $x = 1, y = \pm 3; x = 8, y = \pm 8.5$ .  
 23.  $x = 4, y = \pm 3; x = -4, y = \pm 3$ .

## Section 85. Pages 180-181

13.  $x = 3, y = 2; x = 1, y = -2$ .    15.  $x = 0, y = 1; x = 3, y = 4$ .  
 17.  $x = -3, y = 2; x = -1, y = -2$ .  
 19.  $x = 2, y = 3; x = -17/11, y = 7/11$ .  
 23.  $k = \sqrt{2}; x = y = \sqrt{2}/2$ .



## Section 86. Pages 182-184

1.  $x = -3, y = 2; x = -1, y = -2$ .
3.  $x = -3, y = -2; x = -1, y = 2$ .
5.  $x = 0, y = 0; x = 0, y = 4$ .      7.  $x = \pm 3, y = \pm 2$
9.  $x = \pm 2, y = \pm \sqrt{3}$ .      11.  $x = \pm 3/2, y = \pm \sqrt{2}/2$
13.  $x = \pm 1/3, y = \pm \sqrt{3}/3$ .
15.  $x = 4, y = 3; x = 3, y = 4; x = -4, y = -3; x = -3, y = -4$ .
17.  $x = -4, y = -3; x = 4, y = 3; x = -5, y = 0; x = 0, y = 5$ .
19.  $x = 5, y = 4; x = -5, y = -4; x = 3\sqrt{2}, y = 3; x = -3\sqrt{2}, y = -3$ .
21.  $x = 2, y = 1; x = -2, y = -1; x = 3, y = 2; x = -3, y = -2$ .
23.  $x = 0, y = \pm 1; x = 3, y = -2; x = -3, y = 2$ .
25.  $x = 1, y = 2; x = -1, y = -2; x = 1/2, y = 5/2; x = -1/2, y = -5/2$ .
27.  $x = 2, y = 1; x = 1, y = 2; x = -2, y = -1; x = -1, y = -2$ .

## Section 87. Pages 185-186

1. 11, 13.      3. 15 ft., 8 ft.      5. 16 ft., 9 ft.
7. 30 miles per hour; 60 miles.      9. 7.5 ft., 18 ft.
11. 28 ft., 18 ft.      13. 32 items at 15 cents each.

## Section 88. Pages 189-191

1.  $1/3$ .      3.  $3/2$ .      5.  $2x/3$ .      7.  $(x+1)/2$ .      9.  $3/4$ .      11.  $3/8$
13.  $1/2$ .      15. 3.      17. 7.      19. 0.      21. 85.      23. 56.
25.  $b^2$ .      27. 1.      29.  $\pm 6$ .      31.  $\pm 1/8$ .      33.  $\pm 1/16$ .
35.  $\pm \sqrt{ab}$ .      37.  $\pm (x+3)$ .      39. 27.      41. 18.      43.  $b^2 a$ .
45. 8 in., 12 in.      47. 22 in., 33 in., 77 in.
49.  $6\frac{2}{3}$  in., 8 in.      51. 38 ft.

## Section 89. Pages 193-194

1. (a)  $y = 3x$ ; (b) 18; (c) tripled.      3. (a)  $y = \frac{1}{2}x$ ; (b) 7.
5. (a)  $r = \frac{1}{8}v^2$ ; (b) 41,667 ft.; (c) 600 ft./sec.
7. 95.2 lb. approx.      9. 576 ft.
11. (a)  $V = \frac{1}{3}\pi r^2 h$ ; (b)  $V = 12\pi$  cu. in.      13. 3.1 sec.
15. 300 cu. ft.      17. 13 amp.

## Section 90. Pages 196-197

13.  $-i$ .      15. 1.      17.  $-1$ .      19.  $3 - 4i$ .      21.  $5 + 2i$ .      23.  $-3 - 2i$ .
25.  $-1 - 2i\sqrt{2}$ .      27.  $x = 2, y = -3$ .      29.  $x = 4, y = 0$ .
31.  $x = -3, y = 2$ .      45.  $x = 4, y = 3$ .      47.  $x = -2, y = 2$ .
49.  $x = 1, y = \frac{1}{2}$ .



## Section 91. Pages 198–199

1.  $5i$ . 3.  $5i$ . 5.  $-6$ . 7.  $-6$ . 9.  $-6i$ . 11.  $72i$ .  
 33. 11. 35.  $-1 + 7i\sqrt{2}$ . 37.  $2 - 10i\sqrt{2}$ . 39. 1. 41.  $i$ .  
 43.  $-i$ . 45.  $-i$ . 47.  $\frac{1}{13} + \frac{8}{13}i$ . 49.  $\frac{-18}{13} + \frac{14}{13}i$ . 51.  $\frac{1}{2} - \frac{3}{4}i$ .  
 53.  $-i$ . 55.  $\frac{6}{5} - \frac{3}{5}i$ . 57.  $i$ . 59.  $\frac{2}{7} - \frac{\sqrt{3}}{7}i$ .  
 61.  $\frac{1}{18} - \frac{5\sqrt{5}}{18}i$ . 63.  $\frac{-5}{29} - \frac{2}{29}i$ . 65.  $\frac{3}{11} + \frac{\sqrt{2}}{11}i$ . 67.  $\frac{1}{2} - \frac{1}{2}i$ .  
 69.  $-\frac{3}{13} - \frac{2}{13}i$ . 71.  $\frac{1}{4}i$ . 73.  $\frac{2}{7} - \frac{\sqrt{3}}{7}i$ .

## Section 94. Pages 204–205

1. 1, 2,  $\sqrt{3}/2$ ,  $\sqrt{3}/3$ . 3.  $\sqrt{3}/2$ ,  $1/2$ ,  $\sqrt{3}$ . 5.  $\sqrt{2}/2$ ,  $-\sqrt{2}/2$ , 1.  
 7.  $-1/2$ ,  $-\sqrt{3}/2$ ,  $\sqrt{3}/3$ . 9.  $-\sqrt{3}/2$ ,  $-1/2$ ,  $\sqrt{3}$ .  
 11.  $-\sqrt{2}/2$ ,  $\sqrt{2}/2$ ,  $-1$ . 13.  $1/2$ ,  $\sqrt{3}/2$ ,  $\sqrt{3}/3$ .  
 15.  $1/2$ ,  $-\sqrt{3}/2$ ,  $-\sqrt{3}/3$ . 17. 0, 1, 0. 19. 0,  $-1$ , 0.  
 21. 0, 1, 0. 23. 0.3420, 0.9397, 0.3640.  
 25. 0.9397,  $-0.3420$ ,  $-2.747$ . 27.  $-0.3420$ ,  $-0.9397$ , 0.3640.  
 29.  $-0.9397$ , 0.3420,  $-2.747$ . 31. 0.3420, 0.9397, 0.3640.  
 33.  $-0.7660$ ,  $-0.6428$ , 1.192. 35. 0.4067,  $-0.9135$ ,  $-0.4452$ .  
 37. 0.8192,  $-0.5736$ ,  $-1.428$ . 39.  $-0.9063$ ,  $-0.4226$ , 2.145.

## Section 95. Page 207

1.  $\sqrt{3} + i$ . 3.  $3\sqrt{2} + 3i\sqrt{2}$ . 5.  $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ . 7.  $3i$ . 9.  $-4$ .  
 11. 2. 13.  $-2\sqrt{2} - 2i\sqrt{2}$ . 15.  $-5i$ . 17.  $\sqrt{3} - i$ .  
 19.  $2(\cos 60^\circ + i \sin 60^\circ)$ . 21.  $4\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$ .  
 23.  $4(\cos 120^\circ + i \sin 120^\circ)$ . 25.  $2(\cos 135^\circ + i \sin 135^\circ)$ .  
 27.  $3(\cos 90^\circ + i \sin 90^\circ)$ . 29.  $5(\cos 0^\circ + i \sin 0^\circ)$ .  
 31.  $2(\cos 180^\circ + i \sin 180^\circ)$ . 33.  $3(\cos 270^\circ + i \sin 270^\circ)$ .  
 35.  $4(\cos 330^\circ + i \sin 330^\circ)$ . 37.  $1(\cos 180^\circ + i \sin 180^\circ)$ .  
 39.  $8(\cos 270^\circ + i \sin 270^\circ)$ .  
 41.  $8(\cos [0^\circ + k \cdot 360^\circ] + i \sin [0^\circ + k \cdot 360^\circ])$ ,  $k = 0, \pm 1, \pm 2, \dots$ .  
 43.  $8(\cos [90^\circ + k \cdot 360^\circ] + i \sin [90^\circ + k \cdot 360^\circ])$ ,  $k = 0, \pm 1, \pm 2, \dots$ .  
 45.  $2(\cos [330^\circ + k \cdot 360^\circ] + i \sin [330^\circ + k \cdot 360^\circ])$ ,  $k = 0, \pm 1, \pm 2, \dots$ .  
 47.  $1.8794 + 0.6840i$ . 49.  $-0.9848 - 0.1736i$ .  
 51. (a)  $0^\circ$ ; (b)  $180^\circ$ ; (c)  $90^\circ$ ; (d)  $270^\circ$ .

## Section 96. Page 209

1. (a)  $6(\cos 45^\circ + i \sin 45^\circ)$ ; (b)  $3\sqrt{2} + 3i\sqrt{2}$ .  
 3. (a)  $8(\cos 180^\circ + i \sin 180^\circ)$ ; (b)  $-8$ .



5. (a)  $3(\cos 90^\circ + i \sin 90^\circ)$ ; (b)  $3i$ .

7. (a)  $5(\cos 120^\circ + i \sin 120^\circ)$ ; (b)  $-\frac{5}{2} + \frac{5\sqrt{3}}{2}i$ .

### Section 97. Page 210

1.  $27(\cos 180^\circ + i \sin 180^\circ) = -27$ .

3.  $27(\cos 90^\circ + i \sin 90^\circ) = 27i$ .      5.  $32(\cos 75^\circ + i \sin 75^\circ)$ .

7.  $64(\cos 270^\circ + i \sin 270^\circ) = -64i$ .

9.  $\frac{1}{16}(\cos 120^\circ + i \sin 120^\circ) = -\frac{1}{32} + \frac{\sqrt{3}}{32}i$ .      11. 16.

13.  $-2 - 2i$ .      15. 1.      17.  $-16$ .      19.  $-1$ .

### Section 98. Pages 212-213

1.  $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ .      3.  $\frac{3\sqrt{3}}{2} + \frac{3i}{2}, -\frac{3\sqrt{3}}{2} + \frac{3i}{2}, -3i$ .

5.  $4i, -2\sqrt{3} - 2i, 2\sqrt{3} - 2i$ .

7.  $+1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$ .

9.  $1 + i\sqrt{3}, -2, 1 - i\sqrt{3}$ .

11. 2,  $2(\cos 72^\circ + i \sin 72^\circ)$ ,  $2(\cos 144^\circ + i \sin 144^\circ)$ ,  
 $2(\cos 216^\circ + i \sin 216^\circ)$ ,  $2(\cos 288^\circ + i \sin 288^\circ)$ .

13.  $\sqrt{2} + i\sqrt{2}, 2(\cos 165^\circ + i \sin 165^\circ), 2(\cos 285^\circ + i \sin 285^\circ)$ .

15.  $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ .

17.  $\sqrt{3} + i, -1 + i\sqrt{3}, -\sqrt{3} - i, 1 - i\sqrt{3}$ .      19. 2,  $2i, -2, -2i$ .

21.  $2\sqrt{3} + 2i, -2\sqrt{3} - 2i$ .

### Section 99. Page 213

1. 2,  $-1 + i\sqrt{3}, -1 - i\sqrt{3}$ .      3.  $\sqrt{3} - i, -\sqrt{3} - i, 2i$ .

5.  $\sqrt{2} + i\sqrt{2}, -\sqrt{2} - i\sqrt{2}$ .

7.  $\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -3i$ .

9.  $\sqrt{3} + i, -1 + i\sqrt{3}, -\sqrt{3} - i, 1 - i\sqrt{3}$ .

### Section 101. Pages 217-218

1. 1.      3.  $-1$ .      5. 20.      7.  $-21$ .      9. True.      11. True.      13. False.  
15. True.      17. False.      19. True.

21. (a) True. (b) True. (c) True. (d) True. (e) True. (f) True.

23. No.      25. 3.

### Section 102. Pages 220-221

1. Quotient  $= 4x^2 + 8x + 14$ ; remainder  $= 33$ .

3. Quotient  $= x^2 + x - 3$ ; rem.  $= 2$ .

5. Quotient  $= 2x^2 + 4x + 6$ ; rem.  $= 8$ .



7. Quotient  $= 6x^3 - 12x^2 + 3$ ; rem.  $= 2$ . 9. 22, -18. 11.  $5/8$ , 0.  
 13. -6.976, -0.916. 15. True,  $x^2 + x - 3$ . 17. True,  $x^2 - 3x + 4$ .  
 19. True,  $x^4 - 2x^3 + 4x^2 - 8x + 16$ . 21. False. 23. False.  
 25. True. 27. False. 29. True. 31. True. 33. True. 35.  $k = -7$ .

## Section 103. Pages 222-223

1. 3, 2, -1. 3. -2,  $-1 \pm i\sqrt{3}$ . 5. -3,  $-1/2$ , 2, 3.  
 7. 1,  $-2/3$ ,  $\frac{-5 \pm \sqrt{21}}{2}$ . 9.  $3/2$ ,  $1/3$ ,  $\pm i\sqrt{2}$ .

## Section 104. Page 226

1. 3,  $1/2$ , -3. 3. 1, 1,  $-1 \pm i\sqrt{3}$ . 5. -2,  $-1/2$ ,  $3/2$ .  
 7. 0, -1, 2,  $5/2$ . 9.  $x^3 - 7x + 6 = 0$ .  
 11.  $2x^3 - 3x^2 - 11x + 6 = 0$ . 13.  $x^3 - 4x^2 + 4x = 0$ .  
 15.  $4x^4 + 8x^3 - 11x^2 + 3x = 0$ . 17.  $x^4 - 3x^2 = 0$ .  
 19.  $x^4 - 2x^3 + 2x^2 - 8x - 8 = 0$ .  
 21.  $x^4 - 6x^3 + 14x^2 - 22x + 5 = 0$ .  
 23.  $6x^4 - 7x^3 + 26x^2 - 28x + 8 = 0$ . 25.  $x^3 - 6x^2 + 12x - 8 = 0$ .  
 27.  $x^4 - 3x^2 + 2x = 0$ . 29. 3, 3,  $\pm i$ . 31.  $1/2$ ,  $1/2$ ,  $-\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$ .  
 33.  $x^3 - ix^2 - 4x + 4i = 0$ . 35.  $x^4 - x^3 - x^2 - 4 = 0$ .

## Section 106. Pages 229-230

1.  $3 - 2i$ . 3.  $-3 + 3i$ . 5.  $-3i$ . 7.  $x^3 - 7x^2 + 15x - 25 = 0$ .  
 9.  $4x^4 - 4x^3 + x^2 + 6x + 2 = 0$ . 11.  $1 \pm i$ ,  $\pm i$ .  
 13.  $\pm i$ ,  $\frac{-3 \pm \sqrt{5}}{2}$ . 15.  $\pm i$ ,  $i$ ,  $\frac{-3 \pm \sqrt{5}}{2}$ . 21. False.  
 23.  $x^3 - 5x^2 + 5x - 1 = 0$ . 25.  $x^4 - 4x^3 + 2x^2 + 4x = 0$ .  
 27.  $2 \pm \sqrt{3}$ ,  $\pm i$ .

## Section 107. Pages 234-235

1. 1, 3, -2. 3. 1, 6, -2. 5. 1, 5, -2, -3. 7.  $2/3$ ,  $-1/3$ ,  $\pm i$ .  
 9.  $-1/2$ ,  $-1/2$ ,  $\frac{1 \pm \sqrt{5}}{2}$ . 11. None. 13. 3, 3,  $\pm i$ .  
 15. 0, 2,  $\pm i$ . 17. 0,  $-1/2$ ,  $-1 \pm i$ . 19.  $3/2$ ,  $3/2$ ,  $\pm i$ .  
 21. 2,  $-3/2$ ,  $\pm 2i$ . 23.  $-3/2$ ,  $2 \pm \sqrt{3}$ . 25.  $(x + 2)(x - 3)(x + 4)$ .  
 27.  $(x - 2)(x + 2)(2x - 1)$ .  
 29.  $(2x + 3)(x - 2 - \sqrt{3})(x - 2 + \sqrt{3})$ .  
 31.  $(2x + 1)(2x + 1)(x + 1 - i)(x + 1 + i)$ .  
 33.  $(2x - 1)(x^2 + 4)$ . 35.  $(x - 2)(x - 2)(x^2 + x + 1)$ .  
 37.  $(2x - 1)(3x + 2)(x^2 + 2x + 4)$ . 39. False.

## Section 108. Pages 235-236

21.  $\pm 1.4$ . 23. 1.3. 25. No.



## Section 109. Page 236

1.  $x^3 + 2x^2 - 5x - 6 = 0$ .      3.  $x^3 + 4x^2 + x - 6 = 0$ .  
 5.  $9x^4 + 3x^3 + 7x^2 + 3x - 2 = 0$ .      7.  $3x^4 + 2x^2 + 5 = 0$ .  
 9.  $-1, -2$ .      11.  $-2$ .

## Section 110. Pages 239–240

1. 2,  $-2$ .      3. 2,  $-1$ .      5. 2,  $-1$ .      7.  $\pm 2$ .      9. 3,  $-1$ .  
 11. None.      13. None.      15.  $\pm 1$ .

## Section 111. Pages 243–244

1. (a) 1 pos., 2 imag. (b) Same.  
 3. (a) 2 pos., 2 imag.; or no pos., 4 imag. (b) Same.  
 5. (a) 1 pos., 1 neg., 2 imag.; or 4 imag. (b) 1 pos., 1 neg., 2 imag.  
 7. (a) 4 imag. (b) Same.  
 9. (a) 1 pos., 1 neg., 2 imag.; or 4 imag. (b) 1 pos., 1 neg., 2 imag.  
 11. (a) 3 pos.; or 1 pos., 2 imag. (b) Same.  
 13. (a) 2 pos., 2 neg.; or 2 pos., 2 imag.; or 2 neg., 2 imag.; or 4 imag.; or 1 pos., 1 neg., 2 imag. (b) Same except 1 pos., 1 neg., 2 imag. now ruled out.  
 15. (a) 1 neg., 4 imag. (b) Same.  
 17. (a) 1 pos., 1 neg., 4 imag.; or 6 imag. (b) 1 pos., 1 neg., 4 imag.  
 19. (a) one zero root, 2 imag. (b) Same.      27. 2,  $-1$ .

## Section 112. Page 247

1. 1.41.      3. 1.15.      5. 2.91.      7.  $-1.24$ .      9. 1.59.

## Section 113. Pages 248–249

11. 1.236.      13. 1.817.      15. 2.080.

## Section 114. Pages 251–252

1.  $x^3 - 7x + 6 = 0$ .      3.  $x^3 + 2x^2 - 3x = 0$ .  
 5.  $2x^3 - 17x^2 + 40x - 25 = 0$ .      7.  $x^3 - 1.4x^2 - 1.68x + 1.728 = 0$ .  
 9.  $x^3 - 3.94x^2 + 0.8412x + 6.018408 = 0$ .  
 11.  $x^3 - 6x^2 - 16x + 96 = 0$ .      13.  $x^3 - 40x^2 + 100x + 6,000 = 0$ .  
 15.  $x^4 + 78x^3 + 2,144x^2 + 21,922x - 7,249 = 0$ .

## Section 115. Page 255

17. 3.732.      19. 1.236, 1.414.      21. 3.684.      23.  $-2.844$ .  
 25. 0.268, 2.414, 3.732,  $-0.414$ .      27. 0.92.      29.  $-0.4, -40.6$ .

## Section 116. Pages 257–258

1.  $5/2, -1/2$ .      3. Roots 2,  $-1, -3$ ;  $k = -5$ .  
 5. Roots 4,  $(-1 \pm i\sqrt{3})/2$ ;  $k = -3$ .      7. Roots 1, 1,  $-2, -2$ ;  $k = -4$ .  
 9. Roots 3,  $-2, 1/2$ ;  $h = -11, k = 6$ .



## Section 118. Page 260

1.  $-1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ .      3.  $1 + \omega, 1 + \omega, 2\omega^2$ .

## Section 119. Page 262

1.  $\frac{1}{2}(-\sqrt{2} \pm \sqrt{2\sqrt{2}-2}), \frac{1}{2}(\sqrt{2} \pm i\sqrt{2\sqrt{2}+2})$ .

## Section 120. Page 265

1. 5.      3. -2.      35. 6.      37. 2, 3.

## Section 121. Pages 266-267

1. 1.      3. 10.      5. 0.      7. 2.      9. 1, -11/3.  
11.  $a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$ .

## Section 122. Page 269

11.  $i_1 = 2$  amp.,  $i_2 = 2$  amp.,  $i_3 = 0$  amp.

## Section 123. Page 271

1. 14 square units.      3. 13 square units.      7.  $y = x + 1$ .      9.  $x = 2$ .

## Section 124. Page 273

1.  $a_1b_2 - a_2b_1$ .

## Section 125. Pages 277-278

3.  $x = 6$ .      5.  $\begin{vmatrix} 0 & -5 & -11 \\ 0 & 17 & 21 \\ 1 & 3 & 4 \end{vmatrix}$       7.  $\begin{vmatrix} 2 & -5 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & -6 & 5 & 0 \\ 4 & -5 & 5 & 14 \end{vmatrix}$

## Section 126. Page 282

7. -17.      9. -5.      11. 0.      13. 132.      15. 17.

## Section 127. Page 285

1.  $x = 1, y = -1, z = 2, t = 1/3$ .      3.  $x = 1/2, y = 3/2, z = 0, t = 1$ .  
5.  $x = -2, y = 1, z = -1, t = 3$ .      7.  $x = 3, y = 1/2, z = -1, t = 2$ .

## Section 128. Page 287

1.  $x = m, y = 0, z = -m; x = 1, y = 0, z = -1$ ; or  $x = 2, y = 0, z = -2$ .  
3.  $x = m, y = -2m, z = 2m; x = 1, y = -2, z = 2$ ; or  $x = 2, y = -4, z = 4$ .  
5. No non-trivial solutions.      7.  $k = 1$ .

## Section 129. Pages 289-290

3.  $x = 5, y = 2$ .      5. Inconsistent.      7. Inconsistent.  
9.  $x = a + 2, y = 4a - 2, z = a$ , where  $a$  is arbitrary.  
11. Inconsistent.



## Section 130. Page 292

1.  $b_0a_1^2 - b_1a_0a_1 + b_2a_0^2$ .

## Section 131. Pages 293–294

1.  $(x = 0, y = 0), (x = -1, y = 1), (x = 9, y = 6), (x = 2, y = -1)$ .  
 3.  $(x = 0, y = 0), (x = -2, y = 2), (x = 3, y = -1)$ .  
 5.  $(x = 0, y = 0), (x = 1, y = 1)$ .

## Section 132. Pages 297–298

1. 120.      3. 20.      5. (a) 60; (b) 125.      7. 6,840.      9. 216.  
 11. 180.      13. (a) 48; (b) 72.      15. (a) 604,800; (b) 10,000,000.  
 17. (a) 840; (b) 120; (c) 240.      19. 36.  
 21. (a) 325; (b) 130; (c) 282.      23. 47.

## Section 133. Page 300

1. 720.      3. 72.      5. 72.      7. 210.      9. 15.      11. 60.      13. 14,400.  
 15. 576.      17. 64.      19. 195.      21. 162.

## Section 134. Pages 301–302

1. 15,120.      3. 12,600.      5. 35.      7. 210.

## Section 135. Pages 302–303

1. 120.      3. 60.      5. 12.      7. (a) 48; (b) 72.

## Section 136. Pages 304–305

1. 10.      3. 35.      5. 190.      7. 120.      9. 35.      11. 4.      13. 20.  
 15. 350.      17. (a) 3,003; (b) 1,001; (c) 2,002.      19. (a) 112; (b) 196.  
 21. 403,200.      23. 5,040.      25. (a) 45; (b) 9.      27. 2,520.  
 29. 7,200.      31. 10.

## Section 137. Page 308

1.  $1/6$ .      3.  $1/36$ .      5. (a)  $1/4$ ; (b)  $1/13$ ; (c)  $1/52$ .  
 7. (a)  $1/4$ ; (b)  $1/2$ ; (c)  $1/4$ .      9.  $1/2$ .      11.  $2/3$ .      13.  $1/4$ .  
 15.  $4/35$ .      17. (a)  $2/5$ ; (b)  $1/2$ .      19. (a)  $2/13$ ; (b)  $4/13$ .

## Section 138. Page 309

1. 93.3%.      3. 46.7%.      5. 50.1%.

## Section 139. Pages 312–314

1.  $1/36$ .      3. (a)  $1/28,561$ ; (b)  $1/270,725$ .      5. (a)  $3/11$ ; (b)  $2/33$ .  
 7.  $625/3,888$ .      9. (a)  $15/128$ ; (b)  $121/128$ .  
 11. (a)  $125/3,888$ ; (b)  $69/1,944$ ; (c)  $3,875/3,888$ .  
 13. (a)  $1/7,776$ ; (b)  $1/60,466,176$ ; (c) No.      15.  $11/12$ .      17. 27 to 8.  
 19.  $7/10$ .      21.  $61/125$ .      23. (a)  $91/216$ ; (b) 125 to 91.  
 25. 0.057395628 or about 6%.  
 27. (a)  $53,760/9,765,625$  or about  $1/2$  of 1%; (b)  $62,201/9,765,625$  or about  $2/3$  of 1%.



## Section 140. Page 316

1. (a) 2, 4, 6, 8, 10; (b) 20; (c)  $2k$ ; (d)  $2(k+1) = 2k+2$ .
3. (a) 1, 4, 9, 16, 25; (b) 100; (c)  $k^2$ ; (d)  $(k+1)^2 = k^2 + 2k + 1$ .
5. (a) 2, 6, 12, 20, 30; (b) 110; (c)  $k(k+1) = k^2 + k$ ;  
(d)  $(k+1)(k+2) = k^2 + 3k + 2$ .
7. (a)  $\frac{1}{2}, \frac{-1}{6}, \frac{1}{12}, \frac{-1}{20}, \frac{1}{30}$ ; (b)  $\frac{-1}{110}$ ; (c)  $\frac{(-1)^{k+1}}{k(k+1)}$ ; (d)  $\frac{(-1)^{k+2}}{(k+1)(k+2)}$ .

## Section 143. Pages 326–327

1.  $x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7$ .
3.  $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$ .
5.  $32a^{15} + 240a^{12}b^2 + 720a^9b^4 + 1,080a^6b^6 + 810a^3b^8 + 243b^{10}$ .
7.  $a^3 - 3a^2b + 3ab^2 - b^3$ .
9.  $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ .
11.  $x^4/y^4 - 4x^2/y^3 + 6/y^2 - 4/x^2y + 1/x^4$ .
13.  $x^8/16 - x^5 + 6x^2 - 16/x + 16/x^4$ . 15.  $680x^{14}h^3$ . 17.  $3,003x^{10}h^5$ .
19.  $\frac{15}{2}a^{14}b^6$ . 21.  $70a^4b^4$ . 23.  $-63x^8$ . 25.  $-198/x^4$ .
27.  $105x^6/32$ . 29.  $-14x^7$ . 31.  $6x^3$ . 33.  $90,720x^4$ .
35. 1.094. 37. 0.942. 39. 1.243. 41. 0.868.

## Section 145. Page 329

1.  $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{8}{48}x^3 + \dots$
3.  $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \dots$
5.  $1 + x + x^2 + x^3 + \dots$  7. 4.12. 9. 4.02. 11. 5.29.
13. 7.071. 15. 2.924.

## Section 146. Pages 332–333

1.  $d = 4, l = 39, s_{10} = 210$ . 3.  $d = -3, l = -11, s_8 = -4$ .
5.  $d = 5/3, l = 61/3, s_{12} = 134$ . 7. 12,960. 9. (a) 27; (b) 1,026.
11. (a) 19; (b) 2,508. 13. 5, 7.1, 9.2, 11.3, 13.4.
15. -3, -1, 1, 3, 5, 7, 9.
17.  $n(n+1)$ . 19.  $2n(n+1)$ . 21.  $a = 4, l = 40$ .
23.  $d = 4, s_n = 738$ .
25.  $n = 11, s_n = 165$ . 27.  $a = 3, s_n = 429$ . 29.  $d = 4, n = 18$ .
31.  $d = \frac{l^2 - a^2}{2s_n - a - l}$ . 33.  $d = \frac{2(nl - s_n)}{n^2 - n}$ . 35.  $a = \frac{2s_n - n(n-1)d}{2n}$ .
37. 73. 39. \$6,750. 41. (a) \$3,040; (b) \$23,200.
43.  $a = p + q, d = p, l = pn + q, s_n = \frac{n}{2}(2q + [n+1]p)$ .

## Section 147. Pages 335–336

1.  $r = 3, l = 162, s_5 = 242$ . 3.  $r = -2, l = -96, s_6 = -63$ .
5.  $r = -1/2, l = 3/16, s_5 = 33/16$ . 7. 10, 20, 40, 80.
9. -54, 18, -6, 2. 11.  $n = 6, s_n = 1,008$ . 13.  $n = 5, s_n = 53\frac{7}{8}$ .
15. 32. 17. 8. 19. 2. 21. 256. 23. 372 ft. 25. \$2,342.56.
27. The geometric mean or 40%. 29. 64/729.



## Section 148. Page 337

1.  $1/23$ . 3.  $1/9, 1/14, 1/19$ . 5.  $1/10, 1/6, 1/2$ . 7.  $20/7$ . 9.  $72/59$ .

## Section 149. Pages 338–339

1.  $3/2$ . 3.  $2/3$ . 5. 9. 7.  $2/3$ . 9.  $2/11$ . 11.  $1/6$ .  
13.  $3/13$ . 15.  $23/11$ . 17. 120 ft. 19. 160 ft.

## Section 151. Pages 342–343

1.  $n(n+1)(n+2)/3$ . 3.  $n(n+1)(n+2)/6$ .  
5.  $n(n+1)(2n+1)(3n^2+3n-1)/30$ .  
7. (a)  $n(n+1)(n+2)/6$ ; (b) 1,140.  
9. (a)  $n(n+1)(2n+1)/6$ ; (b) 2,109.  
11. (a)  $n^2 - n + 1$ ;  
(b)  $n^2 - n + 1 + (n-1)(n-2)(n-3)(n-4)(n-5)$ ; (c) No.  
13. (a)  $n^3 - n^2 + 1$ ;  
(b)  $n^3 - n^2 + 1 + (n-1)(n-2)(n-3)(n-4)(n-5)$ ; (c) No.

## Section 154. Pages 346–347

1.  $x < 4$ . 3.  $x < 3$ . 5.  $x > 13/14$ .  
7.  $-2 < x < 2$ , or  $|x| < 2$ . 9.  $x > 4$ ,  $x < -4$ , or  $|x| > 4$ .  
11.  $2 < x < 3$ .  
13.  $x > 4$ ,  $x < 2$ . 15.  $-2 < x < 1$ . 17.  $2 < x < 3$ .  
19.  $x < -1$ ,  $3 < x < 4$ .

## Section 155. Page 348

21.  $x > 1$ ,  $x < -4$ . 23.  $-3 < x < 1$ .

## Section 156. Pages 349–350

1.  $\log_5 25 = 2$ . 3.  $\log_{10} 100 = 2$ . 5.  $\log_8 2 = 1/3$ .  
7.  $\log_5 \frac{1}{25} = -2$ .  
9.  $\log_5 1 = 0$ . 11.  $2^4 = 16$ . 13.  $10^{-2} = 0.01$ . 15.  $10^0 = 1$ .  
17.  $5^3 = 125$ . 19.  $10^{-1} = 0.1$ . 21. 5. 23.  $2/3$ . 25. 1. 27. 0.  
29.  $3/2$ . 31. 4. 33. 5. 35.  $4/3$ . 37.  $-1/3$ . 39.  $-1/3$ .  
41. 4. 43.  $1/64$ . 45. 2. 47. 2. 49. 4.

## Section 157. Page 351

1. 1.1461. 3. 0.5441. 5. 1.0791. 7.  $-0.6990$ . 9. 0.1505.  
11.  $-0.0416$ . 13. 0.5411. 15. 0.6990. 17. 1.1128. 19. 0.9065.

## Section 158. Page 354

1. 2.0934. 3.  $8.0934 - 10$ . 5. 1.5977. 7.  $8.6385 - 10$ .  
9.  $6.5391 - 10$ . 11. 2.5563. 13.  $7.6990 - 10$ . 15. 33.2. 17. 0.332.  
19. 49.8. 21. 0.545. 23. 9,370.  
25. (a)  $10^{2.9355} = 862$  approx.; (b)  $\sqrt[10,000]{10^{29,355}} = 862$  approx.



## Section 159. Page 356

1. 2.3824.      3. 0.4709.      5. 8.9301 - 10.      7. 9.6301 - 10.  
 9. 0.6037.      11. 155.5.      13. 3.447.      15. 0.6303.  
 17. 0.08008.      19. 0.02044.

## Section 160. Pages 357-358

1. 3.21.      3. 14.20.      5. 5.000.      7. One.      9. Two.  
 11.  $3.15 \leq 3.2 < 3.25$ ,  $3.195 \leq 3.20 < 3.205$ .

## Section 161. Pages 359-361

1. 1.72.      3. 2.59.      5. 0.783.      7. 40.7.      9.  $128 \times 10$ .  
 11. 0.205.      13. 1.092.      15. 6.321.      17.  $-3,168 \times 10$ .      19. 2.663.  
 21. 1.93.      23. 0.0545.      25. \$229.70.      27. \$675.60.  
 29. 18 years approx.      31. 197,000,000 sq. mi.  
 33. \$10,720,000. approx.      35. 2 secs.      37. (a) \$328; (b) \$107.  
 39. 120,000 sq. in. approx.      41. 5.337.

## Section 162. Page 362

1. 9.6925 - 10.      3. 0.0501.      5. 7.2532 - 10.

## Section 163. Page 363

1. (a) 1.099; (b) 0.477.      3. (a) 0.405; (b) 0.176.  
 5.  $\log_e x = 1.386$ .      7.  $\log_{10} x = 0.8407$ .

## Section 164. Page 365

3. 1.43.      5. -0.690.      7. 175.      9.  $n = \frac{\log A - \log P}{\log(1 + r)}$ .  
 11.  $n = \frac{\log(si + 1)}{\log(1 + i)}$ .  
 13.  $n = \frac{\log c - \log p}{\log v}$ .      17. 1.48 mg.      19. 0.0604 grams.

## Section 165. Pages 367-368

1. (a) \$60; (b) \$560.      3. (a) \$23.44; (b) \$773.44.  
 5. (a) \$237.50; (b)  $5 \frac{26}{100}\%$  approx.  
 7. (a) \$932.50; (b)  $4 \frac{82}{100}\%$  approx.  
 9. \$208.33.      11. \$1,239.      13. (a)  $P = A/(1 + rn)$ ; (b)  $(A - P)/Pn$ ;  
 (c)  $(A - P)/Pr$ .

## Section 166. Pages 370-371

1. (a) \$657.95; (b) \$659.75.      3. (a) \$339.96; (b) \$346.07.  
 5. (a) \$337.78; (b) \$336.49.      7. (a) \$489.78; (b) \$488.10.      9. 27.  
 11. 17.      13. 32.      15. 4% per annum.      17. 5% per annum.

## Section 167. Page 372

1. 4.04%.      3. 6.09%.      5. 8.16%.      7. 4.06%.      9. 6.14%.



## Section 168. Pages 374-375

1. (a) \$3,519.42; (b) \$2,494.98.      3. \$5,236.52.      5. \$8,113.62.  
 7. \$241.67.      11. \$679.34.

## Section 169. Pages 378-379

1. 20.      3. 12.      5. 1.      7. 6.

## Section 170. Page 380

1.  $x - 3$ .      3.  $x - 2$ .      5.  $2x - 1$ .

## Section 171. Pages 383-387

1.  $\frac{5}{2x-1} - \frac{2}{x+2}$ .      3.  $\frac{3}{2x+3} - \frac{2}{x-2}$ .  
 5.  $\frac{1}{x+1} + \frac{2}{x-2} - \frac{1}{x+3}$ .  
 7.  $x^2 + 1 + \frac{3}{x+2} - \frac{1}{x+3}$ .      9.  $5 + \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3}$ .  
 11.  $\frac{3}{x+1} + \frac{2}{(x+1)^2}$ .  
 13.  $\frac{2}{x-2} - \frac{1}{(x-2)^2} + \frac{1}{(x-2)^3}$ .      15.  $\frac{1}{2x+1} + \frac{2}{x+3} - \frac{1}{(x+3)^2}$ .  
 17.  $\frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{x-2} + \frac{1}{(x-2)^2}$ .  
 19.  $3 + \frac{6}{2x+4} + \frac{1/2}{x-4} - \frac{1/2}{(x-4)^2}$ .  
 21.  $\frac{2x-3}{x^2+x+1} - \frac{1}{x-1}$ .      23.  $\frac{3x-1}{x^2-x+1} + \frac{2}{x+1}$ .  
 25.  $\frac{3}{x} + \frac{2x-1}{x^2+x+3}$ .  
 27.  $\frac{2}{x^2+1} - \frac{1}{2x^2-x+3}$ .      29.  $x + 2 + \frac{2x-1}{x^2+3} - \frac{x-3}{x^2-x+2}$ .  
 31.  $4 + \frac{2}{x} - \frac{3}{x^2} + \frac{x-2}{x^2+x+2}$ .      33.  $\frac{1}{x^2+1} + \frac{2x-3}{(x^2+1)^2}$ .  
 35.  $\frac{2}{x} + \frac{x}{x^2+3} + \frac{3}{(x^2+3)^2}$ .      37.  $\frac{1}{(x^2+1)^2} + \frac{3x+1}{(x^2+2)^2}$ .  
 39.  $\frac{1}{x-1} + \frac{2x}{x^2+1} - \frac{1}{(x^2+1)^2}$ .

## Section 174. Pages 392-393

1. (a) 3.1, 3.01, 3.001, 3.0001; (b) 3.      3. (a)  $\frac{3}{2}, \frac{9}{4}, \frac{15}{8}, \frac{33}{16}$ ; (b) 2.  
 5. (a)  $\frac{1}{2}, \frac{5}{12}, \frac{5}{12}, \frac{17}{48}$ ; (b)  $\frac{1}{3}$ .      7. (a)  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ ; (b) 1.  
 9. (a)  $\frac{1}{3}, \frac{3}{5}, \frac{7}{9}, \frac{15}{17}$ ; (b) 1.      11. (a)  $\frac{4}{5}, \frac{9}{7}, \frac{16}{9}, \frac{25}{11}$ ; (b) diverges.



# Index

## A

Abel, N. H., 258, 328  
 Abscissa, 78  
 Absolute convergence, 402  
 Absolute value, 27, 344  
   of complex number, 206  
 Ahmes, 7  
 Algebraic expression, 59  
 Algebraic sum, 30, 60  
 Algorithm, 91  
   division, 94  
   Euclidean, 376  
 Al-jabr, 7  
 Al-Khowarizmi, 91  
 Allowable values, 67  
 Alternating series, 401  
 Alternation, 188  
 Amicable numbers, 17  
 Amount, 366  
   compound, 368  
   of annuity, 372  
 Amplitude of complex number, 206  
 Angle of complex number, 206  
 Annuity, 372  
   amount of, 372  
   present value of, 372  
 Apollonius of Perga, 177  
 Arabic notation, *see* Hindu-Arabic  
 Argument of complex number, 206  
 Arithmetic means, 331  
 Arithmetic progression, 330  
   of  $n$ th order, 341  
 Associative law:  
   for addition, 9  
   for multiplication, 10  
Axiom of mathematical induction, 315  
 Axioms of algebra, 51ff.  
 Axis, 77

## B

Binomial, 61  
 Binomial formula, 324  
 Binomial series, 328  
 Binomial theorem, 323  
 Biquadratic equation, 72  
 Biquadratic polynomial, 65  
 Briggs, H., 351, 363  
 Bürgi, J., 363

## C

Cancellation, 18  
 Cantor, G., 41  
 Cardano, 225, 258  
 Characteristic, 352  
 Ch'in Chiu-shao, 252  
 Circle, 177  
 Closed system under an operation, 12  
 Closure, law of:  
   for addition, 8  
   for multiplication, 10  
 Coefficient, 61  
   leading, 214  
 Coefficients of polynomial, 65  
 Cofactor of an element, 282  
 Cologarithm, 361  
 Combination, 303  
 Common difference, 330  
 Common ratio, 333  
 Commutative law:  
   for addition, 8  
   for multiplication, 10  
 Complex fraction, 110  
 Complex number, 195  
 Complex numbers, 48  
   difference of, 197  
   equal, 196  
   product of, 197  
   quotient of, 198  
   sum of, 197  
 Composition, 188  
 Conclusion, 1  
 Condition, necessary and sufficient, 2  
 Conditional equation, 67  
 Conic sections, 177  
 Conjugate complex numbers, 196  
 Conjugate surd, 153  
 Consistent system of equations, 287  
 Consistent units, 189  
 Constant, 59  
   of proportionality, 191  
 Constant function, 76  
 Constant term, 214  
 Continuous function, 82  
 Converge:  
   absolutely, 402  
   conditionally, 402  
 Convergent sequence, 391  
 Convergent series, 393



Converse, 2  
 Conversion period, 368  
 Coordinates, 78  
   polar, 205  
   rectangular, 78  
 Cramer, G., 283  
 Cramer's rule, 264, 268, 283  
 Cube, 34  
 Cube root, 35, 57  
 Cubic equation, 72, 214  
 Cubic polynomial, 65  
 Curve fitting, 416 ff.

## D

Decimal expression, 42, 397 ff.  
   periodic, 43  
   recurring, 43  
 Dedekind, R., 41  
 Deduction, 1  
 Deductive reasoning, 1  
 Defined function, 73  
 Degree:  
   of equation, 72  
   of monomial, 61  
   of polynomial, 64  
 del Ferro, 258  
 de Moivre, A., 209  
 de Moivre's theorem, 209, 321  
 Denominator, 17  
 Dependent equations, 128  
 Dependent variable, 73  
*Depends*, 73, 76  
 Depressed equation, 222  
 Descartes, R., 78, 145, 177, 225, 243  
 Descartes' rule of signs, 240  
   extended, 242  
 Determinant:  
   elements of, 263, 265  
   expansion of, 265, 272  
   of order 2, 263  
   of order  $n$ , 271  
   principal diagonal of, 272  
   value of, 265, 272  
 Deviation, 422, 425  
 Difference of natural numbers, 12  
 Differences of  $r$ th order, 339  
 Digits, significant, 356  
 Dimensionless number, 189  
 Diophantus, 138, 225  
 Directed numbers, 24  
 Directly proportional, 191  
 Discount, 370  
   compound, 370  
   simple, 367  
 Discriminant, 160  
 Distributive law, 10  
 Divergent sequence, 391  
 Divergent series, 393

Divisible, 13  
   exactly, 93, 94  
 Division, 188  
   algorithm, 91 ff., 94  
   by monomials, 90 ff.  
   by zero, 32 ff.  
   of natural numbers, 12  
   synthetic, 219  
   with remainder:  
     for integers, 92  
     for polynomials, 93  
 Double root, 225  
   of quadratic, 156

## E

Effective rate, 371  
 Elements of determinant, 263, 265  
 Elimination theory, 263 ff.  
 Ellipse, 177  
 Equal complex numbers, 196  
 Equation:  
   conditional, 67  
   cubic, 214  
   depressed, 222  
   empirical, 416  
   identical, 67  
   irrational, 168  
   linear, 114, 214  
   polynomial, 214  
   quadratic, 214  
   quartic, 214  
   rational integral, 214  
 Equations:  
   dependent, 128  
   incompatible, 128  
   inconsistent, 128  
   polynomial, 71  
   rational integral, 71  
   simultaneous, 127  
 Equivalent statements, 2  
 Eratosthenes, sieve of, 14  
 Error, 422, 425  
 Essentially the same polynomials, 379  
 Euclid, 187  
 Euclidean algorithm, 376  
 Even integer, 33  
 Even natural number, 14  
 Expansion, binomial, 322  
 Expectation, mathematical, 312  
 Exponent, 34, 56  
 Exponential form, 427  
 Extraneous root, 71, 167  
 Extrapolation, 430  
 Extremes, 331, 334, 336  
   of proportion, 187

## F

Factor, 13, 93, 94, 96  
   completely, 97



Factor (*cont.*):  
 of an integer, 33  
 of proportionality, 189  
 theorem, 217  
 Fermat, 177, 307  
 Ferrari, 258  
 Fourth proportional, 187  
 Fraction, 17  
   complex, 110  
   improper, 381  
   proper, 381  
   simplest form of, 18  
 Fractional expression, 60  
 Fractions:  
   difference of, 22  
   equal, 18  
   product of, 19  
   quotient of, 20  
   sum of, 21  
 Franklin, Benjamin, 53  
 Function, 73  
   continuous, 82  
 Fundamental theorem of algebra, 215

G

Galois, E., 259  
 Gauss, K. F., 215  
 General equation, 72  
 General polynomial, 65  
 General term, 391  
   of sequence, 315  
 Geometric mean, 334  
 Geometric means, 334  
 Geometric progression, 333  
 Geometric series, 337, 394  
 Goldbach's conjecture, 16  
 Graph:  
   of a function, 80  
   of an equation, 79  
 Gravitational constant, 192  
 Greater than:  
   for directed numbers, 28  
   for fractions, 22  
   for natural numbers, 12  
 Gregory, J., 413, 416

H

H.C.F., 15, 103, 379 footnote  
 Hamilton, W. R., 50  
 Harmonic mean, 336  
 Harmonic means, 336  
 Harmonic progression, 336  
 Harmonic series, 395  
 Highest common factor (H.C.F.), 15,  
   103, 379 footnote  
 Hindu-Arabic notation, 86

Homogeneous linear equation, 285  
 Horner, W. G., 252  
 Hyperbola, 177  
 Hypothesis, 1

I

*i*, 195  
 Identity, 67  
 Imaginary number, 49, 195  
 Imaginary part, 48, 195  
 Imaginary unit, 48, 195  
 Implies, 1  
 Incompatible equations, 128  
 Inconsistent equations, 128  
 Inconsistent system of equations, 287  
 Independent events, 295, 310  
 Independent variable, 73  
 Index:  
   of a radical, 147  
   of root, 36  
 Indirect method, 40  
 Induction, 3  
   axiom of mathematical, 316  
 Inductive logic, 319  
 Inductive reasoning, 3  
 Inequality, 344  
   absolute, 344  
   conditional, 344  
   sense of, 344  
 Infinite, becomes, 84  
 Infinite discontinuity, 85  
 Infinite sequence, 391  
 Infinite series, 328, 337, 391  
 Infinity of a function, 85  
 Integer, 32  
 Integers, 24  
   positive, 8  
 Interest, 366  
   compound, 368  
   period, 368  
   simple, 366  
 Interpolation, 407  
   by proportional parts, 248, 355, 407  
   formula:  
   Gregory-Newton, 412  
   Lagrange's, 411  
   linear, 248, 407  
 Inversely proportional, 192  
 Inversion, 188, 271  
 Irrational equation, 168  
 Irrational expression, 60  
 Irrational numbers, 41  
 Irreducible, 380

K

Kronecker, L., 52



## L

L.C.D., 109  
 L.C.M., 15, 104  
 Lagrange, J. L., 411  
 Leading coefficient, 214  
 Least common denominator, 109  
 Leibnitz, G. W., 273  
 Length of complex number, 206  
*Less than*, for natural numbers, 12  
 Like terms, 61  
Limit of sequence, 391  
 Linear equation, 72, 114, 214  
 Linear function, 74  
 Linear interpolation, 248, 407  
 Linear polynomial, 65  
 $\ln x$ , 363  
 Logarithm:  
     Briggsian, 351  
     common, 351  
     natural, 362  
     to base  $b$ , 349  
 Logical consequence, 1  
 Lower bound for roots, 239  
 Lowest common multiple (L.C.M.), 15, 104  
 Lowest terms, 105  
     reduction of fraction to, 18

## M

Mantissa, 352  
 Many-valued function, 73  
 Mathematical expectation, 312  
 Mathematical induction, axiom of, 316  
 Maximum, 172  
 Mean:  
     arithmetic, 331  
     geometric, 334  
     harmonic, 336  
     proportional, 187  
 Means:  
     arithmetic, 331  
     geometric, 334  
     harmonic, 336  
     of proportion, 187  
 Member of an equation, 10  
 Minimum, 172  
 Minor of an element, 278  
 Mixed expression, 111  
 Modulus, 362  
     of complex number, 206  
 Monomial, 61  
 Multinomial, 61  
 Multiple, 13  
 Multiplicity of a root, 224  
 Multiplier in synthetic division, 249  
 Mutually exclusive events, 296, 310

## N

Napier, J., 363  
 Natural logarithm, 362  
 Natural numbers, 8  
 Necessary condition, 2  
 Negative numbers, 24  
 Negative of a directed number, 29  
 Newton, Isaac, 145, 255, 328, 413, 416  
 Nominal rate, 371  
 Normal equation, 423, 424, 425  
 $n$ th term, 330, 333, 391  
     of a sequence, 315  
 Numbers:  
     amicable, 17  
     complex, 48, 195  
     dimensionless, 189  
     directed, 24  
     imaginary, 49, 195  
         pure, 48, 195  
     irrational, 41  
     natural, 8  
     perfect, 17  
     pure, 189  
     real, 41, 195  
     signed, 24  
 Numerator, 17

## O

Odd natural number, 14  
 Odds, 312  
 Order of a radical, 147  
 Ordinate, 78  
 Origin, 77

## P

Parabola, 177  
 Partial fractions, 381  
 Partial sums, 393  
 Pascal, B., 307  
 Pascal's triangle, 322  
 Payment period, 372  
 Perfect  $n$ th power, 148  
 Perfect number, 17  
 Periodic decimal, 338  
 Permutation, 298  
 Plotting, 78  
 Polar coordinates, 205  
 Polar form, 206  
 Polynomial, 60  
     equation, 71, 214  
     function, 74  
     irreducible, 380  
     prime, 97, 380  
 Polynomials:  
     essentially the same, 379  
     relatively prime, 380



Positional notation, 87  
 Positive numbers, 24  
 Power:  
     form, 427  
      $n$ th, 34, 56  
     perfect  $n$ th, 148  
     series, 403  
 Present value, 370  
     of annuity, 372  
 Prime, 14, 378  
     polynomial, 97, 380  
     relatively, 15, 378  
 Principal, 366  
 Principal  $n$ th root, 36, 57, 147  
 Probability, 306  
     *a posteriori*, 309  
     *a priori*, 309  
     empirical, 309  
     statistical, 309  
 Product of natural numbers, 9  
 Progression:  
     arithmetic, 330  
         of  $r$ th order, 341  
     geometric, 333  
     harmonic, 336  
 Proportion, 187  
 $p$ -series, 395  
 Pure equation of  $n$ th degree, 213  
 Pure imaginary number, 195  
 Pure number, 189  
 Pure quadratic equation, 155  
 Pythagoras, 40

Q

Quadratic equation, 72, 214  
 Quadratic function, 74  
 Quadratic polynomial, 65  
 Quadratic surd, 148  
 Quartic equation, 72, 214  
 Quartic polynomial, 65  
 Quintic equation, 72  
 Quintic polynomial, 65  
 Quotient of natural numbers, 12

R

Radical, 147  
     sign, 36  
     simplest form of, 149  
 Radicand, 147  
 Range, 73  
 Ratio, 187  
 Rational expression, 60  
 Rational fractional expression, 60  
 Rational function, 74  
 Rational integral equation, 71, 214  
 Rational integral expression, 60  
 Rational integral function, 74

Rational number, 32  
 Rationalizing the denominator, 148  
 Real number, 41, 195  
 Real part, 48, 195  
 Reciprocal, 33  
 Rectangular coordinates, 78  
 Rectangular form of complex number, 195  
*Reductio ad absurdum*, 40  
 Relative frequency, 307  
 Relatively prime, 15, 378, 380  
 Remainder theorem, 216  
 Rent, 372  
 Residual, 422, 425  
 Resolvent cubic, 261  
 Resultant of 2 polynomials, 291  
 Reversible steps, 68  
 Root:  
     double, 156  
     extraneous, 71, 167  
      $n$ th, 35, 57, 147  
     of equation, 67, 82, 214  
     of multiplicity  $k$ , 224  
     principal  $n$ th, 147  
 Rounded off, 44, 357  
 Ruffini, 252  
 Rule of signs, 28  
     for division, 31  
     for multiplication, 30  
 Rule of three, 188

S

Satisfy, 67  
 Seki-Kowa, 273  
 Sense of inequality, 344  
 Sequence, 315  
     infinite, 391  
 Series:  
     alternating, 401  
     geometric, 337, 394  
     harmonic, 395  
     infinite, 391 ff.  
 Sieve of Eratosthenes, 14  
 Signed numbers, 24  
 Significant figures, 356  
 Similar radicals, 150  
 Simple root, 225  
 Simplest form:  
     of a fraction, 105  
     of a radical, 149  
 Simplify a rational integral expression, to,  
     □  
 Simultaneous equations, 127  
 Single-valued function, 73  
 Slope, 120  
 Solution:  
     of equation, 67, 123, 214  
     of system of equations, 123



Solve an equation, 67, 114, 214

Square, 34

Square root, 35, 57

Standard form:

of a positive real number, 142

of complex number, 195

of polynomial, 64

Statistical probability, 309

Stifel, 225

Substitution, principle of, 10

Subtraction of directed numbers, 28

Sufficient condition, 2

Sum, algebraic, 30

Surd, 148

conjugate, 153

Sylvester, J. J., 293

Synthetic division, 219

## T

Tabular difference, 355

Tartaglia, 258

Term, 30, 60

Third proportional, 187

Total number of sets, 324

Trigonometric form, 206

Trinomial, 61

Triple root, 225

Trivial solution, 285

## U

Undetermined coefficients, method of,  
409

Unique factorization theorem, 15

Unknown, 59

Unsigned numbers, 25

Upper bound for roots, 239

## V

Valid, 1

Variable, 59

Variation in sign, 240

*Varies directly as*, 191*Varies directly as the square*, 191*Varies inversely as*, 192*Varies inversely as the square*, 192*Varies jointly as*, 192

Vieta, 225

Vinculum, 58

Vlacq, A., 363

## W

Wallis, 177

Whole numbers, 24

## X

*x*-axis, 77*x*-coordinate, 78*x*-intercept, 122

## Y

*y*-axis, 77*y*-coordinate, 78*y*-intercept, 120

## Z

Zero, 24

division by, 32 ff.

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195